10.11 Suppose that a stock price S follows geometric Brownian motion with expected return μ and volatility σ:

$$dS = \mu S\,dt + \sigma S\,dz$$

What is the process followed by the variable S^n? Show that S^n also follows geometric Brownian motion. The expected value of S_T, the stock price at time T, is $Se^{\mu(T-t)}$. What is the expected value of S_T^n?

***10.12.** Suppose that x is the yield to maturity with continuous compounding on a discount bond that pays off \$1 at time T. Assume that x follows the process

$$dx = a(x_0 - x)\,dt + sx\,dz$$

where a, x_0, and s are positive constants and dz is a Wiener process. What is the process followed by the bond price?

***10.13.** Suppose that x is the yield on a perpetual government bond that pays interest at the rate of \$1 per annum. Assume that x is expressed with continuous compounding, that interest is paid continuously on the bond, and that x follows the process

$$dx = a(x_0 - x)\,dt + sx\,dz$$

where a, x_0, and s are positive constants and dz is a Wiener process. What is the process followed by the bond price? What is the expected instantaneous return (including interest and capital gains) to the holder of the bond?

APPENDIX 10A: DERIVATION OF ITO'S LEMMA

In this appendix we show how Ito's lemma can be regarded as a natural extension of other, simpler results. Consider a continuous and differentiable function G of a variable x. If Δx is a small change in x and ΔG is the resulting small change in G, it is well known that

G: G(x)

$$\Delta G \approx \frac{dG}{dx}\Delta x \qquad \text{(10A.1)}$$

In other words, ΔG is approximately equal to the rate of change of G with respect to x multiplied by Δx. The error involves terms of order Δx^2. If more precision is required, a Taylor series expansion of ΔG can be used:

Taylor series of G

$$\Delta G = \frac{dG}{dx}\Delta x + \frac{1}{2}\frac{d^2G}{dx^2}\Delta x^2 + \frac{1}{6}\frac{d^3G}{dx^3}\Delta x^3 + \cdots$$

$G(x) = G(a) + \mathrm{Sum}_{n=1}^{\mathrm{inf}} \frac{d^nG(a)}{dx^n} \frac{(x - a)^n}{n!}$

For a continuous and differentiable function G of two variables, x and y, the result analogous to equation (10A.1) is

G: G(x, y)

$$\Delta G \approx \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial y}\Delta y \qquad \text{(10A.2)}$$

and the Taylor series expansion of ΔG is

Taylor series G(x,y)

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial y}\Delta y + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\Delta x^2 + \frac{\partial^2 G}{\partial x\,\partial y}\Delta x\,\Delta y + \frac{1}{2}\frac{\partial^2 G}{\partial y^2}\Delta y^2 + \cdots \tag{10A.3}$$

In the limit as Δx and Δy tend to zero, equation (10A.3) gives

dx, dy —> 0

$$dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy \tag{10A.4}$$

A derivative is a function of a variable that follows a stochastic process. We now extend equation (10A.4) to cover such functions. Suppose that a variable x follows the general Ito process in equation (10.4):

Increment of Wiener process
dz = N(0, dt)
cov(dz(t), dz(τ)) = 0

Ito process dx

$$dx = a(x,\, t)\,dt + b(x,\, t)\,dz$$

and that G is some function of x and of time, t. By analogy with equation (10A.3), we can write

Price of derivative G: G(x, t)

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\Delta x^2 + \frac{\partial^2 G}{\partial x\,\partial t}\Delta x\,\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial t^2}\Delta t^2 + \cdots \tag{10A.6}$$

Equation (10A.5) can be discretized to

$$\Delta x = a(x,\, t)\Delta t + b(x,\, t)\underbrace{\epsilon\sqrt{\Delta t}}_{dz}$$

ε = N(0, 1)

or if arguments are dropped,

$$\Delta x = a\Delta t + b\epsilon\sqrt{\Delta t} \tag{10A.7}$$

This equation reveals an important difference between the situation in equation (10A.6) and the situation in equation (10A.3). When limiting arguments were used to move from equation (10A.3) to equation (10A.4), terms in Δx^2 were ignored because they were second-order terms. From equation (10A.7),

$$\Delta x^2 = b^2\epsilon^2\Delta t + \text{ terms of higher order in } \Delta t \tag{10A.8}$$

which shows that the term involving Δx^2 in equation (10A.6) has a component

var(ε) = E[ε – E(ε)]²
= E[ε² – 2ε E(ε) + [E(ε)]²]
= E(ε²) – 2 E(ε) E(ε) + [E(ε)]²
= E(ε²) –[E(ε)]²

order Δt and cannot be ignored.

e variance of a standardized normal distribution is 1.0. This means that

$$E(\epsilon^2) - [E(\epsilon)]^2 = 1$$

var(ε² dt) = dt² —> 0 as dt —> 0.
ε² dt —> E(ε² dt) = E(ε²) dt = dt as dt —> 0.

where E denotes expected value. Since $E(\epsilon) = 0$, it follows that $E(\epsilon^2) = 1$. The expected value of $\epsilon^2\Delta t$ is therefore Δt. It can be shown that the variance of $\epsilon^2\Delta t$ is of order Δt^2 and that as a result of this, $\epsilon^2\Delta t$ becomes nonstochastic and equal to its expected value of Δt as Δt tends to zero. It follows that the first term on the right-hand side of equation (10A.8) becomes nonstochastic and equal to $b^2\,dt$ as Δt tends to zero. Taking limits as Δx and Δt tend to zero in equation (10A.6), and

Ito's multiplication rule:

x	dz	dt
dz	dt	0
dt	0	0

Itô's Regel

•	$dz = \epsilon\sqrt{dt}$	dt
dz	dt	0
dt	0	0

using this last result, we therefore obtain

dt —> 0

$$dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial t}dt + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\,dt \tag{10A.9}$$

This is Ito's lemma. Substituting for dx from equation (10A.5), equation (10A.9) becomes

Ito's lemma

$$dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}b\,dz$$

$$\begin{aligned}
V(\varepsilon^2\Delta t) &= E\{[\varepsilon^2\Delta t - \underbrace{E\{\varepsilon^2\Delta t\}}_{\Delta t}]^2\} \\
&= E\{[\varepsilon^2\Delta t - \Delta t]^2\} \\
&= E\{\Delta t^2\cdot[\varepsilon^2-1]^2\} \\
&= \Delta t^2\, E\{[\varepsilon^2-1]^2\} \\
&= \Delta t^2\, E\{\varepsilon^4 - 2\varepsilon^2 + 1\} \\
&= \Delta t^2\{\underbrace{E\varepsilon^4}_{2} - 2\underbrace{E\varepsilon^2}_{1} + 1\} \\
&= \Delta t^2\{2 - 2\cdot 1 + 1\} = \Delta t^2
\end{aligned}$$

$$E\varepsilon^4 = E\{\varepsilon^2\cdot\varepsilon^2\} = \underbrace{cov(\varepsilon^2,\varepsilon^2)}_{1} + \underbrace{E\varepsilon^2}_{1}\cdot\underbrace{E\varepsilon^2}_{1} = 2$$

$$cov(x,y) = E(x\cdot y) - Ex\cdot Ey$$

1. **Ito process:** $dx = a(x, t)\,dt + b(x, t)\,dz$

2. **Taylor series** with **two** terms for $G(x, t)$:

$$dG = \frac{dG}{dx}dx + \frac{dG}{dt}dt + \frac{1}{2}\frac{d^2G}{dx^2}dx^2 + \frac{d^2G}{dx\,dt}dx\,dt + \frac{1}{2}\frac{d^2G}{dt^2}dt^2$$

3. Insert Ito process into Taylor series, apply **Ito's multiplication rule**:

$dx^2 = [a\,dt + b\,dz]^2 = a^2\,dt^2 + 2\,a\,b\,dt\,dz + b^2\,dz^2 = b^2\,dt$

$dx\,dt = [a\,dt + b\,dz]\,dt = a\,dt^2 + b\,dz\,dt = 0$

$$dG = \frac{dG}{dx}[a\,dt + b\,dz] + \frac{dG}{dt}dt + \frac{1}{2}\frac{d^2G}{dx^2}dx^2 + \frac{d^2G}{dx\,dt}dx\,dt + \frac{1}{2}\frac{d^2G}{dt^2}dt^2$$

Ito's multiplication rule:

x	dz	dt
dz	**dt**	**0**
dt	**0**	**0**

***13.11.** Using risk-neutral valuation arguments, show that an option to exchange one IBM share for two Kodak shares in six months has a value that is independent of the level of interest rates.

13.12. Consider a commodity with constant volatility, σ. Assuming that the risk-free interest rate is constant, show that in a risk-neutral world,

$$\ln S_T \sim \phi\left[\ln F - \frac{\sigma^2}{2}(T-t),\ \sigma\sqrt{T-t}\right]$$

where S_T is the value of the commodity at time T and F is the futures price for a contract maturing at time T.

***13.13.** What is the formula for the price of a European call option on a foreign index when the strike price is in dollars and the index is translated into dollars at a predetermined exchange rate? What difference does it make if the index is translated into dollars at the exchange rate prevailing at the time of exercise?

APPENDIX 13A: GENERALIZATION OF ITO'S LEMMA

Ito's lemma as presented in Appendix 10A provides the process followed by a function of a single stochastic variable. Here we present a generalized version of Ito's lemma for the process followed by a function of several stochastic variables.

Suppose that a function, f, depends on the n variables $x_1, x_2, \ldots, x_n$ and time, t. Suppose further that x_i follows an Ito process with instantaneous drift a_i and instantaneous variance b_i^2 $(1 \le i \le n)$, that is,

Ito processes:
$a_i(t, x_1, \ldots, x_n)$
$b_i(t, x_1, \ldots, x_n)$
$\rho_{ij} = \text{corr}(dz_i, dz_j)$

Derivative price f: $f(t, x_1, x_2, \ldots, x_n)$

$$dx_i = a_i\,dt + b_i\,dz_i \tag{13A.1}$$

where dz_i is a Wiener process $(1 \le i \le n)$. Each a_i and b_i may be any function of all the x_i's and t. A Taylor series expansion of f gives

Taylor series

$$\Delta f = \sum_i \frac{\partial f}{\partial x_i}\Delta x_i + \frac{\partial f}{\partial t}\Delta t + \frac{1}{2}\sum_i\sum_j \frac{\partial^2 f}{\partial x_i\,\partial x_j}\Delta x_i\,\Delta x_j + \frac{1}{2}\sum_j \frac{\partial^2 f}{\partial x_i\,\partial t}\Delta x_i \Delta t + \cdots \tag{13A.2}$$

Equation (13A.1) can be discretized as

$$\Delta x_i = a_i\,\Delta t + b_i \overbrace{\epsilon_i\sqrt{\Delta t}}^{dz_i}$$

where ϵ_i is a random sample from a standardized normal distribution. The correlation, ρ_{ij}, between dz_i and dz_j is defined as the correlation between ϵ_i and ϵ_j. In Appendix 10A it was argued that

$$\lim_{\Delta t \to 0} \Delta x_i^2 = b_i^2\,dt$$

$\cdot$	dt	dz_i	dz_j
dt	0	0	0
dz_i	0	dt	$\rho_{ij}\,dt$
dz_j	0	$\rho_{ij}\,dt$	dt

Similarly,

$$\lim_{\Delta t \to 0} \Delta x_i \, \Delta x_j = b_i b_j \rho_{ij} \, dt$$

As $\Delta t \to 0$, the first three terms in the expansion of Δf in equation (13A.2) are of order Δt. All other terms are of higher order. Hence

$$df = \sum_i \frac{\partial f}{\partial x_i} dx_i + \frac{\partial f}{\partial t} dt + \frac{1}{2} \sum_i \sum_j \frac{\partial^2 f}{\partial x_i \, \partial x_j} b_i b_j \rho_{ij} \, dt$$

This is the generalized version of Ito's lemma. Substituting for dx_i from equation (13A.1) gives

Ito's lemma

$$df = \left(\sum_i \frac{\partial f}{\partial x_i} a_i + \frac{\partial f}{\partial t} + \frac{1}{2} \sum_i \sum_j \frac{\partial^2 f}{\partial x_i \, \partial x_j} b_i b_j \rho_{ij} \right) dt + \sum_i \frac{\partial f}{\partial x_i} b_i \, dz_i \tag{13A.3}$$

APPENDIX 13B: DERIVATION OF THE GENERAL DIFFERENTIAL EQUATION SATISFIED BY DERIVATIVES

θ_j : n state variables (snow, securities)□
f_j: $f_j(t, \theta_1, \ldots, \theta_n)$, (n+1) traded securities

Consider a certain derivative security that depends on n state variables and time, t. We make the assumption that there are a total of at least $n + 1$ traded securities (including the one under consideration) whose prices depend on some or all of the n state variables. In practice, this is not unduly restrictive. The traded securities may be options with different strike prices and exercise dates, forward contracts, futures contracts, bonds, stocks, and so on. We assume that no dividends or other income is paid by the $n + 1$ traded securities.[10] Other assumptions are similar to those made in Section 11.4 to derive the Black–Scholes equation.

The n state variables are assumed to follow continuous-time Ito diffusion processes. We denote the ith state variable by θ_i $(1 \le i \le n)$ and suppose that

$m_i = m_i(\theta_1, \ldots, \theta_n, t)$
$s_i = s_i(\theta_1, \ldots, \theta_n, t)$

$$d\theta_i = m_i \theta_i \, dt + s_i \theta_i \, dz_i \tag{13B.1}$$

state variables

where dz_i is a Wiener process and the parameters, m_i and s_i, are the expected growth rate in θ_i and the volatility of θ_i. The m_i and s_i can be functions of any of the n state variables and time. Other notation used is as follows:

- ρ_{ik}: correlation between dz_i and dz_k $(1 \le i, k \le n)$
- f_j: price of the jth traded security $(1 \le j \le n + 1)$
- r: instantaneous (i.e., very short-term) risk-free rate may be a state variable

[10]This is not restrictive. A non-dividend-paying security can always be obtained from a dividend-paying security by reinvesting the dividends in the security.

One of the f_j is the price of the security under consideration. The short-term risk-free rate, r, may be one of the n state variables.

Since the $n + 1$ traded securities are all dependent on the θ_i, it follows from Ito's lemma in Appendix 13A that the f_j follow diffusion processes:

$j = 1, \dots, n+1$

$$df_j = \mu_j f_j \, dt + \sum_{i=1}^{n} \sigma_{ij} f_j \, dz_i \tag{13B.2}$$

where nach (12A.3)

$$\mu_j f_j = \frac{\partial f_j}{\partial t} + \sum_i \frac{\partial f_j}{\partial \theta_i} m_i \theta_i + \frac{1}{2} \sum_{i,k} \rho_{ik} s_i s_k \theta_i \theta_k \frac{\partial^2 f_j}{\partial \theta_i \partial \theta_k} \tag{13B.3}$$

jth security

ith state variable

$$\sigma_{ij} f_j = \frac{\partial f_j}{\partial \theta_i} s_i \theta_i \tag{13B.4}$$

$\ominus$ falls $\frac{\partial f_i}{\partial \theta_i} < 0$

In these equations, μ_j is the instantaneous mean rate of return provided by f_j and σ_{ij} is the component of the instantaneous standard deviation of the rate of return provided by f_j, which may be attributed to the θ_i.

Since there are $n + 1$ traded securities and n Wiener processes in equation (13B.2), it is possible to form an instantaneously riskless portfolio, Π, using the securities. Define k_j as the amount of the jth security in the portfolio, so that

$$d\Pi = \sum_{j=1}^{n+1} k_j \mu_j f_j \, dt + \sum_{j=1}^{n+1} \sum_{i=1}^{n} k_j \sigma_{ij} f_j \, dz_i$$

$$\Pi = \sum_{j=1}^{n+1} k_j f_j \Rightarrow d\Pi = \sum_{j=1}^{n+1} k_j \, df_j \tag{13B.5}$$

The k_j must be chosen so that the stochastic components of the returns from the securities are eliminated. From equation (13B.2) this means that

Bedingung, damit $\sum_{j=1}^{n+1} \sum_{i=1}^{n} k_j \sigma_{ij} f_j \, dz_i = 0$

$$\sum_{j=1}^{n+1} k_j \sigma_{ij} f_j = 0 \tag{13B.6}$$

$i = 1, \dots, n$ state variables

n Gleich.

for $1 \le i \le n$. The return from the portfolio is then given by

$$d\Pi = \sum_{j=1}^{n+1} k_j \mu_j f_j \, dt$$

The cost of setting up the portfolio is $\sum_j k_j f_j$. If there are no arbitrage opportunities, the portfolio must earn the risk-free interest rate, so that

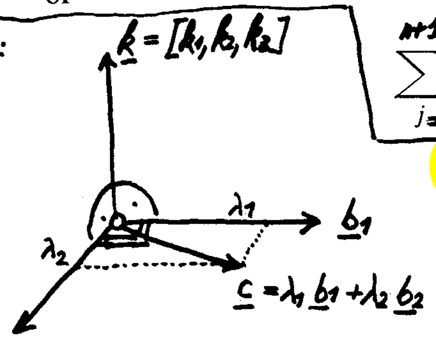
$d\Pi = r \Pi \, dt$

$$\underbrace{\sum_{j=1}^{n+1} k_j \mu_j f_j \, dt}_{d\Pi} = r \underbrace{\sum_{j=1}^{n+1} k_j f_j}_{\Pi} \cdot dt \tag{13B.7}$$

or

1 Gleich.

$n+1 = 3$:



$$\sum_{j=1}^{n+1} k_j f_j (\mu_j - r) = 0 \tag{13B.8}$$

ZURMÜHL, Satz 4: Ein lineares, homogenes System von n Gleichungen in n Unbekannten hat dann und nur dann nichttriviale Lösungen, falls seine $n \times n$-Matrix singulär ist, d.h. $\det \underline{A} = 0$.

Equations (13B.6) and (13B.8) can be regarded as $n+1$ homogeneous linear equations in the k_j's. The k_j's are not all zero. From a well-known theorem in linear algebra, equations (13B.6) and (13B.8) can be consistent only if *

$$f_j(\mu_j - r) = \sum_{i=1}^{n} \lambda_i \sigma_{ij} f_j \tag{13B.9}$$

or

Parameter aus Linearkombination: Marktpreis des Faktorrisikos

$$\mu_j - r = \sum_{i=1}^{n} \lambda_i \sigma_{ij} \qquad j=1,\dots,n+1 \tag{13B.10}$$

for some λ_i ($1 \le i \le n$), which are dependent only on the state variables and time. This proves the result in equation (13.13).

Substituting from equations (13B.3) and (13B.4) into equation (13B.9), we obtain

$$\frac{\partial f_j}{\partial t} + \sum_i \frac{\partial f_j}{\partial \theta_i} m_i \theta_i + \frac{1}{2} \sum_{i,k} \rho_{ik} s_i s_k \theta_i \theta_k \frac{\partial^2 f_j}{\partial \theta_i \partial \theta_k} - r f_j = \sum_i \lambda_i \frac{\partial f_j}{\partial \theta_i} s_i \theta_i$$

which reduces to

$$\frac{\partial f_j}{\partial t} + \sum_i \theta_i \frac{\partial f_j}{\partial \theta_i}(m_i - \lambda_i s_i) + \frac{1}{2} \sum_{i,k} \rho_{ik} s_i s_k \theta_i \theta_k \frac{\partial^2 f_j}{\partial \theta_i \partial \theta_k} = r f_j$$

Dropping the subscripts to f, we deduce that any security whose price, f, is contingent on the state variables $\theta_i (1 \le i \le n)$ and time, t, satisfies the second-order differential equation $f = f(\theta_1, \theta_2, \dots, \theta_n, t)$ fundamentale DGL

$\varphi(t, \theta_i)$

$$\frac{\partial f}{\partial t} + \sum_i \theta_i \frac{\partial f}{\partial \theta_i}(m_i - \lambda_i s_i) + \frac{1}{2} \sum_{i,k} \rho_{ik} s_i s_k \theta_i \theta_k \frac{\partial^2 f}{\partial \theta_i \partial \theta_k} = r f \tag{13B.11}$$

The particular derivative security that is obtained is determined by the boundary conditions that are imposed on equation (13B.11).

Gleichungssystem (13B.6) & (13B.8)

$$\begin{array}{l} (13B.6) \\ \\ \\ (13B.8) \end{array} \begin{bmatrix} (\sigma_{11} f_1) & (\sigma_{12} f_2) & \cdots & (\sigma_{1,n+1} f_{n+1}) \\ (\sigma_{21} f_1) & (\sigma_{22} f_2) & \cdots & (\sigma_{2,n+1} \cdot f_{n+1}) \\ \vdots & \vdots & & \vdots \\ (\sigma_{n1} \cdot f_1) & (\sigma_{n2} f_2) & \cdots & (\sigma_{n,n+1} \cdot f_{n+1}) \\ \hdashline (\mu_1 - r) f_1 & (\mu_2 - r) f_2 & \cdots & (\mu_{n+1} - r) f_{n+1} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \\ \hdashline k_{n+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \hdashline 0 \end{bmatrix} \Rightarrow \underset{(n \times (n+1))}{\underline{B}} \equiv \begin{bmatrix} \underline{b}_1 \\ \underline{b}_2 \\ \vdots \\ \underline{b}_n \\ \hdashline \underline{c} \end{bmatrix} \cdot \underline{k} = \underline{0}$$

Wenn $\underline{B}_{n \times (n+1)} \cdot \underline{k} = 0$, dann steht jeder Zeilenvektor $\underline{b}_1, \underline{b}_2, \dots, \underline{b}_n$ senkrecht auf der Lösung $\underline{k}$. Die Matrix $\underline{B}_{n \times (n+1)}$ spannt einen n-dimensionalen Unterraum auf. Da $\underline{c} \cdot \underline{k} = 0$, steht auch $\underline{c}$ senkrecht auf $\underline{k}$ und liegt deshalb im selben Unterraum wie $\underline{B}$. Deshalb kann $\underline{c}$ als Linearkombination dargestellt werden: $\underline{c} = \sum_{i=1}^{n} \lambda_i \cdot \underline{b}_i \Rightarrow c_j = \sum_{i=1}^{n} \lambda_i \cdot b_{ij} \Rightarrow (\mu_j - r) f_j = \sum_{i=1}^{n} \lambda_i \cdot \sigma_{ij} f_j$

$\Leftarrow$