Lecture Notes in Economics and Mathematical Systems

247

Optimization and Discrete Choice in Urban Systems

Proceedings, Waterloo, Canada, 1983

Edited by
Bruce G. Hutchinson, Peter Nijkamp and Michael Batty

247

Optimization and
Discrete Choice
in Urban Systems

Proceedings of the International Symposium
on New Directions in Urban Systems Modelling
Held at the University of Waterloo, Canada
July 1983

Edited by
Bruce G. Hutchinson, Peter Nijkamp and Michael Batty

PREFACE

The papers contained in this volume were originally presented at the International Symposium on New Directions in Urban Systems Modelling held at the University of Waterloo in July, 1983. The papers have been reviewed and rewritten since that time. The exception is the introductory paper written specially by Manfred Fischer and Peter Nijkamp as an introduction to this volume.

The manuscript was prepared in the word processing unit in the Department of Civil Engineering, University of Waterloo. The sustained work of Mrs. I. Steffler in preparing this manuscript is gratefully acknowledged. Mr. R. K. Kumar provided excellent assistance with the editorial process.

The Symposium and the preparation of this manuscript were supported financially by the Natural Sciences and Engineering Research Council of Canada, The Academic Development Fund and the Department of Civil Engineering, University of Waterloo.

TABLE OF CONTENTS

A COMBINED LINEAR/NON-LINEAR PROGRAMMING MODEL OF EMPLOYMENT, TRANSPORTATION, AND HOUSING IN AN URBAN ECONOMY

Hans-Jürg Büttler
Swiss National Bank
8022 Zurich
Switzerland

Abstract. The purpose of this paper is to describe a model which is normative, allocates resources efficiently, incorporates all economic features which are prerequisite to the existence of cities, and serves as a tool for town planning. The model is organized as a linear program when population distributions are known and as a non-linear program when these distributions are to be predicted. Relationships to existing macro-economic models such as Mills' (1972) model are sketched, and suggestions are made for further development of this model framework.

INTRODUCTION

If a well organized spatial structure is the outcome of town planning alone, then one can say that town planning is as old as cities themselves. For instance, spatially well organized cities can be found in Mesopotamia, in the Roman Empire, and existed throughout the Middle Ages. Many would argue that modern cities have lost this clear-cut spatial structure. If this is the case, the question of why this happened has to be answered. There appear to be two reasons for this view. Firstly, the town planning of modern cities is subject to more uncertainties than in ancient cities. In particular, the considerable growth of population and transportation since World War II, rapid changes in technology, and structural changes in demand for housing are very difficult to forecast. Predicting these variables, however, is essential, if town planning is to provide a plan which is viable in the long run where the cost of structural change is kept to a minimum. Secondly, in democratically organized countries, town planning in the final outcome is a political process. This follows readily from the fact that any town-planning policy has an immediate impact on both the allocation and the distribution of resources. For instance, Hochman (1979) has shown that there are at least two environmental policies against air pollution: either a Pigouvian tax or land-use zoning. While both rules, if correctly applied, are efficient, they exhibit an entirely different distribution of resources. In the former case, revenues from the Pigouvian tax can, in principle, be distributed equally to all residents of the city, while in the latter case, these revenues go to landlords who benefit from higher land

rents. As far as town planning deals with the distribution of resources, it is mainly predicted as the outcome of a political process (Henderson, 1977). Therefore, it is not surprising that theoretical approaches to town planning deal mostly with the question of efficient resource allocation, leaving the problem of distribution aside.

Recent analytical approaches to town planning have three bases. Firstly, models which are descendants of the spatial interaction tradition such as the Lowry model. In these models, the spatial allocation of employment within an urban area is explained by means of fixed labour ratios, with economic variables such as prices and incomes usually absent (Batty, 1976; Anas, 1983; Echenique, 1983). Secondly, models which primarily focus on the housing market. Both empirical models and simulation models are used. However, some of these models lack economic variables and some are not suited to application. Thirdly, models which tackle the land use and trip distribution of an urban area by minimizing net economic costs (Sharpe, this volume). All three attempts are directed towards the question of how resources can be efficiently allocated within an urban area given some social consensus about the distribution of resources among residents.

Other attempts to approach the normative task of town planning have been made in urban economics literature, but without widespread application to real cities. Perhaps the first attempt in this field was that by Herbert and Stevens (1960). They considered the equilibrium in the land and housing market of a two-dimensional city by means of a linear programming model. Due to the enormous amount of data required, computer simulations to date have not been performed. Mills (1972a, 1972b) considered both the normative aspect and the computer application of a two-dimensional city model, while Kim (1979) primarily addressed the question of how different transportation modes should be efficiently used within an urban area. Both Mills and Kim minimize total resource costs of a city, while household's preferences are ignored. Following Kanemoto's approach (1981) to a symmetrical and featureless city, the welfare of city residents will be maximized here, while being subject to resource constraints. The purpose of this paper is to describe a model which is normative, allocates resources efficiently, incorporates all economic features which are prerequisite to the existence of cities, and serves as a tool for town planning. To make the model operational, a combined linear/non-linear programming model is chosen.

The outline of the paper is as follows. The following section describes the basic model. The model is that of the closed city with city population, capital stock, and export demand given. A specific town is divided into rectangular zones which serve as plane coordinates. The objective function to

be maximized is the Benthamite welfare function of the urban economy. Consequently, the individual utility functions are stepwise linear. For each zone, relationships between housing, employment, production, and transportation are described. Two important features which are prerequisite to the existence of cities are introduced: increasing returns to scale in the production of commodities and comparative advantages of certain locations. In order to consider increasing returns to scale, activities are distinguished by both input-output technology and land use intensity. On the other hand, export nodes exhibit comparative advantages. Moreover, the model considers several transport modes: private vehicles, buses, tramways, subways, etc. It is shown that an efficient resource allocation requires maximization of the non-linear welfare function with regard to the populations to be allocated to each zone. This maximization is subject to two conditions: firstly, the sum of all populations residing in each zone must be equal to the exogenously given city population; secondly, the value of the welfare function must be evaluated by a linear programming model. This combined linear/non-linear approach has the advantage that all endogenous variables except the populations residing in each zone can be determined by a linear programming model and therefore makes a computer application possible. At the optimum, there exists a dual program to the linear subprogram which determines shadow prices of resources and shadow incomes of households. The model serves as a tool for town planning when varying exogenous variables. For instance, the forecast of the city population may be varied since it is a random variable, ie., a point estimate with confidence interval. Further, it may be desirable to study the sensitivity of land use zoning when new export nodes are developed in the city.

In the final section, several extensions are briefly discussed: the method of finite elements, traffic congestion, imports, local public goods, different resource endowments of the city residents, and application to the small, open city.

THE MODEL

Discrete Plane Coordinates

Based on urban economics literature, a city is defined as its metropolitan area. A city may therefore consist of several political communities. In this context, the jurisdictional fragmentation of the city must not influence the efficient resource allocation within the urban area, but it naturally plays a crucial role in the allocation of local public goods. To make the model operational, discrete plane coordinations are introduced. Fig. 1 considers the map of a hypothetical city, for which rectangular or squared zones are defined. The rectangular coordinates allow a convenient presentation of the model but in

the final section it is shown that a triangular map could reduce the number of variables considerably. In Fig. 1, there are $j=1,...,J$ segments on the abscissa and $i=1,...,I$ segments on the ordinate. The segments can be of different size and their number is arbitrary but should be chosen so as to allow for all activities within the urban area. Unless the city is symmetric, the origin of the map can arbitrarily be chosen. The topographical features of the city are considered by technical coefficients of the model which are defined for each rectangular zone. Shaded zones in Fig. 1 denote export nodes where specific commodities are exported to and imported from other cities. The location of these export nodes, which may exist or be planned, are exogenous to the model but should be chosen so as to exhibit locational comparative advantages in terms of transportation costs such as a harbour, railway station, airport, or highway junction.
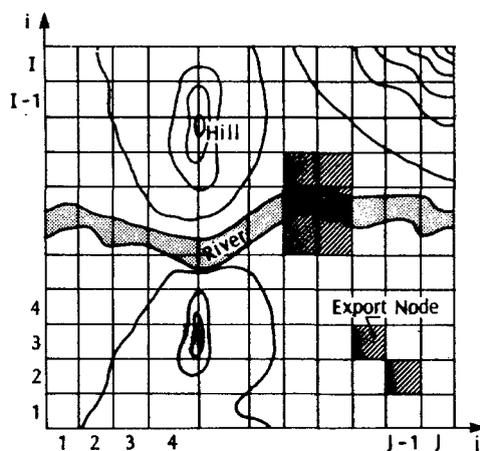
Figure 1 - Rectangular Network

Production and Transportation

There are $r=1,...,R$ industries in the city which each produce a single commodity where the Rth good is floor space. Since all production activities are assumed to be linear, the definition of distinct industries should preferably be taken from an input-output table of the city in question, if it exists. Commodities are produced in buildings with a floor-to-land ratio of g_s, where the index runs from 1 to S. Hence, each industry can be described by a set of linear production activities, the single activities of which are distinguished by different building densities. In other words, increasing returns to scale are considered by introducing building densities. There are $m=1,...,M$ transportation modes, eg., automobiles, buses, tramways, subways, etc. While people can use all transport modes, it is assumed that commodities can only be trans-

ported by automobiles (including trucks). If necessary, the latter assumption can easily be relaxed. The model will not only assign transport modes to residents and commodities efficiently but also compute commuting flows.

List of Variables

A list of all exogenous variables is given below. Many of them, however, may be varied due to uncertainty so as to evaluate the sensitivity of the optimal resource allocation.

E_r = export of commodity r (r=1,...,R) where the export of locally consumed goods is zero,

P = city population,

$d(i,j)$ = mean distance to travel within zone (i,j),

$A(i,j)$ = available land for construction and transportation activities in zone (i,j).

K = available physical capital stock of the city, except building capital,

q_s = building density (floor-to-land ratio), where s=1,...,S and $g_{s+1} > g_s$; for instance, g_1 denotes the density of a one-story, single-family house, while g_s denotes the density of the tallest building. (The maximum height of a building to be considered depends both on structure and soil),

$a_{r,q,s}$ = quantity of commodity r (r=1,...,R) required to produce one unit of commodity q (q=1,...,R-1) when production of commodity q takes place in a building with density g_s (input-output coefficients and factor ratios could depend on zone (i,j) in order to consider topographical differences),

$a_{r,R}$ = quantity of commodity r (r=1,...,R-1) required to produce one unit of floor space in commercial buildings or residences, respectively,

$a_{R+1,q}$ = number of person equivalences for one unit of good q (q=1,...,R-1),

$b_{q,r,s}$ = quantity of factor q (q=1,...,3) required to produce one unit of commodity r (r=1,...,R-1) when production of commodity r takes place in a building with density g_s; the factors are labour (q=1) capital (q=2) and land (q=3),

$b_{q,R+m}$ = quantity of factor q (q=1,...,3) required to produce one unit of transportation service with transport mode m (m=1,...,M),

α_r^u = marginal utility over interval u (u=1,...,3) of commodity r (r=1, ...,R-1),

$\alpha_{R,s}^u$ = marginal utility of floor space (r=R) in a building with density g_s (s=1,...,S) where u=1,...,3,

$\overline{Y}_r^u$ = lower boundary of interval u (u=1,...,3) for commodity r (r=1,...,R).

Endogenous variables are determined by the programming model, namely

$Y_r^u(i,j)$ = quantity of commodity r (r=1,...,R-1) over interval u=1,...,3, consumed by a household residing in zone (i,j),

$Y_{R,s}^u(i,j)$ = floor space (r=R), over interval u=1,...,3, in a building with density g_s, which is consumed by a household residing in zone (i,j),

$E_r(i^*,j^*)$ = export of commodity r (r=1,...,R-1) in an export node (i^*,j^*),

$Z_{r,s}(i,j)$ = production of commodity r (r=1,...,R-1) in a building with density g_s which is located in zone (i,j),

$Z_R(i,j)$ = production of floor space in commercial buildings and residences in zones (i,j),

$F_r^b(i,j)$ = quantity of commodity r (r=1,...,R-1) shipped across bth boundary (b=1,...,4) of zone (i,j),

$P(i,j)$ = population residing in zone (i,j) and this is an exogenous variable for the linear programming model,

$D_r(i,j)$ = final demand for commodity r (r=1,...,R-1) produced in zone (i,j).

$D_{R+m}(i,j)$ = number of persons moved by transport mode m (m=1,...,M) in zone (i,j),

$Z_{R+m}(i,j)$ = transportation services, as measured by persons times distance, of mode m (m=1,...,M) in zone (i,j), where m=1 is automobile,

$H^b(i,j)$ = number of persons moved across bth boundary (b=1,...,4) of zone (i,j), or the commuting flow.

Objective Function

The objective is to maximize the Benthamite welfare function of city residents which is defined as the sum of individual utilities. In a first stage, it is assumed that all individuals are identical in that they have the same utility function. In order to obtain a linear objective function, the monotonically increasing and strictly concave utility function is linearized. This situation is depicted in Fig. 2a for the first R-1 commodities. The domain of each commodity is divided into three intervals u=1(1,...,3) over which the marginal utility is constant. The lower boundary of the first interval $\bar{Y}_r^1$ is assumed to be nil. Accordingly, total demand for each commodity is divided into three variables $Y^u(u=1,...,3)$. The situation is different in case of demand for floor space by residents. In Fig. 2b, it is assumed that the utility function depends not only on floor space but also on housing attributes (Büttler and Beckmann, 1980). Naturally, a resident prefers a flat in a low-density building to the same flat in a high-density building. Hence, marginal utilities as depicted in Fig. 2b tend to fall with increasing building density g_s. Although all individuals (or households in this context) are

treated as if they were identical, it is well known that at the optimum they are treated unequally (Mirrlees, 1972; Kanemoto, 1980). The model, however, can be modified by using the Rawlsian welfare function so that equal individuals are treated equally at the optimum. Given the assumptions above, the objective function to be maximized is

$$W[P(i,j)] = \max \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{r=1}^{R-1} \sum_{u=1}^{3} \alpha_r^u Y_r^u(i,j) P(i,j)$$

$$+ \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{u=1}^{3} \sum_{s=1}^{S} \alpha_{R,s}^u Y_{R,s}^u(i,j) P(i,j) \qquad (1)$$
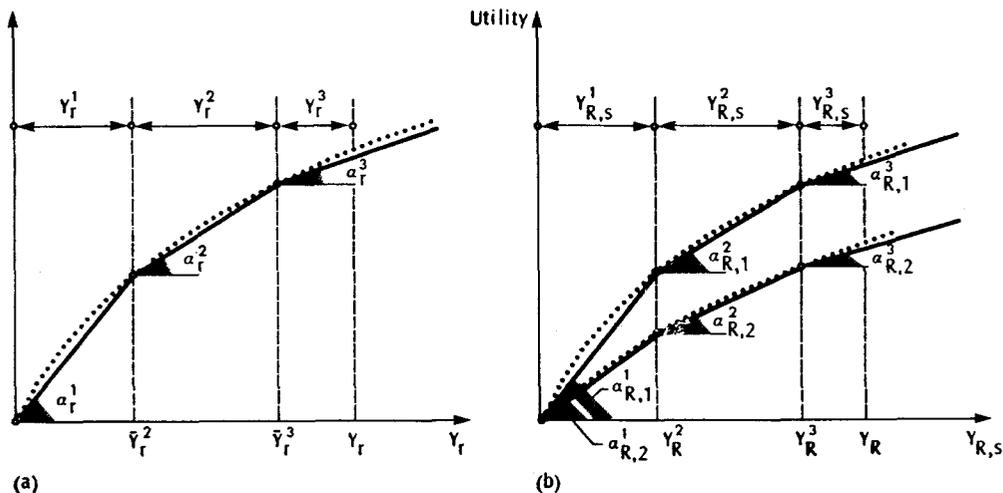
Figure 2 - Linearized Utility Functions

The first term on the right-hand side is the total utility of all city re-residents obtained from the first R-1 commodities, and the second term is the utility from floor space in buildings with different densities. For the linear programming model (LPM), the welfare is maximized with regard to all endogenous variables except the populations P(i,j) residing in zone (i,j). The latter are treated as constant parameters in the LPM. Hence, the welfare W evaluated at the optimum of the LPM is a function of populations P(i,j). The sum of all populations P(i,j) residing in zones (i,j) must be equal to the city population

$$P = \sum_{i=1}^{I} \sum_{j=1}^{J} P(i,j) \qquad (2)$$

There are constraints on the demand components $Y_r(r=1,\ldots,R-1)$ due to the linearized utility function in Fig. 2a

$$0 \leq Y_r^1(i,j) \leq \bar{Y}_r^2 \qquad (3a)$$

$$0 \leq Y_r^2(i,j) \leq \overline{Y}_r^3 - \overline{Y}_r^2 \tag{3b}$$

$$0 \leq Y_r^3(i,j) \tag{3c}$$

$$\alpha_r^u > \alpha_r^{u+1}, \quad u=1,2 \tag{3d}$$

Inequality eqn. (3d) tells us that the marginal utility tends to fall with increasing consumption. Similar constraints are obtained for floor space demand

$$0 \leq Y_{R,s}^1(i,j) \leq \overline{Y}_R^2 \tag{4a}$$

$$0 \leq Y_{R,s}^2(i,j) \leq \overline{Y}_R^3 - \overline{Y}_R^2 \tag{4b}$$

$$0 \leq Y_{R,s}^3(i,j) \tag{4c}$$

$$\alpha_{R,s}^u > \alpha_{R,s}^{u+1}, \quad u=1,2 \tag{4d}$$

$$\alpha_{R,s}^u > \alpha_{R,s+1}^u, \quad s=1,\ldots, S-1; \; u=1,\ldots, 3 \tag{4e}$$

$$\sum_{s=1}^{S} Y_{R,s}^1(i,j) \leq \overline{Y}_R^2 \tag{4f}$$

$$\sum_{s=1}^{S} Y_{R,s}^2(i,j) \leq \overline{Y}_R^3 - \overline{Y}_R^2 \tag{4g}$$

Eqns. (4a)-(4d) are the same as (3a)-(3d). Eqn. (4e) tells us that the marginal utility tends to fall with increasing building density. Eqns. (4f) and (4g) are introduced to avoid the LPM choosing the first or second component of total floorspace demand from different building densities. Since $Y_{R,1}^1$ yields the greatest possible utility, the LPM will first exhaust, if feasible, the first interval in Fig. 2b. Condition (4f) impedes the LPM from choosing $Y_{R,2}^1$, if $\alpha_{R,2}^1 > \alpha_{R,1}^2$, because the same apartment cannot be located in a building with different densities. It is possible, however, that a solution still exists for which components of floorspace demand are chosen from different densities. In this case, the LPM could be rerun excluding some densities.

Export

The export of commodity r in prespecified export nodes (i*,j*) must be at least as great as the exogenous demand for exports

$$\sum_{i=i*} \sum_{j=i*} E_r(i,j) \geq E_r, \quad r=1, \ldots, R-1 \tag{5}$$

For locally consumed commodities E_r is zero.

Production

To consider transportation cost in terms of resources, each zone (i,j) is distinguished by four boundaries as shown in Fig. 3. The numbering of the boundaries follows in a clockwise rotation manner starting with the southbound boundary. To economize on notation, all zones are equally treated but boundary

conditions of the urban area are given separately. The sum of final demand for, outflow of, and use as an input in other productions of a good in zone (i,j) cannot exceed the sum of inflow and production of that good; this is production constraint (6) for commodity r

$$[F_r^1(i+1,j) + F_r^2(i,j+1) + F_r^3(i-1,j) + F_r^4(i,j-1)] + \sum_s z_{r,s}(i,j)$$

$$- \sum_s \sum_{q=1}^{R-1} a_{r,q,s} z_{q,s}(i,j) - a_{r,R} z_R(i,j) - \sum_{b=1} F_r^b(i,j) - E_r(i,j)$$

$$- D_r(i,j) \geq 0 \ , \ r=1, \ \dots, \ R-1; \ i=1, \ \dots, \ I; \ j=1, \ \dots, \ J \quad (6)$$

The first term in brackets denotes the inflow of commodity r and the second term is production. The next two terms denote the use of commodity r as an input in the production of all other commodities. It has been assumed that floor-space production is independent of the building density because the construction industry is mobile. This allows one to distinguish the production of floorspace by building densities on the demand side, ie., in connection with the demand for floor space by residents. The last terms are outflow, export, and final demand. Exports are neglected in eqn. (6) if zone (i,j) is not an export node (i^*,j^*). Most of the input-output coefficients are not sensitive to changes in the building density and can therefore be kept constant, ie., $a_{r,q,s} = a_{r,q}$. However, in the absence of public goods in this model, at least one industry should exhibit increasing returns to scale in the sense that either its input-output coefficients $a_{r,q,s}$ or its factor ratios $b_{r,q,s}$, or both, depend on the building density. For the boundaries of the urban area, it is assumed that

$$F_r^b(I+1,j) = F_r^b(0,j) = 0 \ , \ j=0,\dots, J+1; \ b=1,\dots, 4 \quad (7a)$$

$$F_r^b(i,J+1) = F_r^b(i,0) = 0 \ , \ i=1,\dots, I; \ b=1,\dots, 4 \quad (7b)$$

$$F_r^3(I,j) = F_r^1(1,j) = 0 \ , \ j=1,\dots, J \quad (7c)$$

$$F_r^2(i,1) = F_r^4(i,J) = 0 \ , \ i=1,\dots, I \quad (7d)$$

Eqns. (7) ensure that there is no inflow or outflow of goods at the boundaries of the urban area.

Final Demand (other than housing)

The model determines commuting trip flows but not shopping trip flows. Therefore, it simply requires total final demand for commodity r to be at least as great as total consumption by households

$$\sum_{i=1}^{I} \sum_{j=1}^{J} D_r(i,j) \geq \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{u=1}^{3} P(i,j) \ Y_r^u(i,j) \ , \ r=1, \ \dots, \ R-1 \quad (8)$$

At the expense of additional endogenous variables the model can be modified to include shopping trips in the same way as is shown for commuting flows below.
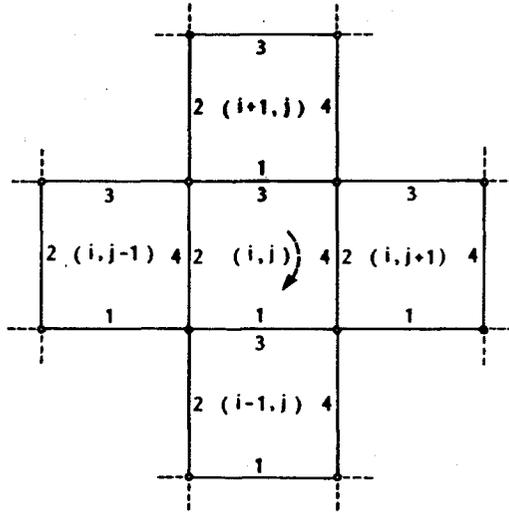
Figure 3 - Internal Numbering of Zones

Production of Floor Space

Total production of floor space in zone (i,j) must be at least as great as total demand for floor space by the R-1 industries and residents in that zone

$$Z_R(i,j) - \sum_{s=1}^{S} \sum_{q=1}^{R-1} a_{R,q,s}\, Z_{q,s}(i,j) - \sum_{u=1}^{3} \sum_{s=1}^{S} P(i,j) Y_{R,s}^{u}(i,j) \geq 0 ,$$

$$i=1, \ldots, I; \quad j=1, \ldots, J \tag{9}$$

To economize on variables, total floor space production could be substituted into eqn. (6).

Commuting and Employment

The model considers both commuting trip flows and commodity flows. It requires that the demand for labour by all industries which are located in zone (i,j) does not exceed the supply of labour in that zone. The latter is given by the sum of net inflow of labourers and residents in that zone

$$\sum_{r=1}^{R-1} \sum_{s=1}^{S} [b_{1,r,s} + b_{1,R,s}\, a_{R,r,s}]\, Z_{r,s}(i,j) + \sum_{u=1}^{3} \sum_{s=1}^{S} b_{1,R,s}\, P(i,j) Y_{R,s}^{u}(i,j)$$

$$\leq [H^1(i+1,j) + H^2(i,j+1) + H^3(i-1,j) + H^4(i,j-1)] + P(i,j) - \sum_{b=1}^{4} H^b(i,j) ,$$

$$i = 1, \ldots, I; \quad j = 1, \ldots, J \tag{10}$$

In order to account for the building density the production of floor space has been substituted from eqn. (9). The first term denotes the demand for labour by all industries including construction of commercial buildings. The demand

for labour by housing construction is given in the second term. Population $P(i,j)$ residing in zone (i,j) plus inflow minus outflow of the labour force is defined as the supply of labour. Once again, increasing returns to scale can be considered by the labour ratio $b_{1,r,s}$ which should fall, at least in a certain range, with increasing building density. In high-rise buildings, however, the input of labour per unit of output could rise due to increasing communication within that building. For a closed urban economy it is required that

$$H^b(I+1,j) = H^b(0,j) = 0 \ , \ j=0,\ldots,J+1; \ b=1,\ldots,4 \tag{11a}$$

$$H^b(i,J+1) = H^b(i,0) = 0 \ , \ i=1,\ldots,I; \ b=1,\ldots,4 \tag{11b}$$

$$H^3(I,j) = H^1(1,j) = 0 \ , \ j=1,\ldots,J \tag{11c}$$

$$H^4(i,J) = H^2(i,1) = 0 \ , \ i=1,\ldots,I \tag{11d}$$

which ensures that there is no inflow or outflow of labour at the boundary of the city. Moreover, employment in transportation, which is not assigned to specific zones, is considered by eqn. (12).

$$\sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{m=1}^{M} b_{1,R+m} \ Z_{R+m}(i,j)$$

$$+ \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{r=1}^{R-1} \sum_{s=1}^{S} [b_{1,r,s} + b_{1,R,s} \ a_{R,r,s}] \ Z_{r,s}(i,j)$$

$$+ \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{u=1}^{3} \sum_{s=1}^{S} b_{1,R,s} \ P(i,j) \ Y^u_{R,s}(i,J) \leq P \tag{12}$$

The demand for labour by transportation and industries cannot exceed total city population.

Transportation

The LPM chooses those transport modes which are efficient in terms of resource costs, while tripmakers are indifferent to various modes, ie., individual utility functions are independent of commuting costs. It is assumed that commodities can be shipped by automobiles only. In constraint eqn. (13) the supply of transportation services of automobiles (ie., mode 1) must be at least as great as the demand for transportation services in zone (i,j)

$$Z_{R+1}(i,j) \geq d(i,j) \ [\sum_{q=1}^{R-1} a_{R+1,q} \ [F^1_q(i+1,j) + F^2_q(i,j+1) + F^3_q(i-1,j) + F^4_q(i,j-1)]$$

$$+ \sum_{q=1}^{R-1} \sum_{b=1}^{4} a_{R+1,q} \ F^b_q(i,j) + D_{R+1}(i,j)] \ , \ i=1,\ldots,I; \ j=1,\ldots,J \tag{13}$$

The same applies to the remaining modes in constraint eqn. (14). Total demand for transportation services in zone (i,j) is given by the sum of inflow and outflow of commuters. This is constraint eqn. (15).

$$Z_{R+m}(i,j) \geq d(i,j) \ D_{R+m}(i,j) \ , \ m = 2,\ldots,M; \ i = 1,\ldots,I; \ j = 1,\ldots,J \tag{14}$$

$$\sum_{m=1}^{M} D_{R+m}(i,j) = H^1(i+1,j) + H^2(i,j+1) + H^3(i-1,j) + H^4(i,j-1)$$

$$+ \sum_{b=1}^{4} H^b(i,j) \ , \ i = 1,\ldots,I; \ j = 1,\ldots,J \tag{15}$$

Capital Services

The city is given a fixed capital stock. The demand for capital services which depends on transportation services (first term), commodity production (second term), and housing construction (third term) must not exceed the fixed capital stock

$$\sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{m=1}^{M} b_{2,R+m} \ Z_{R+m}(i,j)$$

$$+ \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{r=1}^{R-1} \sum_{s=1}^{S} [b_{2,r,s} + b_{2,R,s} \ a_{R,r,s}] \ Z_{r,s}(i,j)$$

$$+ \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{u=1}^{3} \sum_{s=1}^{S} b_{2,R,s} \ P(i,j) \ Y_{R,s}^u(i,j) \leq K \tag{16}$$

Land Constraints

In each zone, the use of land for transportation (first term), commercial buildings (second term), and housing (third term) cannot exceed the supply of land in that zone

$$\sum_{m=1}^{M} b_{3,R+m} \ Z_{R+m}(i,j) + \sum_{q=1}^{R-1} \sum_{s=1}^{S} \frac{a_{R,q,s}}{g_s} \ Z_{q,s}(i,j)$$

$$+ \sum_{u=1}^{3} \sum_{s=1}^{S} \frac{P(i,j)}{g_s} \ Y_{R,s}^u(i,j) \leq A(i,j) \ , \ i=1,\ldots,I; \ j=1,\ldots,J \tag{17}$$

In contrast to earlier attempts (Mills, 1972a; Kim, 1979), this model distinguishes clearly between the demand for floor space and land by introducing the floor-to-land ratio. It is clear that at the optimum not all transport modes will necessarily be used. Both technology and topography of a specific city will decide whether a subway system, say, is efficient for a given endowment of resources.

Combined Linear/Non-Linear Programming

The objective function eqn. (1) and constraint eqns. (2)-(17) describe the LPM for fixed populations $P(i,j)$ residing in zone (i,j). For each assignment of the city population to various zones, there exists an optimum welfare W which is given by the LPM. The final goal is to maximize this non-linear function

$$\{P(i,j)\} \quad \max \quad W[P(i,j)] \tag{18a}$$

$$\text{s.t.} \quad \sum_{i=1}^{I} \sum_{j=1}^{J} P(i,j) = P \tag{18b}$$

$$W[P(i,j)] = LPM[\text{eqns.}(1)-(17)] \tag{18c}$$

where the welfare function W is defined by the LPM of eqns. (1)-(17). Naturally the model in eqns. (1)-(17) can be solved directly as a non-linear optimization problem assuming a computer is available to deal with this number of dependent variables. Given a CDC-6000 computer for example, the number of dependent variables is restricted to about one hundred in non-linear optimizations. This is far too low a capacity when compared to the number of dependent variables of this model, the latter being: $IJ[6+2M+8(R-1)+(R+2)S]+(R-1)I*J*$. For instance, with 50 zones, 4 modes, 4 industries (eg., retail, manufacturing, services, and construction), 5 densities, and 2 export nodes, the number of dependent variables is 3406. In the combined linear/non-linear programming model defined by eqn. (18), the number of endogenous variables of the non-linear optimization is equal to the number of zones (IJ), whereas the remaining dependent variables are determined by the LPM.

At the optimum, the efficient allocation of labour, capital, buildings, and land in each zone is given. In particular, the assignment of various activities to each zone is determined together with the efficient building density for each activity and zone. Moreover, the solution will decide which transport modes should be used by how much and if at all. The dual program to the primal LPM of eqns. (1)-(17) yields the economic evaluation of resources, ie., shadow incomes, capital rents, and land rents at the social optimum. Since populations P(i,j) are exogenous to the LPM, it is clear that shadow incomes (hence utilities) are not constant between zones which is due to the Benthamite welfare function.

Land Use Policies

The present model could serve as a tool for town planning. Naturally, many exogenous variables are subject to forecast and measurement errors. Since land use zoning is quite expensive in terms of resources if binding, such regulations should be viable in the long run so that misallocations are avoided. Land use zoning should provide a framework within which residents and investors are free to choose from alternative activities, information is given to everybody at no or minimal cost, long run investments are not subject to unnecessary uncertainty, and future changes in land use regulations should be possible at minimal cost. With this in mind, the sensitivity of the model is evaluated when key variables are changed. For instance, changes in city population or

production technology might have a big impact on the allocation of resources. In addition, it might be of interest to evaluate the effect of new export nodes to be developed in the city, ie., the model can be used as a basis for cost-benefit analysis. Similarly, restrictions on building densities or on building heights can be evaluated.

The practicality of implementing the model obviously depends on the specifications of the master plan in question. As implied earlier, the master plan of a particular city should be viable for a long time horizon, thus making exogenous variables subject to uncertainty. As a consequence, it is not desirable to collect extensive statistical data on input-output coefficients, factor ratios, floor-to-land ratios, and so forth. What is required are best guesses of possible, future technologies based on present knowledge and a few statistical samples, as well as imagination about the uncertainty involved in estimating future trends. Hence, it seems to me that the model can be implemented for real cities at low costs, considering the fact that modifications such as an irregular zoning system, zone-dependent coefficients to capture the topography of the city, etc., can easily be done by simply attaching locational coordinates to corresponding coefficients.

A Technical Digression

It is not intended to deal here with the problem of which numerical algorithm is best suited for the LPM described above (eg., a particular decomposition method as described in Sharpe, this volume). The structure of the linear/non-linear model, however, deserves some comments. Denote the number of endogenous variables (excluding $P(i,j)$) as ν and that of constraints as μ. Let $\underline{P}$ be the $(\nu \times 1)$ row vector whose first I J elements are $P(i,j)$ with the remaining elements equal to zero; $\underline{\alpha}$ the $(\nu \times \nu)$ matrix of marginal utilities; $\underline{x}$ the $(\nu \times 1)$ column vector of endogenous variables; $\underline{A}$ a $(\mu \times \nu)$ matrix of constants, some elements of which contain $P(i,j)$; $\underline{B}$ a $(\mu \times \nu)$ matrix of constants; and finally $\underline{b}$ a $(\mu \times 1)$ vector of resource constraints. Then the LPM can be written in matrix notation:

$$W(\underline{P}) = \max_{\underline{x}} F = \underline{P} \, \underline{\alpha} \, \underline{x} \tag{19a}$$

$$\text{s.t.} \quad \underline{A} \, \underline{x} - \underline{B} \, \underline{P}' \leq \underline{b} \tag{19b}$$

A solution of the LPM exists if the opportunity set of $\underline{x}$, as described by eqn. (19b), is bounded and nonempty. By eqns. (2)-(17), consumption, production, and transportation ultimately use labour, capital, and land, total amounts of which are fixed. Hence, the opportunity set is bounded. Of course, the values of the exogenous variables must be set such that the opportunity set is nonempty. For instance, if there is a positive demand for exports, it must be possible to

assign all production and residents to export nodes in order to satisfy at least that export demand while total transport costs and final demands are zero. The opportunity set is nonempty if there is no demand for exports because a solution with no activity ($\underline{x} = \underline{0}$) is feasible. If the number of endogenous variables is greater than the number of constraints, then the solution occurs at a vertex and is unique. In our case

$$\nu = IJ[5+2M+8(R-1) + (R+2)S] + (R-1)I*J* \qquad (20a)$$

$$\mu = IJ[4+M+4(R-1)+3\ S] + 14 + 4R + 5S + 10(I+J) \qquad (20b)$$

With the same values as in the last section, $\nu = 3,356$ and $\mu = 1,955$, given that $I = 5$, $J = 10$, and $I\ J = 50$. Therefore, it is very likely that in real applications the LPM solution is unique with regard to $\underline{x}$.

Let the constraints (19b) be partitioned and renumbered such that the first μ_1 constraints are satisfied as equalities at the optimum, while the remaining constraints are satisfied as strict inequalities. Then the welfare function can be written, when assuming a vertex solution, as

$$W(\underline{P}) = \underline{P}^1 \underline{\alpha}^{11} (\underline{A}^{11})^{-1} (\underline{b}^1 + \underline{B}^{11} \underline{P}'^1) \qquad (21)$$

where superscripts denote partitioning as described above. Since $\underline{A}^{11}$ may contain $\underline{P}^1$, the welfare function is at least quadratic, ie., non-linear. By the Weierstrass theorem, the global maximum of W exists (on its domain), if the opportunity set

$$\Pi = \left\{ P(i,j) \mid \Sigma\Sigma P(i,j) = P \right\} \qquad (22)$$

is nonempty and compact, and if the welfare function is continuous. In our case, both conditions are fulfilled. Moreover, the solution is unique, if the opportunity set is also convex, and if the welfare function is also strictly concave. While the former condition applies to eqn. (22), the latter is not easy to show. However, there are weaker conditions for a unique solution to hold.

EXTENSIONS

Finite Elements

In this section, several possible extensions are briefly discussed. The first extension of the model concerns the geometry of the zones. In order to reduce the number of variables, the topography of the city can possibly better be approximated by using triangular rather than rectangular zones. In Fig. 4, a hypothetical city map is divided into triangular zones where regions of no construction activity are not included in the network. The modification of the above equations to triangular zones is straightforward. Further, by analogy to the method of finite elements in mechanical engineering it is possible to

change the method. The tentative outline of the finite element approach when applied to this situation is the following. Firstly, the city is divided into triangular zones. Secondly, the welfare maximization problem in its general, non-linear form is formulated for each zone independently and solved in closed form. If a closed form solution is not possible, find a best, parameterized approximation. The solution for each zone depends on both boundary conditions and dependent variables of the objective function. Thirdly, all zones are linked together by boundary conditions. The latter are known at the outskirts of the city. Hence, dependent variables not included in the objective function can be obtained by solving a linear equation system. Fourthly, optimal utilities are summed over all zones and maximized when subject to the linear equation system. It is left to future research, however, to answer the question whether this approach brings about numerical advantages.
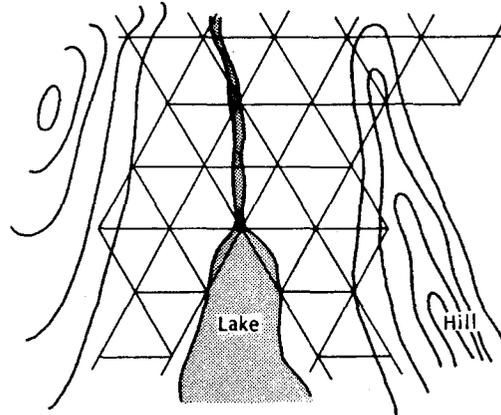

Figure 4 - Triangular Network

Traffic Congestion

It is a well known fact that traffic congestion in cities causes considerable social costs. Mills (1972a) has shown how traffic congestion can be introduced into a LPM by considering a stepwise linear marginal cost function. Of course, this requires the knowledge of the marginal cost function of a typical city (see also Hutchinson, 1974).

Imports

Imports can be considered in two ways in this model. Firstly, in the basic model exports were treated as variables net of imports. Secondly, imports can be dealt with as separate inputs whereas total imports are given exogenously. Moreover, import nodes need not be identical to export nodes.

Public Goods

Introducing public goods is not a straightforward matter. Initially, public goods should be distinguished by their degree of localness. While some of them provide services to all residents in the city, eg., a theatre, opera, etc. others are strictly local, ie., do not provide services to residents residing in adjacent zones. On the other hand, individual utilities now depend on public-goods consumption. Hence, the efficient allocation of households depends crucially on the spatial allocation of public goods if these are local in degree.

Different Resource Endowment

In the basic model, all households are equally endowed. To consider an unequal distribution of resources, different classes with regard to ability to work, capital, and land can be introduced. Define a joint density function

$$f(A_a, K_k, B_b) \tag{23a}$$

where a, k, and b denote indices for different classes of work ability (A), capital endowment (K), and land endowment (B). The sum of densities over all classes is equal to one

$$\sum_a \sum_k \sum_b f(A_a, K_k, B_b) = 1 \tag{23b}$$

The supply of labour, capital, and land in the city is given by:

$$\sum_a A_a \ P \sum_k \sum_b f(A_a, K_k, B_b) \tag{23c}$$

$$\sum_k K_k \ P \sum_a \sum_b f(A_a, K_k, B_b) \tag{23d}$$

$$\sum_b B_b \ P \sum_a \sum_k f(A_a, K_k, B_b) \tag{23e}$$

For each class, work places, living places, and commuting flows have to be determined in the same way a was shown in the basic model. The number of variables, however, will increase considerably even with a moderate number of classes.

The Small, Open City

In the basic model, the city was considered to be closed with both labour and capital supply fixed. This is one of two polar cases. In the other polar case the city is assumed to be open but small, ie., migration of labour and capital between the city and the rest of the economy does not affect the state of the rest of the economy. In other words, the utility of city residents is given by that obtained outside the city. In the case of a small open economy it is appropriate to maximize the city production net of resources costs when

subject to the condition that all city residents attain the given utility level (Kanemoto, 1980). If neither of these polar cases is a good approximation to the city in question, a model should be formulated in which migration of labour depends on the welfare attainable in the city.

REFERENCES

Anas, A., 1983, From Physical to Economic Urban Models: The Lowry Framework Revisited, a paper presented to the International Symposium on New Directions in Urban Modelling, University of Waterloo, Waterloo, Ontario.

Batty, M., 1976, Urban Modelling, Cambridge University Press, London.

Büttler, H.-J. and Beckmann, M.J., 1980, Design Parameters in Housing Construction and the Market for Urban Housing, Econometrica, 48, 201-225.

Echenique, M., 1983, The Practice of Modelling in Developing Countries, a paper presented to the International Symposium on New Directions in Urban Modelling, University of Waterloo, Waterloo, Ontario.

Henderson, J., 1977, Economic Theory and the Cities, Academic Press, New York.

Herbert, J.D. and Stevens, B.H., 1960, A Model for the Distribution of Residential Activity in Urban Areas, Journal of Regional Science, 2, 21-36.

Hochman, O. and Ofek, H., 1979, A Theory of the Behaviour of Municipal Governments: The Case of Internalizing Pollution Externalities, Journal of Urban Economics, 6, 416-431.

Hutchinson, B.G., 1974, Principles of Urban Transport Systems Planning, McGraw-Hill, New York.

Kanemoto, Y., 1980, Theories of Urban Externalities, North Holland, Amsterdam.

Kim, T.J., 1979, Alternative Transportation Modes in an Urban Land Use Model: A General Equilibrium Approach, Journal of Urban Economics, 6, 197-215.

Mills, E.S., 1972a, Markets and Efficient Resource Allocation in Urban Areas, Swedish Journal of Economics, 74, 100-113.

Mills, E.S., 1972b, Studies in the Structure of the Urban Economy, The Johns Hopkins University Press, Baltimore, MD.

Mirrlees, J.A., 1972, The Optimum Town, Swedish Journal of Economics, 74, 114-135.

Sharpe, Ron, 1984, An Optimum Economic/Energy Land-Use Transportation Model, this volume.