Term Structure Models

- Black’s model is concerned with describing the probability distribution of a single variable at a single point in time.
- A term structure model describes the evolution of the whole yield curve.
Use of Risk-Neutral Arguments

• The process for the instantaneous short rate, $r$, in the traditional risk-neutral world defines the process for the whole zero curve in this world.

• If $P(t, T)$ is the price at time $t$ of a zero-coupon bond maturing at time $T$

$$ P(t, T) = \hat{E}\left[ e^{-\tilde{r}(T-t)} \right] $$
Equilibrium Models

Rendleman & Bartter:

\[ dr = \mu r \, dt + \sigma r \, dz \]

Vasicek:

\[ dr = a(b - r) \, dt + \sigma \, dz \]

Cox, Ingersoll, & Ross (CIR):

\[ dr = a(b - r) \, dt + \sigma \sqrt{r} \, dz \]
Mean Reversion
(Figure 23.1, page 539)

HIGH interest rate has negative trend

LOW interest rate has positive trend

Reversion Level
Alternative Term Structures in Vasicek & CIR
(Figure 23.2, page 540)
Equilibrium vs No-Arbitrage Models

- In an equilibrium model today’s term structure is an output
- In a no-arbitrage model today’s term structure is an input
Developing No-Arbitrage Model for $r$

A model for $r$ can be made to fit the initial term structure by including a function of time in the drift.
Ho and Lee

\[ dr = \theta(t)dt + \sigma dz \]

- Many analytic results for bond prices and option prices
- Interest rates normally distributed
- One volatility parameter, \( \sigma \)
- All forward rates have the same standard deviation
Hull and White Model

\[ dr = [\theta(t) - ar]dt + \sigma dz \]

• Many analytic results for bond prices and option prices
• Two volatility parameters, \( a \) and \( \sigma \)
• Interest rates normally distributed
• Standard deviation of a forward rate is a declining function of its maturity
Diagrammatic Representation of Hull and White

Short Rate

Forward Rate Curve

Time

Options, Futures, and Other Derivatives, 5th edition © 2002 by John C. Hull
Options on Coupon Bearing Bonds

• A European option on a coupon-bearing bond can be expressed as a portfolio of options on zero-coupon bonds.

• We first calculate the critical interest rate at the option maturity for which the coupon-bearing bond price equals the strike price at maturity.

• The strike price for each zero-coupon bond is set equal to its value when the interest rate equals this critical value.
Interest Rate Trees vs Stock Price Trees

• The variable \( \delta \) at each node in an interest rate tree is the \( \delta t \)-period rate

• Interest rate trees work similarly to stock price trees except that the discount rate used varies from node to node
Two-Step Tree Example
(Figure 23.6, page 551))

Payoff after 2 years is $\text{MAX}[100(r - 0.11), 0]$ $p_u=0.25; p_m=0.5; p_d=0.25$; Time step=1yr

\[
\begin{align*}
\text{0.35}\text{**} & \quad 1.11\text{*} \\
0.23 & \quad 0.12 \quad 1 \\
0.00 & \quad 0.10 \quad 0 \\
0.08 & \quad 0 \\
0.06 & \quad 0
\end{align*}
\]

*: $(0.25 \times 3 + 0.50 \times 1 + 0.25 \times 0)e^{-0.12 \times 1}$

**: $(0.25 \times 1.11 + 0.50 \times 0.23 + 0.25 \times 0)e^{-0.10 \times 1}$
Alternative Branching Processes in a Trinomial Tree
(Figure 23.7, page 552)
An Overview of the Tree Building Procedure

\[ dr = [\theta(t) - ar]dt + \sigma dz \]

1. Assume \( \theta(t) = 0 \) and \( r(0) = 0 \)
2. Draw a trinomial tree for \( r \) to match the mean and standard deviation of the process for \( r \)
3. Determine \( \theta(t) \) one step at a time so that the tree matches the initial term structure
Example

\[ \sigma = 0.01 \]
\[ \alpha = 0.1 \]
\[ \delta t = 1 \text{ year} \]

The zero curve is as shown in Table 23.1 on page 556
The Initial Tree
(Figure 23.8, page 554)

<table>
<thead>
<tr>
<th>Node</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
<th>I</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>0.000%</td>
<td>1.732%</td>
<td>0.000%</td>
<td>-1.732%</td>
<td>3.464%</td>
<td>1.732%</td>
<td>0.000%</td>
<td>-1.732%</td>
<td>-3.464%</td>
</tr>
<tr>
<td>$p_u$</td>
<td>0.1667</td>
<td>0.1217</td>
<td>0.1667</td>
<td>0.2217</td>
<td>0.8867</td>
<td>0.1217</td>
<td>0.1667</td>
<td>0.2217</td>
<td>0.0867</td>
</tr>
<tr>
<td>$p_m$</td>
<td>0.6666</td>
<td>0.6566</td>
<td>0.6666</td>
<td>0.6566</td>
<td>0.0266</td>
<td>0.6566</td>
<td>0.6666</td>
<td>0.6566</td>
<td>0.0266</td>
</tr>
<tr>
<td>$p_d$</td>
<td>0.1667</td>
<td>0.2217</td>
<td>0.1667</td>
<td>0.1217</td>
<td>0.0867</td>
<td>0.2217</td>
<td>0.1667</td>
<td>0.1217</td>
<td>0.8867</td>
</tr>
</tbody>
</table>
The Final Tree
(Figure 23.9, Page 556)

<table>
<thead>
<tr>
<th>Node</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
<th>I</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>3.824%</td>
<td>6.937%</td>
<td>5.205%</td>
<td>3.473%</td>
<td>9.716%</td>
<td>7.984%</td>
<td>6.252%</td>
<td>4.520%</td>
<td>2.788%</td>
</tr>
<tr>
<td>$p_{u}$</td>
<td>0.1667</td>
<td>0.1217</td>
<td>0.1667</td>
<td>0.2217</td>
<td>0.8867</td>
<td>0.1217</td>
<td>0.1667</td>
<td>0.2217</td>
<td>0.0867</td>
</tr>
<tr>
<td>$p_{m}$</td>
<td>0.6666</td>
<td>0.6566</td>
<td>0.6666</td>
<td>0.6566</td>
<td>0.0266</td>
<td>0.6566</td>
<td>0.6666</td>
<td>0.6566</td>
<td>0.0266</td>
</tr>
<tr>
<td>$p_{d}$</td>
<td>0.1667</td>
<td>0.2217</td>
<td>0.1667</td>
<td>0.1217</td>
<td>0.0867</td>
<td>0.2217</td>
<td>0.1667</td>
<td>0.1217</td>
<td>0.8867</td>
</tr>
</tbody>
</table>
Extensions

The tree building procedure can be extended to cover more general models of the form:

\[ df(r) = \left[ \theta(t) - a f(r) \right] dt + \sigma dz \]
Other Models

Black, Derman, and Toy:

\[ d \ln r = \left[ \theta(t) + \frac{\sigma'(t)}{\sigma(t)} \ln(r) \right] dt + \sigma(t) dz \]

Black and Karasinski:

\[ d \ln r = \left[ \theta(t) - a(t) \ln(r) \right] dt + \sigma(t) dz \]

- These models allow the initial volatility environment to be matched exactly.
- But the future volatility structure may be quite different from the current volatility structure.
Calibration: $a$ and $\sigma$ constant

- The volatility parameters $a$ and $\sigma$ are chosen so that the model fits the prices of actively traded instruments such as caps and European swap options as closely as possible.
- We can choose a global best fit value of $a$ and imply $\sigma$ from the prices of actively traded instruments. This creates a volatility surface for interest rate derivatives similar to that for equity option or currency options (see Chapter 15).
Calibration: \( a \) and \( \sigma \) functions of time

- We minimize a function of the form

\[
\sum_{i=1}^{n} (U_i - V_i)^2 + P
\]

where \( U_i \) is the market price of the \( i \)th calibrating instrument, \( V_i \) is the model price of the \( i \)th calibrating instrument and \( P \) is a function that penalizes big changes or curvature in \( a \) and \( \sigma \)