COMPARATIVE STATICS OF A RESIDENTIAL CITY WITH SEVERAL INCOME CLASSES *

by

Hans-Jürg Büttler

Institut für Bauplanung und Baubetrieb

Swiss Federal Institute of Technology (ETH)

8093 Zürich, Switzerland

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Abstract

On the basis of common land-use models, comparative-statics results turn out to be counter-intuitive with respect to an income change. For instance, lowering the income tax rate of the richest class in an urban economy ceteris paribus would not only make people of this richest class better off but also all of the other, poorer classes. This will be shown to be incorrect, if a more realistic land-use model is used, the extensions of which are: First, both time cost and money cost of commuting are considered. Second, housing is a multi-dimensional good. Third, the supply of housing is based on a structural frame cost function which implies that people can explicitly reside in multi-storey buildings. Fourth, equilibrium is concerned with the housing market rather than the land market. It will be shown that some effects are ambiguous but some are clearly not. Hence policy recommendations, e.g. for a tax-subsidy schedule, should be based on either simulation or empirical investigations. Finally, the simulations carried out exhibit results which are consistent with economic intuition.

1. Introduction and Summary

The recent discussion in comparative statics of an urban area is about the question on the welfare implication of rich households on poor households. Suppose that there were n classes of people in a residential city extending between the Central Business District (CBD) radius r_0 and the city radius r_n. The city is assumed to be of the standard type, i.e. it is symmetrical, located on a flat and featureless plane, and people commute to work in the CBD. The i-th class consists of P_i people with the utility functions U_i and income y_i. The money cost of commuting rises with increasing distance. Time costs are neglected. Furthermore, preferences and incomes are related in such a way that a household in class i occupies more land than a household in class i-1. In equilibrium, people in different classes will, therefore, reside in concentric rings around the CBD. Households in class i occupy the ring $[r_{i-1}, r_i]$ and live farther from the CBD than people in class i-1, i.e. $r_0 < r_1 < r_2 <\ldots < r_n$. While the CBD radius r_0 is fixed, the city radius r_n is determined by the condition that housing buys land at the opportunity cost which is given exogeneously as the agricultural land rent p_a. In equilibrium, both the ring boundaries r_j and the utility levels U_j are, among other things, functions of all class populations P_i and all incomes y_i. Hartwick et al. (1976) first presented two sets of results under the assumptions briefly described above. The first set is:

$$\partial r_j/\partial P_i < 0 \quad \text{if } j < i, \quad \partial r_j/\partial P_i > 0 \quad \text{if } j \geqslant i \tag{1a}$$

$$\partial U_j/\partial P_i < 0 \quad \text{all } j \tag{1b}$$

If the population of the i-th class increases,then the inner classes are squeezed towards the CBD whereas the outer classes, including the i-th class, are pushed away from it. Everyone's welfare, or real income, falls. The second set is:

$$\partial r_j/\partial y_i > 0 \quad \text{all } j \tag{2a}$$

$$\partial U_j/\partial y_i > 0 \quad \text{if } j \leqslant i,\ \partial U_j/\partial y_i < 0 \quad \text{if } j > i \tag{2b}$$

If the income of the i-th class increases, then all classes are pushed away from the CBD. The outer classes suffer a reduction in their welfare whereas the inner classes, including the i-th class, enjoy an increase. As Hartwick et al. (1976) pointed out, these results must hold, if all households have the same utility function and rich people live farther away from the CBD than poor people, i.e. $y_1 < y_2 < \ldots < y_n$. Therefore, from equation (2b) it follows that increasing the income of the i-th class rises the welfare of those classes with incomes lower or equal to it, but reduces the welfare of those classes with higher incomes. This counter-intuitive result has strong policy implications: subsidizing a particular middle-income class would tend to level off the welfare disparity between the poorest and the richest classes. Or, levying taxes on the richest class would not only make the richest class worse off but also all the other classes.

Special cases of the analysis carried out by Hartwick et al. (1976) have been proved by Wheaton (1974,1976) and Miyao (1975). Their particular results fit into equations (1) and (2). In a recent article, Arnott et al. (1978) extend Wheaton's analysis (1976) for two income classes in the sense that the transportation cost is allowed to be a function of the income, i.e. the time cost of commuting is introduced. In this case, increasing the income of the rich class can make the poor class worse off, but not necessarily vice versa. The authors note that the results of both Wheaton (1976) and Hartwick et al. (1976) do then not carry through.

In this paper a more realistic model is used, the extensions of which are: First, time spent on commuting, or leisure, is an argument of the household's utility function. Second, housing is a multi-dimensional good, its elements being: floor

The first term in equation (4) represents rental revenues, the second and third term the structural frame cost of the building, the fourth term the finishing cost of the building, and the last term the land cost of the land lot. Profits are to be maximized with regard to design parameters (F,H) and housing attributes (I,u,H). The builder is a price taker but he chooses the housing attributes to be supplied. The housing rent implicitly prices both distance and housing attributes. The number of dwellings per building is defined as the floor surfaces divided by the housing space demanded per household:

$$D \equiv F(H/t)s^{-1} \tag{5}$$

Equations (3), (4), and (5) imply: First, the profit and utility maximizations are simultaneous through the housing attributes. Second, constant elasticity demand and supply functions in terms of both housing rent and land rent are obtained.

On a competitive housing market, land rents are determined by the condition that the builders' profit is zero throughout the city:

$$\pi(r) = 0 \tag{6}$$

Using equations (3) to (6), all demand and supply functions can be expressed in terms of the so far unknown bid housing-rent functions on their own.[5]

The shape of the residential zone of the city is considered to be a square ring extending between distances r_0 and r_n from the CBD. Assume a rectangular road grid. Distances are then, say, the absolute sum of East-West and North-South distances. The area in a strip of width dr between distances r and r + dr from the CBD is, approximately, 4 r dr. Suppose that all housing attributes are supplied at every location. The supply of housing space per unit of area is $F(H/t)(F + G)^{-1}$. The demand for housing space per household is s. The ratio of housing space supplied per unit of area to housing space demanded per household is therefore the number of households per unit of area. Define this ratio as the household or population density Φ:

$$\Phi \equiv F(H/t)(F + G)^{-1} s^{-1} \tag{7}$$

The population density Φ in the ring $[r_{i-1}, r_i]$ of class i is a function of the bid housing-rent function q_i. Equilibrium on the housing market requires all households of the i-th class P_i to be accommodated within the ring $[r_{i-1}, r_i]$:

$$P_i = 4\int_{r_{i-1}}^{r_i} \Phi\left(q_i(r, U_i)\right) r\, dr, \; r_i < \min\,(y_i/k_0,\, T/k)$$
$$i = 1 \text{ to } n \tag{8}$$

The equilibrium conditions (8) are general in the sense that they do not depend on particular demand and supply functions. If the ring boundaries are known, then equations (8) determine the utility levels U_i of the n classes, since they are the only unknowns in the bid housing-rent functions q_i. [6]

The long-run equilibrium requires the land rent of the n-th class, evaluated at the edge of the city, to be equal to the agricultural land rent p_a:

$$p_n(r_n) = p_a \tag{9}$$

The remaining ring boundaries are determined by the condition that the housing rents of two neighbouring classes must be equal at the respective ring boundary: [7]

$$q_i(r_i) = q_{i+1}(r_i), \qquad i = 1 \text{ to } n - 1 \tag{10}$$

Equations (8) to (10) determine the long-run equilibrium of the residential city. We proceed as follows. In the first step the ring boundaries are assumed to be known. Housing rents, land rents, and utility levels can then be derived from equations (8) in terms of the unknown ring boundaries. Second, equations (9) and (10) form a non-linear equation system that determines the ring boundaries.

Using equations (3) to (8), the utility levels in equilibrium become:

$$U_i = A - \alpha_3 \ln(\kappa + \delta) + \alpha_1 T - \alpha_2 \ln t - \{2(\varepsilon - 1)(\alpha_2 - \alpha_5) - \alpha_4\gamma\}$$

$$(2\gamma)^{-1} \ln C - (\alpha_2 - \alpha_5)\gamma^{-1} \ln C_0 + \alpha_2(\gamma - \varepsilon)(\gamma\lambda)^{-1}$$

$$(\ln V_i - \ln P_i), \quad i = 1 \text{ to } n \tag{11}$$

The following definitions are used:

$$A : \text{function of } \alpha_j (j = 0 \text{ to } 5),\ \gamma, \text{ and } \varepsilon \tag{12a}$$

$$V_i \equiv \int_{r_{i-1}}^{r_i} (y_i - k_0 r)^{\nu - 1} e^{-\mu r} r \, dr, \qquad i = 1 \text{ to } n \tag{12b}$$

$$\lambda \equiv 2\alpha_2(\gamma - \varepsilon)\left[\alpha_4 \gamma + 2\alpha_2(\gamma - \varepsilon) + 2\varepsilon\,\alpha_5\right]^{-1}, \quad 0 < \lambda < 1 \tag{12c}$$

$$\nu \equiv (\alpha_0 + \alpha_2 + \alpha_3)\,\alpha_2^{-1}\,\gamma(\gamma - \varepsilon)^{-1}\lambda, \quad \nu > 1 \tag{12d}$$

$$\mu \equiv \alpha_1\,\alpha_2^{-1}\,\gamma(\gamma - \varepsilon)^{-1}\lambda\, k, \quad \mu > 0 \tag{12e}$$

The term A in equation (11) is a function of both consumption attractions and structural frame cost elasticities. It is a lengthy expression which is omitted here for the sake of brevity. Note that A is the same for all classes and can be treated as a constant that has no effect on our further considerations. The correct expression, however, has been taken into account when we present the computational results in section 4, but any constant, of course, would also do. The equilibrium utility level tends to fall with increasing interest rate κ, depreciation rate δ, storey height t, fixed structural frame cost C, cost coefficient C_0, population P_i, and lower ring boundary r_{i-1}, but it tends to rise with increasing income y_i and upper ring boundary r_i. It also follows from equations (11) and (12b), increasing the money or time cost of commuting tends to lower the utility level.

The housing rent functions in equilibrium become:

$$q_i = 2^{-\frac{4(\gamma-\varepsilon) - (\varepsilon-1)}{\gamma}} V_i^{-\frac{\gamma-\varepsilon}{\gamma}} \gamma^{\frac{\varepsilon}{\gamma}} \alpha_2 (\alpha_0+\alpha_2)^{-\frac{\gamma-\varepsilon}{\gamma}} (\alpha_2-\alpha_3-\alpha_5)^{-\frac{1}{\gamma}}$$

$$\left[(\alpha_2-\alpha_3)(\gamma-\varepsilon) + \varepsilon\ \alpha_5\right]^{-\frac{2(\gamma-\varepsilon)}{\gamma}} \left[\alpha_4\ \gamma + 2(\alpha_2-\alpha_3)(\gamma-\varepsilon) + 2\varepsilon\ \alpha_5\right]^{\frac{2(\gamma-\varepsilon)}{\gamma}}$$

$$\left[2(\varepsilon-1)(\alpha_2-\alpha_3-\alpha_5) - \alpha_4\ \gamma\right]^{-\frac{\varepsilon-1}{\gamma}}\ t\ C^{\frac{\varepsilon-1}{\gamma}}\ C_0^{\frac{1}{\gamma}}\ k_0^{\frac{2(\gamma-\varepsilon)}{\gamma}}\ P_i^{\frac{\gamma-\varepsilon}{\gamma}}$$

$$(y_i - k_0\ r)^{\frac{\alpha_0+\alpha_2+\alpha_3}{\alpha_2}\lambda}\ e^{-\frac{\alpha_1}{\alpha_2}\lambda\ k\ r}, \quad r_{i-1} \leqslant r \leqslant r_i,\ i=1 \text{ to } n \qquad (13)$$

The equilibrium housing rent tends to fall with increasing distance r and upper ring boundary r_i, but tends to rise with increasing storey height t, fixed structural frame cost C, cost coefficient C_0, population P_i, and lower ring boundary r_{i-1}. The direct income effect, however, is ambiguous: The housing rent tends to rise with increasing income y_i at the upper ring boundary r_i, but it could rise or fall at the lower ring boundary r_{i-1}. Note that the housing rent is homogeneous of degree one in terms of money.

Finally, we also need the land rent functions in equilibrium:

$$p_i = 2^{-3}\ \{V_i\gamma\ (\alpha_0 + \alpha_2)\}^{-1}\ \left[\alpha_4\gamma + 2(\alpha_2 - \alpha_3)(\gamma - \varepsilon) + 2\varepsilon\alpha_5\right]$$

$$P_i\ (y_i - k_0 r)^{\nu}\ e^{-\mu r}, \quad r_{i-1} \leqslant r \leqslant r_i, \quad i=1 \text{ to } n \qquad (14)$$

We first establish the equilibrium conditions for the ring boundaries r_1 and $r_2 \equiv r_n$ upon substituting equations (13) into (10) and equations (14) into (9):

$$\Gamma_1(r_1,r_2) \equiv -\ell n\, V_1(r_0,r_1,y_1) + \ell n\, V_2(r_1,r_2,y_2) + \ell n\, P_1 - \ell n\, P_2 + \nu \ell n(y_1 - k_0 r_1) - \nu \ell n(y_2 - k_0 r_1) = 0 \tag{15a}$$

$$\Gamma_2(r_1,r_2) \equiv \ell n\, \bar{p} - \ell n\, V_2(r_1,r_2,y_2) + \ell n\, P_2 + \nu \ell n(y_2 - k_0 r_2) - \mu r_2 - \ell n\, p_a = 0 \tag{15b}$$

where

$$\bar{p} \equiv -3\ell n 2 - \ell n \gamma - \ell n(\alpha_0 + \alpha_2) + \ell n\left[\alpha_4 \gamma + 2(\alpha_2 - \alpha_3)(\gamma - \varepsilon) + 2\varepsilon\alpha_5\right] \tag{15c}$$

Differentiating system (15) yields the following vector equation in order to determine the variation of the ring boundaries dr_1 and dr_2:

$$c^{-1}\,\{B\}\{dr\} = \{m\} \tag{16}$$

where

$$\{B\} \equiv \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \equiv c \begin{bmatrix} \partial\Gamma_1/\partial r_1 & \partial\Gamma_1/\partial r_2 \\ \partial\Gamma_2/\partial r_1 & \partial\Gamma_2/\partial r_2 \end{bmatrix} \tag{17a}$$

$$c \equiv V_1\, V_2 (y_1 - k_0 r_1)(y_2 - k_0 r_1)(y_2 - k_0 r_2) > 0 \tag{17b}$$

$$\{dr\} \equiv [dr_1, dr_2]' \quad \text{(column vector)} \tag{17c}$$

$$\{m\} \equiv -\left[(\partial\Gamma_1/\partial y_1)dy_1 + (\partial\Gamma_1/\partial y_2)dy_2 + (\partial\Gamma_1/\partial P_1)dP_1 + (\partial\Gamma_1/\partial P_2)dP_2,\; (\partial\Gamma_2/\partial y_2)dy_2 + (\partial\Gamma_2/\partial P_2)dP_2\right]' \tag{17d}$$

The elements of the matrix {B} are the partial derivatives of the equilibrium conditions (15) with regard to the ring boundaries, whereas the "loading" vector {m} represents their variation with regard to incomes and populations. Note that the scalar c is positive by (12b) and by the inequality in (8). The solution of (16) is:

$$\{dr\} = c\ \{B\}^{-1}\{m\} \tag{18}$$

For the present case of two classes, the inverse matrix can easily be found as:

$$\{B\}^{-1} = \frac{1}{b}\begin{bmatrix} b_{22} & -b_{12} \\ -b_{21} & b_{11} \end{bmatrix} \tag{19}$$

where

$$b \equiv b_{11}\ b_{22} - b_{12}\ b_{21} > 0 \tag{20}$$

The scalar b is positive which can be seen from equations (21), since the elements of the matrix {B}, written in full, are:

$$b_{11} = -\{V_2(y_2-k_0r_1)(y_2-k_0r_2)(y_1-k_0r_1)^{\nu}\ e^{-\mu r_1}\ r_1+V_1(y_1-k_0r_1)(y_2-k_0r_2)(y_2-k_0r_1)^{\nu}\ e^{-\mu r_1}\ r_1+V_1V_2\nu k_0(y_2-k_0r_1)(y_2-k_0r_2)-V_1V_2\nu k_0(y_1-k_0r_1)(y_2-k_0r_2)\} < 0 \tag{21a}$$

$$b_{12} = V_1(y_1-k_0r_1)(y_2-k_0r_1)(y_2-k_0r_2)^{\nu}\ e^{-\mu r_2}\ r_2 > 0 \tag{21b}$$

$$b_{21} = V_1(y_1-k_0r_1)(y_2-k_0r_2)(y_2-k_0r_1)^{\nu}\ e^{-\mu r_1}\ r_1 > 0 \tag{21c}$$

$$b_{22} = -\{V_1(y_1-k_0r_1)(y_2-k_0r_1)(y_2-k_0r_2)^{\nu}\ e^{-\mu r_2}\ r_2+V_1V_2\nu k_0(y_1-k_0r_1)(y_2-k_0r_1)+V_1V_2\mu(y_1-k_0r_1)(y_2-k_0r_1)(y_2-k_0r_2)\} < 0 \tag{21d}$$

The diagonal elements are negative, whereas the off-diagonal elements are positive. The algebraic product $b_{12}b_{21}$ in (20) cancels out with the algebraic product of the second term in (21a) and the first term in (21d). Next, the variation of the equilibrium utility levels, as given by (11), is considered. We first transform them into:

$$\bar{U}_i \equiv \gamma\ \lambda[\alpha_2(\gamma-\varepsilon)]^{-1}\ U_i \qquad (i = 1,2) \tag{22}$$

Differentiating (22) yields:

$$d\bar{U}_1 = V_1{}^{-1}\{(\partial V_1/\partial r_1)dr_1+(\partial V_1/\partial y_1)dy_1\} - P_1{}^{-1}\ dP_1 \tag{23a}$$

$$d\bar{U}_2 = V_2{}^{-1}\{(\partial V_2/\partial r_1)dr_1+(\partial V_2/\partial r_2)dr_2+(\partial V_2/\partial y_2)dy_2\} - P_2{}^{-1}\ dP_2 \tag{23b}$$

The variation of the equilibrium utility levels depends on the variation of the ring boundaries, incomes, and populations. The partial derivatives of the integrals V_i with regard to income are:

$$\partial V_i/\partial y_i \equiv V_i' = (\nu - 1) \int_{r_{i-1}}^{r_i} (y_i - k_0 r)^{\nu-2} e^{-\mu r}\ r\ dr > 0,$$

$$i = 1,2 \tag{24a}$$

The partial derivatives of the integrals V_i with regard to the ring boundaries are:

$$\partial V_1/\partial r_1 = (y_1-k_0r_1)^{\nu-1}\ e^{-\mu r_1}\ r_1 > 0 \tag{24b}$$

$$\partial V_2/\partial r_1 = -(y_2-k_0r_1)^{\nu-1}\ e^{-\mu r_1}\ r_1 < 0 \tag{24c}$$

$$\partial V_2/\partial r_2 = (y_2-k_0r_2)^{\nu-1}\ e^{-\mu r_2}\ r_2 > 0 \tag{24d}$$

Next, we consider the variation of the housing-rent functions in equilibrium. First, transform equations (13) into:

$$\bar{q}_i \equiv \gamma(\gamma-\varepsilon)^{-1}\ \ell n\ q_i \qquad (i = 1,2) \tag{25}$$

Second, differentiating (25) for i = 1 yields:

$$d\bar{q}_1(r_0) = - d\bar{U}_1 + \nu(y_1 - k_0 r_0)^{-1} dy_1 \tag{26a}$$

$$d\bar{q}_1(r_1 + dr_1) = - d\bar{U}_1 + \nu(y_1 - k_0 r_1)^{-1} dy_1 - \nu k_0 (y_1 - k_0 r_1)^{-1} dr_1 - \mu dr_1 \tag{26b}$$

The variation of the housing-rent function for the rich class need not be considered, since we know that: (i) the variation of both functions must be equal at the separating distance $r_1 + dr_1$, i.e. $d\bar{q}_1(r_1 + dr_1) = d\bar{q}_2(r_1 + dr_1)$, and (ii), the variation of the second function must be zero at the city radius $r_2 \equiv r_n$, i.e. $d\bar{q}_2(r_2 + dr_2) = 0$. The latter is due to the fact that the land rent at the edge of the city is equal to the fixed agricultural land rent and hence, the housing rent at the edge of the city is equal to the opportunity cost of housing, which is a function of the agricultural land rent, i.e. $q_n(r_n) = q(p_a)$ by (6). This is visualized in Figure 1. Both statements can be used for checking calculations.

We now turn to the population effects. The loading vector for P_1 is:

$$\{m\} = [- P_1^{-1} dP_1, 0]' \tag{27}$$

Substituting (27) into (18), the variation of the ring boundaries becomes:

$$\partial r_1 / \partial P_1 = - a P_1^{-1} b_{22} > 0 \tag{28a}$$

$$\partial r_2 / \partial P_1 = a P_1^{-1} b_{21} > 0 \tag{28b}$$

where

$$a \equiv c/b > 0 \tag{29}$$

Since c and b are positive by (17b) and (20), the scalar a in (29) is also positive. The signs of (28) readily follow from (21) and (29). Substituting (24b) and (28a) into (23a), and substituting (24d) and (28) into (23b), the variation of the utility levels becomes:

$$\partial\bar{U}_1/\partial P_1 = (V_1 b P_1)^{-1}\{-c(y_1-k_0 r_1)^{\nu-1} e^{-\mu r_1} r_1 b_{22} - V_1 b\} < 0 \tag{30a}$$

$$\partial\bar{U}_2/\partial P_1 = a(V_2 P_1)^{-1}\{(y_2-k_0 r_1)^{\nu-1} e^{-\mu r_1} r_1 b_{22}+(y_2-k_0 r_2)^{\nu-1} e^{-\mu r_2} r_2 b_{21}\} < 0 \tag{30b}$$

The signs readily follow from substituting (20) and (21) into (30) and some algebra. Using (26) and the fact that $dr_1 > 0$, the variation of the housing-rent function becomes:

$$\partial\bar{q}_1(r_0)/\partial P_1 = \partial\bar{q}_1(r_1)/\partial P_1 = -\ \partial\bar{U}_1/\partial P_1 > 0 \tag{31}$$

Since both the city radius r_2 increases and the relative housing-rent gradients do not change, the housing rent must rise at every distance in the city. Consider now the population effect of the rich class. The loading rector is:

$$\{m\} = [P_2^{-1}\, dP_2,\ -\ P_2^{-1}\, dP_2]' \tag{32}$$

Using (32) and (18), the variation of the ring boundaries becomes:

$$\partial r_1/\partial P_2 = a\ P_2^{-1}\ \{b_{22} + b_{12}\} < 0 \tag{33a}$$

$$\partial r_2/\partial P_2 = -\ a\ P_2^{-1}\ \{b_{21} + b_{11}\} > 0 \tag{33b}$$

The signs of (33) readily follow from substituting (21) into (33) and some algebra. From (23) and (24), we have the variation of the utility levels:

$$\partial\bar{U}_1/\partial P_2 = V_1^{-1}(\partial V_1/\partial r_1)(\partial r_1/\partial P_2) < 0 \tag{34a}$$

$$\partial\bar{U}_2/\partial P_2 = -(P_2V_2b)^{-1}\{(y_2-k_0r_1)^{\nu-1}\ e^{-\mu r_1}\ r_1\ c(b_{22}+b_{12})$$
$$-(y_2-k_0r_2)^{\nu-1}\ e^{-\mu r_2}\ r_2\ c(b_{21}+b_{11})-V_2b\} < 0 \qquad (34b)$$

In (34a), the first two terms are positive and the last term is negative by (33a), hence (34a) is negative. Using (21), the sign of (34b) is negative. Finally, the variation of the housing rents becomes:

$$\partial\bar{q}_1(r_0)/\partial P_2 = -\ \partial\bar{U}_1/\partial P_2 > 0 \qquad (35a)$$

$$\partial\bar{q}_1(r_1+dr_1)/\partial P_2 = -\ \partial\bar{U}_1/\partial P_2\ -\ \nu k_0(y_1-k_0r_1)^{-1}\ (\partial r_1/\partial P_2)$$
$$-\ \mu(\partial r_1/\partial P_2) > 0 \qquad (35b)$$

Equation (35a) is obvious from (34a). All terms in (35b) are negative by (33a) and (34a). Equations (38), (30), (31), (33), (34), and (35) establish the first set of results (i) of the theorem.

In the next step the income effects are considered. The loading rector for a change of the income of the poor class becomes:

$$\{m\} = [-(\partial\Gamma_1/\partial y_1)\ dy_1,\ 0]' \qquad (36a)$$

$$\partial\Gamma_1/\partial y_1 = -V_1'\ V_1^{-1} + \nu(y_1-k_0r_1)^{-1} > 0 \qquad (36b)$$

To show the sign of (36b), we use the first of the two inequalities (37):

$$(y_i-k_0r_i)\ V_i' < (\nu-1)\ V_i \qquad (i = 1,2) \qquad (37a)$$

$$(y_i-k_0r_{i-1})\ V_i' > (\nu-1)\ V_i \qquad (i = 1,2) \qquad (37b)$$

These inequalities follow from (12b) and (24a) when using the smallest and greatest net income in a ring, respectively. By (36) and (18), the variation of the ring boundaries becomes:

$$\partial r_1/\partial y_1 = -\ a\ b_{22}(\partial\Gamma_1/\partial y_1) > 0 \tag{38a}$$

$$\partial r_2/\partial y_1 = a\ b_{21}(\partial\Gamma_1/\partial y_1) > 0 \tag{38b}$$

This follows from (21) and (36b). The variation of the utility levels becomes:

$$\partial\bar{U}_1/\partial y_1 = V_1^{-1}\{(\partial V_1/\partial r_1)(\partial r_1/\partial y_1)+V_1'\} > 0 \tag{39a}$$

$$\partial\bar{U}_2/\partial y_1 = a(\partial\Gamma_1/\partial y_1)\ V_2^{-1}\{(y_2-k_0r_1)^{\nu-1}e^{-\mu r_1}\ r_1\ b_{22}+(y_2-k_0r_2)^{\nu-1}\ e^{-\mu r_2}\ r_2\ b_{21}\} < 0 \tag{39b}$$

The sign of (39a) can be shown when using (24a), (24b), and (38a), whereas that of (39b) when using (21). The variation of the housing rent becomes:

$$\partial\bar{q}_1(r_0)/\partial y_1 = -\ \partial\bar{U}_1/\partial y_1 + \nu(y_1-k_0r_0)^{-1} = ? \tag{40a}$$

$$\partial\bar{q}_1(r_1)/\partial y_1 = (V_1b)^{-1}\{(y_1-k_0r_1)^{\nu-1}\ e^{-\mu r_1}\ r_1\ c\ b_{22}(\partial\Gamma_1/\partial y_1) -V_1'b+\nu V_1\ b(y_1-k_0r_1)^{-1}\} > 0 \tag{40b}$$

The sign of (40a) is ambiguous, even if one uses the exact relationship (41) between the integral V_1 and its partial derivative with regard to income V_1'. Integrating partially V_i (i = 1,2) in two different ways yields:

$$V_i' = k_0^{-1}\{(y_i-k_0r_{i-1})^{\nu-1}e^{-\mu r_{i-1}}\ r_{i-1}-(y_i-k_0r_i)^{\nu-1}e^{-\mu r_i}\ r_i\}$$
$$+(k_0\mu)^{-1}(y_i+k_0\nu/\mu)^{-1}\{(y_i-k_0r_{i-1})^{\nu}e^{-\mu r_{i-1}}-(y_i-k_0r_i)^{\nu}e^{-\mu r_i}\}$$
$$-(\mu/k_0)\left(y_i+k_0(\nu-1)/\mu\right)(y_i+k_0\nu/\mu)^{-1}\ V_i \qquad (i = 1,2) \tag{41}$$

The sign of (40b), however, is unambiguous after substituting (21) and (36b) into (40b), and when using the inequality (37a). Next, we turn to the income effect of the rich class. The loading rector is:

$$\{m\} = -\left[(\partial\Gamma_1/\partial y_2)dy_2\ ,\ (\partial\Gamma_2/\partial y_2)dy_2\right]' \qquad (42a)$$

$$\partial\Gamma_1/\partial y_2 = V_2'\ V_2^{-1} - \nu(y_2-k_0r_1)^{-1} = ? \qquad (42b)$$

$$\partial\Gamma_2/\partial y_2 = -\ V_2'\ V_2^{-1} + \nu(y_2-k_0r_2)^{-1} > 0 \qquad (42c)$$

The sign of (42b) is ambiguous, whereas that of (42c) is positive by (37a). The variation of the ring boundaries becomes:

$$\partial r_1/\partial y_2 = a\{-b_{22}(\partial\Gamma_1/\partial y_2)+b_{12}(\partial\Gamma_2/\partial y_2)\} = ? \qquad (43a)$$

$$\partial r_2/\partial y_2 = a\{b_{21}(\partial\Gamma_1/\partial y_2)-b_{11}(\partial\Gamma_2/\partial y_2)\} > 0 \qquad (43b)$$

Substituting (21) and (42) into (43a), and using the exact relationship (41), the variation of the first ring boundary is ambiguous. This derivative is crucial, since it determines solely the variation of the utility level of the poor class as will be shown in (44a). Substituting (42) and (21) into (43b), and using the inequalities (37), the variation of the ring boundary of the rich class in (43b) turns out to be positive. The variation of the utility levels becomes:

$$\partial\bar{U}_1/\partial y_2 = V_1^{-1}(\partial V_1/\partial r_1)(\partial r_1/\partial y_2) = ? \qquad (44a)$$

$$\partial\bar{U}_2/\partial y_2 = (bV_2)^{-1}\{-(y_2-k_0r_1)^{\nu-1}\ e^{-\mu r_1}\ r_1\ c[-b_{22}(\partial\Gamma_1/\partial y_2)$$
$$+b_{12}(\partial\Gamma_2/\partial y_2)]+(y_2-k_0r_2)^{\nu-1}\ e^{-\mu r_2}\ r_2\ c[b_{21}(\partial\Gamma_1/\partial y_2)$$
$$-b_{11}(\partial\Gamma_2/\partial y_2)]+V_2'b\} > 0 \qquad (44b)$$

Since the first two terms in (44a) are positive by (12b) and (24b), the sign of (44a) depends on the variation of the ring boundary of the poor class. If the poor are squeezed towards the CBD, then their utility level falls, and vice versa. Substituting (42), (20), and (21) into (44b) and using the inequalities (37), equation (44b) turns out to be positive. The variation of the housing rent becomes ambiguous, for

$$\partial\bar{q}_1(r_0)/\partial y_2 = - \partial\bar{U}_1/\partial y_2 = ? \tag{45a}$$

$$\partial\bar{q}_1(r_1+dr_1)/\partial y_2 = \partial\bar{q}_2(r_1+dr_1)/\partial y_2 = - \partial\bar{U}_2/\partial y_2 + \nu(y_2-k_0r_1)^{-1}$$

$$- \nu k_0(y_2-k_0r_1)^{-1}(\partial r_1/\partial y_2) - \mu(\partial r_1/\partial y_2) = ? \tag{45b}$$

Using the exact relationship (41) in (45) does not help to identify unambiguously the variation of the housing rent. Equations (38), (39), (40), (43), (44), and (45) establish the second set of results (ii) of the theorem. This closes the proof.

4. Numerical Solution for Several Classes

The model defined in equations (3) to (10) has been solved numerically for the cases of two, three, and ten income classes. The computations have been performed on the CDC - 6000 computer system of the ETH by means of Newton's approximation method for non-linear equation systems. Convergence follows from the strictly convex housing-rent functions. Note that all results are numerically significant. (The computer uses 14-digits mantissas, the loss of digits is at most four, and the figures differ at least in the fifth digit; hence, the figures are numerically significant.) The outcome of the computations has been presented in equations (1) and (2)''. Figures 1 and 2 depict the situation of comparative statics with three classes where the size and the income of the second class has been increased ceteris paribus. In both cases, the poorest class is squeezed towards the CBD, whereas the richer classes are pushed away from it. Note that the variation of the ring boundaries are significant, specifically that of the first ring, which was shown to be the crucial one.

5. Concluding Remarks

The comparative statics with two classes have been shown to yield ambiguous signs for the crucial derivatives. Of course, this must also apply to the case of several classes. Policy making would have to be based on either simulation or empirical results. The simulation results presented in equations (2)" suggest the following policy recommendations: Levying taxes on the richest class ceteris paribus would tend to level off the welfare disparity between the richest and the poorest classes, since the utility levels of all classes, except that of the richest class, would rise. Or, subsidizing a particular middle-income class ceteris paribus would tend to amplify the welfare disparity between the classes with lower incomes and this middle-income class, whereas it would tend to level off the disparity between the classes with higher incomes and this middle-income class.

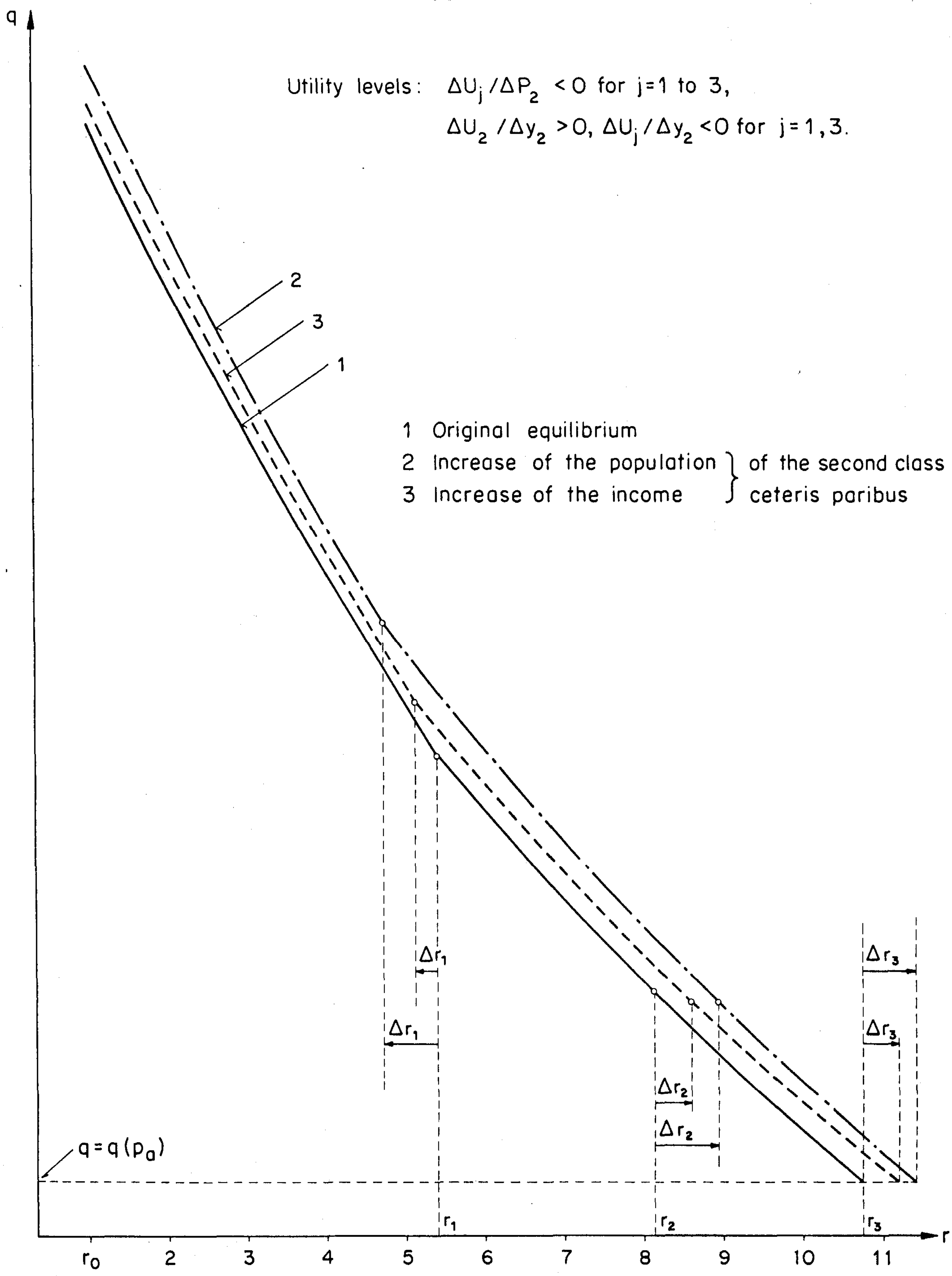

Figure 1: Comparative statics with three classes: Housing rent q

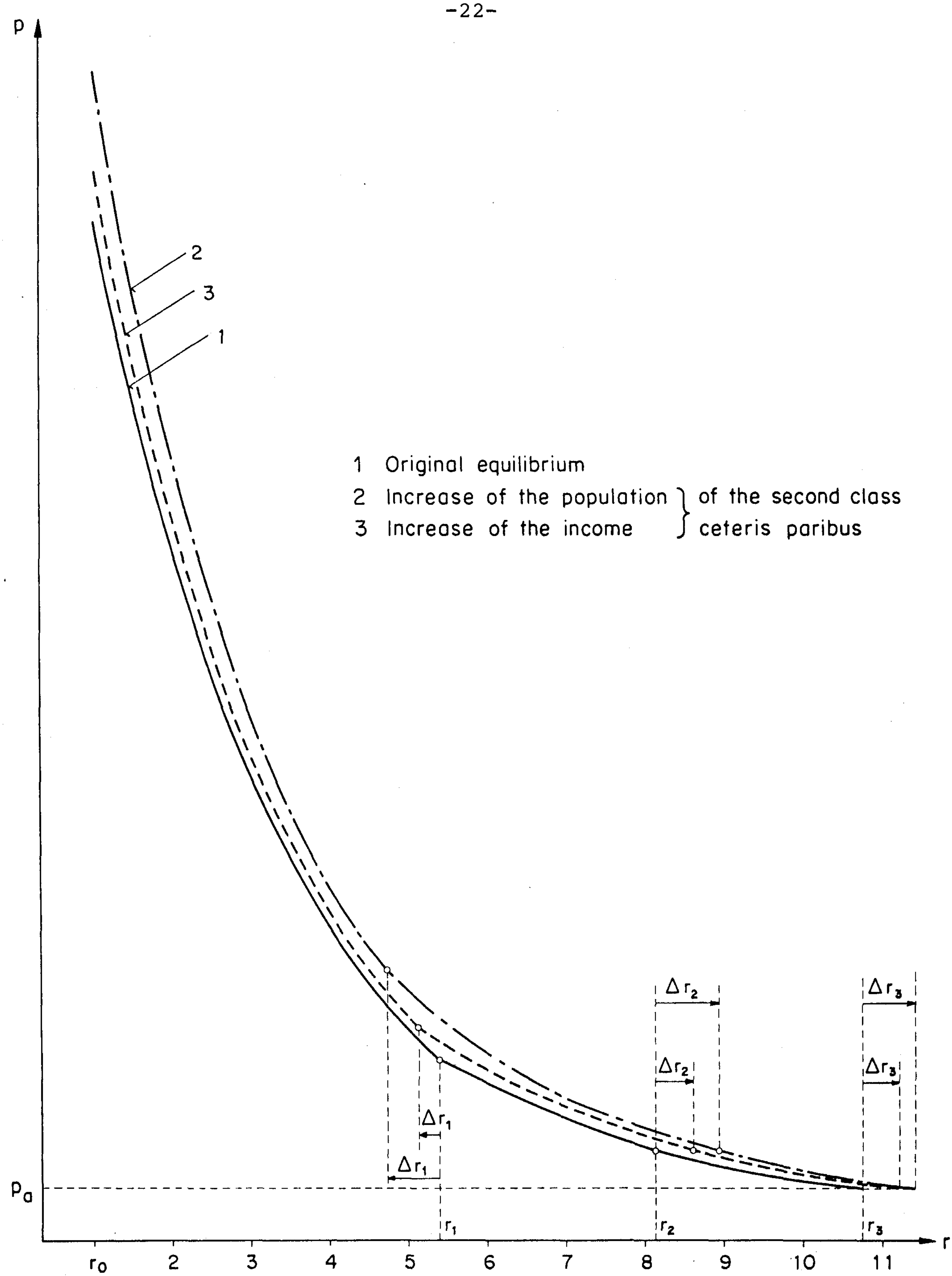

Figure 2: Comparative statics with three classes: Land rent p

Footnotes:

[1] The finishing or non-structural costs include costs for façade, partition walls, false ceilings, sanitary, electrical, and mechanical installations, outside finishing, and inside finishing. It was shown in Büttler (1978) that the finishing cost of a dwelling directly depends on the household's taste and income. However, it should not be included in the composite commodity, since the former is a non-traded good while the latter a traded commodity. All housing attributes are implicitly priced by the housing rent. Moreover, the finishing cost is an important term in the necessary condition of a profit maximum, cf. equation (4).

[2] Note that the garden width u rather than the garden space G is the appropriate design parameter, since the latter is a function of the former and the building area. Due to housing attributes, utility and profit maximizations are to be derived simultaneously. Utility received from garden space, however, can easily be derived from utility received from garden width, once building area and building height are known.

[3] The number of storeys H/t is a proxy for the number of households or dwellings in a building. Since the storey height t is given by engineering design standards or building codes, it is constant and has been deleted. Thus, the building height is the appropriate housing attribute.

[4] Note that the garden space around an apartment building is a non-pure public good, since crowding in consumption of garden space has been taken into account through the negative marginal utility with respect to building height, the latter being a proxy for the number of dwellings. Moreover, tenants of other buildings can be excluded from using that garden. Of course, in the limiting case of a single-family house the garden space becomes a private good by definition.

In addition, it should be noted that buidling area F and garden space G can be interpreted as to be of <u>any rectangular form</u>, though they are treated as quadratic. The quadratic form is consistent with the usual symmetry assumption with regard to the urban area. The consideration of rectangular forms does not alter the results to be derived later but it would involve at least two more parameters, because building areas would have then to be restricted to dimensions which are technically feasible.

5 The bid housing-rent function $q = q(.)$ in a ring is initially a function of distance (r), utility level of class i in that ring (U_i), housing attributes (I, u, H), and income of that class (y_i). From equation (3) we have:

$$q_i = q(r, U_i, I, u, H, y_i), \quad r_{i-1} \leqslant r \leqslant r_i.$$

When solving equation (4) and using (6), the housing attributes turn out to be functions of the bid function on their own, except for the finishing cost which depends on distance and income. Thus the bid function is of the following implicit form:

$$q_i = q(r, U_i, I(r,y_i), u(q), H(q), y_i).$$

It has been shown in Büttler (1978) that this bid function can be solved explicitly to yield the standard form of the bid function:

$$q_i = q(r; U_i, y_i).$$

In what follows the above standard form has been used.

6 The inequality in equation (8) states that neither the time cost nor the money cost of commuting should exhaust incomes on their own.

7 This well-known condition is subject to the standard form of the bid rent function, which has been derived in footnote 5.

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