

The Money Market Account as the Numeraire

Both the financial instrument, f , and the money market account, g , must depend on the same source of risk or factor, respectively. The common factor is the instantaneous interest rate, r , which follows the stochastic differential equation below.

$$dr = m r dt + s r dz \quad (1)$$

The financial derivative, f , is then a function of the interest rate, r , and time, t . From Ito's lemma, the change of the price of the financial derivative can be written as follows:

$$\begin{aligned} df &= \mu_f f dt + \sigma_f f dz, \quad \text{where} \\ \mu_f f &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial r} m r + \frac{1}{2} \frac{\partial^2 f}{\partial r^2} s^2 r^2 \\ \sigma_f f &= \frac{\partial f}{\partial r} s r \end{aligned} \quad (2)$$

The money market account grows according to:

$$\begin{aligned} dg &= r g dt \\ &= \mu_g g dt, \quad (\mu_g = r, \sigma_g \equiv 0) \end{aligned} \quad (3)$$

Next, the relative price of the financial derivative with respect to the money market account is given by:

$$\phi = \frac{f}{g} \quad (4)$$

From Ito's lemma, the expression for the relative change of ϕ can be written as follows (see the document "Option to exchange one asset for another"):

$$\begin{aligned} \frac{d\phi}{\phi} &= [\mu_f - \mu_g - \sigma_f \sigma_g + \sigma_g^2] dt + [\sigma_f - \sigma_g] dz \\ &= [\mu_f - \mu_g] dt + \sigma_f dz, \quad \text{by } \sigma_g \equiv 0 \end{aligned} \quad (5)$$

Setting the growth rates to the ones required by the martingale approach, that is,

$$\begin{aligned} \mu_f &= r + \lambda \sigma_f = r, \quad \text{by } \lambda = \sigma_g \equiv 0 \\ \mu_g &= r + \lambda \sigma_g = r, \quad \text{by } \lambda = \sigma_g \equiv 0 \end{aligned} \quad (6)$$

Finally, upon inserting equation (6) into equation (5), the relative change of ϕ turns out to be a martingale as requested:

$$\frac{d\phi}{\phi} = \sigma_f dz \quad (7)$$

In summary, both financial instruments depend on the common factor, namely the instantaneous risk-free interest rate. The money market account, however, depends directly on the risk-free interest rate through the trend rate.