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INVESTMENT PLANNING OF A ROAD LINK

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Abstract—The purpose of this paper is to determine the optimal investments of an existing road link over time. The problem is formulated in terms of optimal control and solved by Pontryagin's maximum principle. Three state variables of the road are considered: the smoothness of the road pavement surface, the volume of traffic, and the capacity. The control variables are: investment in smoothness and in capacity. Optimality is considered to be that investment programme for smoothness and capacity which maximizes the integral of net benefits over a finite or infinite time horizon.

The time path of the investment in smoothness is uniquely determined by a saddle point solution. There are three possible solutions for the investment in capacity. Either the road will be widened at the initial time of the system, or at a later point in time, or never. This depends on the time path of the shadow price of capacity relative to the constant marginal cost to invest in capacity. Finally, a budget constraint to the Ministry of Transport is imposed. As a result, the pattern of the time paths does not change in general.

NOTATION

- B benefits
- c capacity of road link
- c^m upper bound of capacity
- F discounted scrap value at terminal time
- H Hamiltonian
- i social discount rate
- J objective functional to be maximized
- K cost of investment
- L Lagrangian
- q_1, q_2, q_3 "carried forward" shadow prices of smoothness, volume of traffic, and capacity, respectively.
- s smoothness, riding comfort index, $1 \leq s \leq 10$
- t, t_0, t_1 time, initial and terminal time, respectively.
- u_1 investment in smoothness, measured in Riding Comfort Index.
- u_2 investment in capacity of road
- u_2^m upper bound of investment in capacity
- v volume of traffic
- y_1, y_2, y_3 shadow prices of smoothness, volume of traffic, and capacity, respectively.
- Z budget constraint
- β_0, β_1 coefficients of benefit function
- γ coefficient of cost function, marginal cost due to investment in capacity
- δ_1 depreciation rate of pavement surface
- δ_2, δ_3 coefficients of growth rate of volume of traffic
- λ^*, λ Lagrange multiplier and discounted Lagrange multiplier, respectively.
- μ Lagrange multiplier
- Φ operating costs
- ϕ_0, ϕ_1, ϕ_2 coefficients of operating cost function

1. INTRODUCTION

Managing provincial road systems means allocation of investments over time in order to reach an "optimal" state of the road system. The dynamic aspect of this problem can, in general, be solved by dynamic programming, calculus of variations, or by Pontryagin's maximum principle, the latter being the most general approach to control theory.

The purpose of this paper is to deal with "optimal" investments of a single section of roadway. It is assumed that in rural transport systems individual links are independent of other links in the system in so far as demand is concerned. However, the volume of traffic on the link is responsive to the capacity provided and the degree of congestion. The state variables of the road link are the smoothness of the road which represents the quality of the pavement surface, the volume of traffic, and the capacity. Two control or policy variables are considered: the investment in smoothness and the investment in capacity. Starting from the present state of the three state variables, optimality is considered to be that investment programme for smoothness and capacity which maximizes the integral of net benefits over a finite or infinite time horizon. In the case of a finite time horizon, terminal values of the state variables to be reached can be imposed, or scrap values at the end of the time period. In the case of an infinite time horizon, the system will reach, hopefully, an equilibrium. Finally, investment can be subject to a budget constraint which is assumed to be constant in real terms over time.

The purpose of this paper is to give insight into the dynamics of the system and to derive general investment rules. The problem will be solved in the Section 2 for smoothness alone, and in the Section 3 for volume of traffic and capacity. In the Section 4, the necessary conditions for the combined problem are presented. Finally, the case of a constant budget constraint is considered in the last section.

2. INVESTMENT IN SMOOTHNESS

Time is considered to be continuous, but in numerical calculations the "time period" could be one month, say. The state variable "smoothness" (s) is the usual Riding

Comfort Index which is a subjective measure of riding quality. The index varies between 1 and 10, but we can consider the index to be unbounded due to benefit and cost functions below. The smoothness deteriorates at a rate of $\delta_1 \approx 0.005$ per month which causes an exponential fall from a smoothness of 7, say, to a smoothness of 3 over a period of 14 yr, which is typical for the life of a pavement surfacing. Investments in smoothness u_1 improve the Riding Comfort Index and represent the gross investment in the stock of smoothness. The rate of change in the stock of smoothness is therefore

$$\dot{s} = u_1 - \delta_1 s, \tag{1}$$

which is the equation of motion for the state variable "smoothness". $\dot{s}$ denotes the time derivative of s.

The optimal policy variable "investment in smoothness" u_1 is determined when maximizing the integral of net benefits due to smoothness and investment over a certain time horizon. The benefits are a function of smoothness s and of the volume of traffic to capacity ratio v/c. The benefits rise rapidly with increasing smoothness and then level out at higher smoothness levels in a logarithmic form. The benefit per vehicle for a one mile portion of roadway with an increase in smoothness from 3 to 7, say, is about $\frac{1}{2}$ cent per vehicle mile at a v/c ratio of 0.5 and $\beta_1 = 0.003$. With higher v/c ratios and the corresponding lower speeds the benefits of smoother roads are reduced linearly with increasing v/c ratios. Therefore, the benefits B are:

$$B = \beta_1 c \ln s, \tag{2}$$

where c denotes the capacity of the road (vehicles per month) and β_1 is a coefficient. This is not the only possible formulation, but to date, very little is known about the actual relationship and in the meanwhile, this is a reasonable functional form. The effects of smoothness are more pronounced with rougher roads and also with higher capacity roads. For a reference to this type of work see *Pavement Management Guide* (1977).

Next, the operating costs of the road link are to be considered. These costs represent repair work that does not improve the pavement surfacing, but is necessary to keep the road "operable". They are assumed to increase linearly with the capacity to smoothness ratio c/s. For one mile of a two lane road with a smoothness of 5 the annual pavement costs are \$1400 per year, if $\phi_0 = 1200$ and $\phi_2 = 16.6 \times 10^{-6}$. The operating costs Φ are:

$$\Phi = \phi_0 + \phi_2 \frac{c}{s}. \tag{3}$$

Here, ϕ_0 and ϕ_2 are coefficients.

Next, the maintenance cost or investment in smoothness, respectively, is an increasing function of investment, while the influence of capacity is neglected. For example, for 1 mile of road to go from a smoothness of 3–7 the cost is, approximately, \$40,000 per mile for the two lane road. The following function is assumed for the maintenance cost K,

$$K = (s + u_1)^2 - s^2. \tag{4}$$

Finally, for the case of finite time horizon the discounted scrap value F might be

$$F = s_1^2; \qquad s_1 = s(t_1). \tag{5}$$

That is to say that eqns (3)–(5) all relate to a specific unit of road length. Where numbers are mentioned, 1 mile of road was considered. It all cases these equations relate to a relatively new approach to pavement performance and empirical results are not currently available. However, the general functional forms are thought to be appropriate. With the social discount rate i the objective functional J to be maximized can be written as

$$\max_{\{u_1\}} J = \int_{t_0}^{t_1} \{B(s) - \Phi(s) - K(u_1, s)\} e^{-i(t-t_0)} dt + F(s_1) \tag{6}$$

subject to

$$\dot{s} = u_1 - \delta_1 s$$

$$s(t_0) = s_0$$

given

$$s(t_1) = s_1$$

or

$$F(s_1) = s_1^2$$

given for $t_1 < \infty$. The objective functional J is maximized when the Hamiltonian

$$H = e^{-i(t-t_0)} \{B(s) - \Phi(s) - K(u_1, s) + q_1(u_1 - \delta_1 s)\} \tag{7a}$$

$$y_1 \equiv q_1 e^{-i(t-t_0)} \tag{7b}$$

is maximized with respect to the control variable. Since $B - \Phi - K$ is a concave function, this is a necessary and sufficient condition. Here, the costate variable y_1 has been transformed into q_1 for the sake of convenience: and henceforth we will denote q_1 as the "carried forward" shadow price, though this expression is not exact. The first-order condition for a maximum is

$$\frac{\partial H}{\partial u_1} = e^{-i(t-t_0)} \left\{ -\frac{\partial K}{\partial u_1} + q_1 \right\} = 0 \tag{8a}$$

or, after introducing eqn (4), we get

$$q_1(t) = \frac{\partial K}{\partial u_1} = 2(s + u_1). \tag{8b}$$

The costate variable y_1 can be interpreted as the shadow price of smoothness, because

$$\frac{\partial J}{\partial s(t_0)} = y_1(t_0).$$

Therefore, eqn (8b) means that at any point in time the "carried forward" shadow price must be equal to the instantaneous marginal cost to invest in smoothness. The additional conditions for a maximum are the canonical equations

$$\dot{y}_1 = -\frac{\partial H}{\partial s}, \quad (9a)$$

or equivalently

$$\dot{q}_1 = -\frac{\partial(B-\Phi-K)}{\partial s} + (\delta_1 + i)q_1, \quad (9b)$$

and the equation of motion given by eqn (1). Equation (9b) can be written in the following way

$$\frac{\dot{q}_1}{q_1} + \frac{\partial(B-\Phi-K)/\partial s}{q_1 + \partial K/\partial u_1} - \delta_1 - i = 0, \quad (9c)$$

the economic interpretation of which is straightforward: the net gain of holding a "unit of smoothness" over an interval of time is zero, where the first term in eqn (9c) represents the capital gain or loss, the second term represents the instantaneous marginal internal rate of return due to investment in smoothness, and the last two terms represent the losses due to depreciation of smoothness (δ_1) and interest (i).

Substituting eqn (8b) and the time derivative of (8b) into eqn (9b), we get two differential equations in smoothness and in investment that describe the optimal investment plan:

$$\dot{u}_1 = -\frac{1}{2}\left(\frac{\beta_1 c}{s} + \phi_2 \frac{c}{s^2}\right) + (\delta_1 + i)u_1 + (2\delta_1 + i)s \quad (10)$$

$$\dot{s} = u_1 - \delta_1 s. \quad (11)$$

The kind of solution can be shown in the phase diagram of Fig. 1. The line where there is no tendency for the investment u_1 to change is obtained by setting $\dot{u}_1 = 0$.

$$\dot{u}_1 = 0\colon u_1 = \frac{1}{2}\frac{\left(\frac{\beta_1 c}{s} + \phi_2 \frac{c}{s^2}\right)}{(\delta_1 + i)} - \frac{(2\delta_1 + i)}{(\delta_1 + i)}s \quad (12)$$

and similarly, for smoothness we get

$$\dot{s} = 0\colon u_1 = \delta_1 s. \quad (13)$$

The line $\dot{u}_1 = 0$ given in eqn (12) is convex because we subtract from the convex marginal benefit function a straight line. The equilibrium values for smoothness s^* and for investment in smoothness u_1^* are obtained when substituting eqn (13) into (12):

$$(\delta_1^2 + \delta_1 i + 2\delta_1 + i)s^{*3} - \frac{\beta_1 c}{2}s^* - \frac{\phi_2 c}{2} = 0 \quad (14)$$

$$u_1^* = \delta_1 s^*. \quad (15)$$

The numerical solution of the cubic eqn (14) yields for reasonable values an equilibrium value for smoothness s^* of about 8. Investments in smoothness tend to rise above the $\dot{u}_1 = 0$ line, because

$$\left.\frac{\partial \dot{u}_1}{\partial u_1}\right|_s = (\delta_1 + i) > 0,$$

while smoothness tends to fall to the right of the $\dot{s} = 0$ line, because

$$\left.\frac{\partial \dot{s}}{\partial s}\right|_{u_1} = -\delta_1 < 0.$$

These tendencies are indicated by arrows in Fig. 1. Therefore, the equilibrium is a saddle point. This can be shown by linearizing the system (10) and (11) around the equilibrium values s^*, u_1^*

$$\dot{u}_1 \simeq (\delta_1 + i)(u_1 - u_1^*) + \left(2\delta_1 + i + \frac{\beta_1 c}{2s^{*2}} + \frac{\phi_2 c}{s^{*3}}\right)(s - s^*) \quad (16)$$

$$\dot{s} = (u_1 - u_1^*) - \delta_1(s - s^*). \quad (17)$$

The characteristic values λ of the linearized system (16) and (17) are:

$$\lambda_{1,2} = \frac{1}{2}\left[i \pm \left\{i^2 + 4\left(\delta_1^2 + \delta_1 i + 2\delta_1 + i + \frac{\beta_1 c}{2s^{*2}} + \frac{\phi_2 c}{s^{*3}}\right)\right\}^{1/2}\right].$$

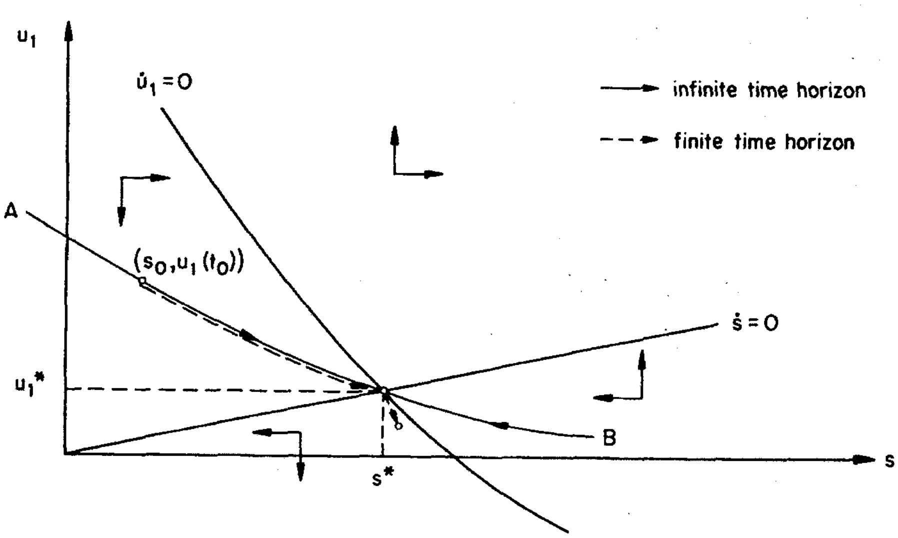

Fig. 1. Phase diagram for s and u_1.

The characteristic values λ_1 and λ_2 are real and of opposite sign which indicates the equilibrium to be a saddle point. The stable branch of this solution is given by the line AB in Fig. 1. The optimal investment programme must, therefore, follow the unique stable branch AB in Fig. 1, i.e. there exists a one-to-one correspondence between degree of smoothness and optimal maintenance of smoothness. Or stated otherwise: a path, not following the stable branch AB in Fig. 1, does not maximise the net benefits in eqn (6) and, in addition, it does not lead to the equilibrium point (u_1^*, s^*). What happens when the system is in equilibrium? The maintenance of the pavement u_1 and the smoothness s are constant over time and are equal to u_1^* and s^*, respectively.

In order to calculate the time paths, it might be more convenient to reduce the system (10) and (11) to one differential equation. Differentiating eqn (11) once with respect to time and substituting for $\dot{u}_1$ and u_1 in eqn (10) we get

$$\ddot{s} - i\dot{s} - (\delta_1^2 + \delta_1 i + 2\delta_1 + i)\, s + \frac{\beta_1 c}{2s} + \frac{\phi_2 c}{2s^2} = 0. \quad (18)$$

The non-linear differential equation of second order (18) determines now the optimal time path of smoothness, while that for maintenance is obtained from eqn (11), once eqn (18) is solved. It seems that a closed-form solution of eqn (18) does not exist. Of course, if we had reduced the control problem stated in eqn (6) to a problem of calculus of variations, then we would have obtained eqn (18) directly, but not the system (10) and (11).

Next, the boundary conditions of eqn (18) are to be considered. Since the present state, s_0, is given, we have

$$s(t_0) = s_0. \quad (19)$$

In the case of a finite time horizon t_1, we either can require that the smoothness reaches a certain level s_1 (or at least a certain level s_1)

$$s(t_1) = s_1 \qquad (\text{or } y_1(t_1)(s(t_1) - s_1) = 0), \quad (20)$$

or that we require a certain scrap value F to be reached. This implies the following terminal boundary condition:

$$y_1(t_1) \equiv q_1(t_1)\, e^{-i(t_1 - t_0)} = \frac{\partial F}{\partial s(t_1)} = 2s(t_1). \quad (21a)$$

Equation (21a) indicates that the shadow price of smoothness must be equal to the marginal scrap value. But by eqn (8b), q_1 is equal to the instantaneous marginal costs due to investment, meaning that the discounted marginal costs due to investment must be equal to the marginal scrap value at the end of the time horizon t_1. Substituting eqn (8b) into eqn (21a) and using eqn (11), the terminal boundary condition can be written in terms of smoothness alone

$$s(t_1)\{e^{-i(t_1 - t_0)}(1 + \delta_1) - 1\} + e^{-i(t_1 - t_0)}\, \dot{s}(t_1) = 0. \quad (21b)$$

In the case of an infinite time horizon, the terminal boundary condition is given by the equilibrium values, for

$$\lim_{t \to \infty} s(t) = s^*; \qquad \lim_{t \to \infty} u_1(t) = u_1^*. \quad (22)$$

Possible time paths for an arbitrary initial state are given in Figs. 1 and 2. For a given initial state of smoothness, the initial investments must lie on the stable branch as displayed as the AB line in Fig. 1. With an infinite time horizon, smoothness and investment reach asymptotically the equilibrium values. With a finite time horizon which is sufficiently long, smoothness and investment will reach the neighbourhood of the equilibrium values, stay there for a certain time interval, i.e. follow the "turnpike", and will leave it to fulfill the terminal boundary conditions.

3. INVESTMENT IN CAPACITY

In so far, we have looked at smoothness while holding capacity constant. Now, we turn to capacity while holding smoothness constant. The volume of traffic is measured in vehicles per month and a typical value of the monthly capacity equivalent to a 8000 Average Daily Traffic (A.D.T.) can be used, i.e. 240,000. The growth rate of volume decreases linearly with increasing volume to capacity ratio v/c. For instance, at a v/c ratio of 0.6 (5000 A.D.T.) the annual growth rate is about 3.6% per year for $\delta_2 = 0.006$ and $\delta_3 = 0.005$. At a v/c ratio of 1.2 (9600 A.D.T.) the volume on the road no longer increases. The equation of motion for the volume v (vehicles per month) is therefore:

$$\dot{v} = \delta_2 v - \delta_3 \left(\frac{v^2}{c}\right), \; \delta_2, \delta_3\text{: coefficients}, \; \frac{\delta_2}{\delta_3} = 1.2. \quad (23)$$

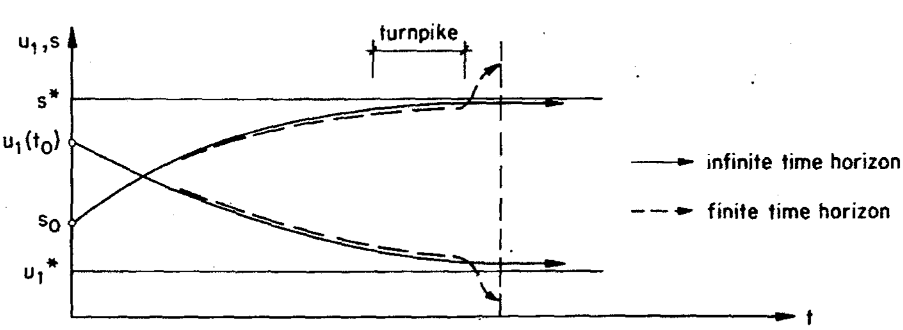

Fig. 2. Possible optimal paths.

Capacity is the monthly volume which can be accommodated under "normal" conditions. For Ontario this can be taken as equivalent to 8000 A.D.T. for a 2 lane rural road. Investment in capacity is considered to be possible in a linear fashion at a linear cost. For instance, if a 24 ft road is widened by 2.4 ft there would be a 10% increase in capacity and the cost would be 10% of that for a 2 lane road (assumed to be $300,000 per mile). These assumptions are not strictly correct but were considered adequate for this purpose. Therefore, the rate of change of capacity c is equal to investment in capacity u_2, which is the policy variable:

$$\dot{c} = u_2, \tag{24}$$

and the costs due to investment are linear:

$$K = \gamma u_2, \quad \gamma\text{: coefficient.} \tag{25}$$

The volume/capacity benefits are benefits relative to "acceptable" speeds. As the daily v/c ratios increase these benefits decrease, becoming negative when the v/c ratio exceeds one. These delay benefits change as the square of the v/c ratio. For instance, for a change in v/c ratio from 1.0 to 0.5 the corresponding benefit is 7.5 c per vehicle mile, or a savings of 1.5 min per vehicle per mile at 5 c per vehicle min (with $\beta_0 = 0.1\$$). This is an approximation for queuing theory results. Again this is an assumed functional form to approximate the main benefit of travel time savings due to reduced congestion. The benefits are

$$B = \beta_0\left(v - \frac{v^3}{c^2}\right), \quad \beta_0\text{: coefficient.} \tag{26}$$

Finally, the operating cost increase linearly with capacity

$$\Phi = \phi_0 + \phi_1 c + \phi_2 \frac{c}{s}, \quad \phi_0, \phi_1, \phi_2\text{: coefficients.} \tag{27}$$

For instance, for snow removal a cost of $3600/yr for a two lane road could be used. With the social discount rate i, the objective functional J to be maximized can be written as

$$\max_{\{u_2\}} J = \int_{t_0}^{t_1} \{B(v, c) - \Phi(c) - K(u_2)\} e^{-i(t-t_0)} dt \tag{28}$$

subject to

$$\dot{v} = \delta_2 v - \delta_3\left(\frac{v^2}{c}\right)$$

$$\dot{c} = u_2$$

$$0 \le u_2 \le u_2^m, \quad c \le c^m, \quad v(t_0) = v_0, \quad c(t_0) = c_0.$$

The approximation of the non-linear cost function through the linear function of eqn (25) is reasonable for low rates of investment per time u_2. We assume the feasible domain of the rates of investment to lie between zero and u_2^m. Investment at a higher speed than this upper bound u_2^m would involve increasing marginal costs. Since all variables are considered to be real, the capacity c is restricted to the upper bound c^m, which is equivalent to an integer multiple of a two-lane road. The value c^m is given exogeneously.

The Hamiltonian of system (28) is now

$$H = e^{-i(t-t_0)}\left\{B - \Phi - K + q_2\left(\delta_2 v - \delta_3 \frac{v^2}{c}\right) + q_3 u_2\right\} \tag{29a}$$

$$y_2 \equiv q_2 e^{-i(t-t_0)}; \qquad y_3 \equiv q_3 e^{-i(t-t_0)}. \tag{29b}$$

Again, instead of the costate variables y_2, y_3 we use the transformed ones q_2, q_3, respectively. Maximizing the Hamiltonian with respect to the policy variable u_2 yields

$$\frac{\partial H}{\partial u_2} = e^{-i(t-t_0)}(q_3 - \gamma) \gtreqless 0. \tag{30}$$

Since the Hamiltonian is linear in the control variable, the solution is of the bang–bang type:

$$u_2(t) = \begin{Bmatrix} 0 \\ u_2^m \end{Bmatrix} \text{ if } q_3 \begin{Bmatrix} \le \\ > \end{Bmatrix} \gamma. \tag{31}$$

Since γ represents the marginal cost due to investment in capacity, eqn (31) states that there will be no investment done if the (carried forward) shadow price of capacity is less than or equal to the marginal cost, but if the shadow price is greater than the marginal cost, then investment will occur at the maximum possible level. We define the Lagrangian L as

$$L \equiv H + \mu(c^m - c). \tag{32}$$

Here μ is a Lagrange multiplier which represents the shadow price of the capacity constraint; now the canonical equations are

$$\dot{q}_2 = -\beta_0\left(1 - 3\left(\frac{v}{c}\right)^2\right) + \left(2\delta_3 \frac{v}{c} + i - \delta_2\right) q_2 \tag{33a}$$

$$\dot{v} = \delta_2 v - \delta_3 \frac{v^2}{c} \tag{33b}$$

$$\dot{q}_3 = -\left(2\beta_0\left(\frac{v}{c}\right)^3 - \phi_1 - \frac{\phi_2}{s}\right) - \delta_3\left(\frac{v}{c}\right)^2 q_2 + i q_3 - \mu \tag{33c}$$

$$\dot{c} = u_2 \tag{33d}$$

$$c^m - c \ge 0 \tag{33e}$$

$$(c^m - c)\mu = 0. \tag{33f}$$

Equations (33a) and (33c) can be written in a way that an economic interpretation is possible:

$$\frac{\dot{q}_2}{q_2} + \frac{\partial(B - \Phi)/\partial v}{q_2} - \left(2\delta_3 \frac{v}{c} - \delta_2\right) - i = 0 \tag{34a}$$

$$\frac{\dot{q}_3}{q_3} + \frac{\partial(B - \Phi)/\partial c}{q_3} + \delta_3\left(\frac{v}{c}\right)^2 \frac{q_2}{q_3} - i + \frac{\mu}{q_3} = 0. \tag{34b}$$

In eqn (34a) the first term represents the "capital gain" of volume, the second an equivalent of the marginal internal rate of return due to volume, while the last two terms are the losses due to "depreciation of volume" and interest rate. In eqn (34b), the first term again represents the "capital gain" of capacity, the second term an equivalent of the marginal internal rate of return due to capacity, the third term represents a "gain" due to volume, the fourth the loss due to the interest rate and the last a "gain" due to the capacity constraint.

The equilibrium values are obtained when setting eqns (33a)–(33d) equal to zero:

$$q_2^* = -\frac{\beta_0\left(3\left(\frac{\delta_2}{\delta_3}\right)^2 - 1\right)}{(\delta_2 + i)} < 0 \tag{35a}$$

$$\frac{v^*}{c^*} = \frac{\delta_2}{\delta_3} = 1.2 \tag{35b}$$

$$q_3^* = \left\{ 2\beta_0\left(\frac{\delta_2}{\delta_3}\right)^3 - \phi_1 - \frac{\phi_2}{s} - \frac{\delta_2{}^2}{\delta_3}\frac{\beta_0\left[3\left(\frac{\delta_2}{\delta_3}\right)^2 - 1\right]}{\delta_2 + i} + \mu \right\} / i \gtreqless 0 \tag{35c}$$

$$u_2^* = 0. \tag{35d}$$

Despite the ambiguity of the sign in eqn (35c), the "carried forward" shadow price of capacity q_3^* is expected to be positive. Linearizing the system (33) around the equilibrium, we get the following derivatives:

$$\frac{\partial \dot{q}_2}{\partial q_2} = \delta_2 + i; \quad \frac{\partial \dot{v}}{\partial v} = -\delta_2; \quad \frac{\partial \dot{q}_3}{\partial q_3} = i; \quad \frac{\partial \dot{c}}{\partial c} = 0. \tag{36}$$

With eqn (36) we can calculate the trace of the linearized system's matrix which is

$$2i > 0.$$

Therefore, according to the Routh–Hurwitz conditions system (33) is locally unstable, but still a saddle point solution could exist. The phase diagram for the costate variables, given the equilibrium volume to capacity ratio v^*/c^*, is shown in Fig. 3. The equilibrium point (q_2^*, q_3^*) is an unstable node. Indeed, the characteristic values of the linearized system (33a) and (33c), when $v/c = v^*/c^*$, are both real and positive:

$$\lambda_1 = \delta_2 + i > 0; \qquad \lambda_2 = i > 0.$$

Further, $\partial \dot{q}_3/\partial q_3$ does not change the sign for all v/c ratios, nor does $\partial \dot{q}_2/\partial q_2$, unless $i < \delta_2$. But, this would happen for very small v/c ratios, only. Although the volume of traffic tends to stabilize according to eqn (36), Fig. 3 suggests that a saddle point solution does not exist for the total system (33).

Several possible paths of the costate variables are given in Fig. 3 and are labeled A, B and C. The paths A and A′ both start with an initial value of the capacity shadow price q_3 which is greater than the marginal cost to invest γ. According to the bang–bang principle of eqn (31), investment $u_2{}^m$ occur right from the beginning until either the maximum capacity, c^m, or the value $q_3 = \gamma$ is reached. In the latter case, i.e. in the case of no restriction on the capacity, eqn (34b) can be rewritten and interpreted as a modified investment rule:

$$\frac{\dot{q}_3}{q_3} + \frac{\partial(B-\Phi)/\partial c}{q_3 = \partial K/\partial u_2} + \delta_3\left(\frac{v}{c}\right)^2 \frac{q_2}{q_3} = i. \tag{34b′}$$

Here $\mu = 0$, because the constraint is not binding. At the time when $q_3(t)$ crosses the horizontal line $q_3 = \gamma$ in Fig. 3, $q_3 = \partial K/\partial u_2 = \gamma$ by definition. Therefore, investment in capacity occur until the "capital loss" (first term of eqn (34b)′; $\dot{q}_3 < 0$ when $q_3 = \gamma$) plus the instanteneous marginal internal rate of return (second term) plus the "gain" from the volume of traffic (third term) is equal to the social discount rate i. The difference between static and dynamic investment approach is visualized in Fig. 4.

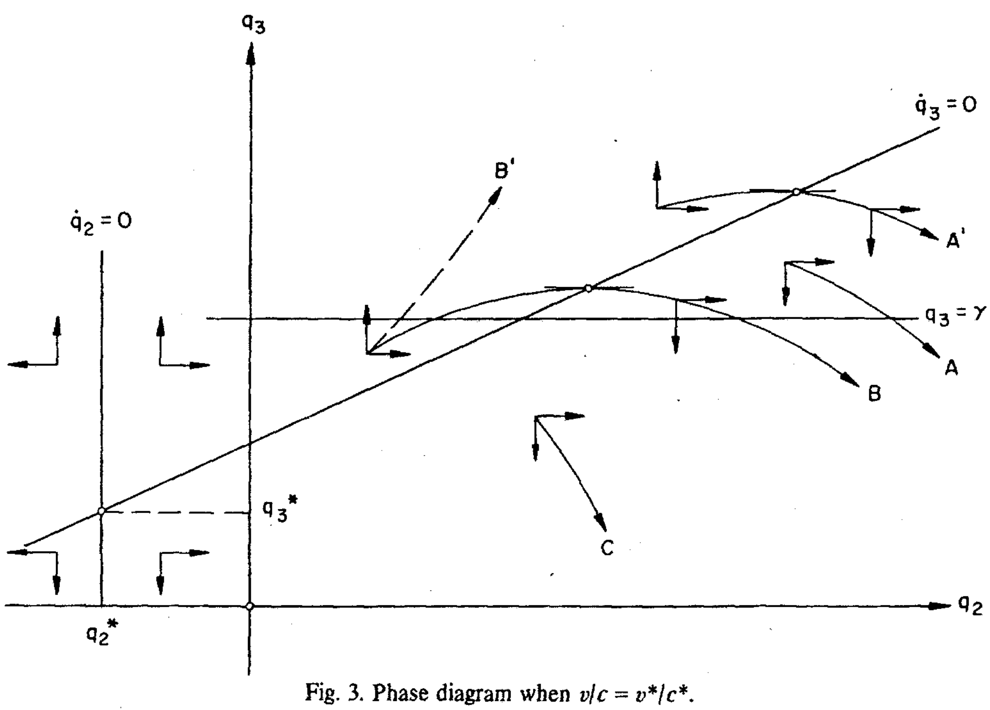

Fig. 3. Phase diagram when $v/c = v^*/c^*$.

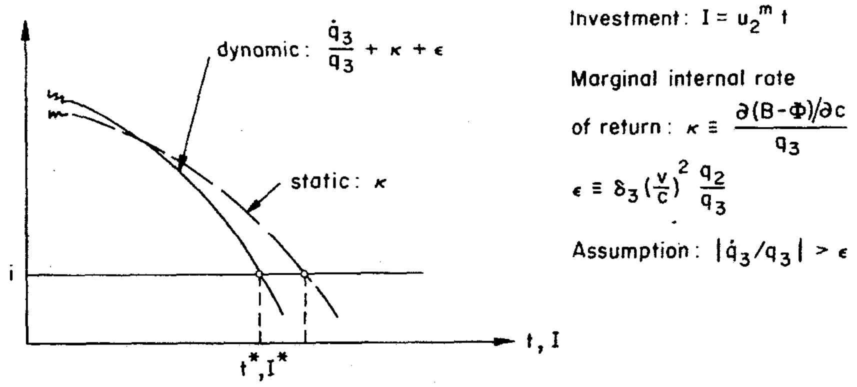

Fig. 4. Static and dynamic investment approach.

Since the rate of investment per time is constant over time and equal to u_2^m, the time axis in Fig. 4 can also be used to represent total investments in capacity $I = u_2^m \times t$, if $t_0 = 0$. The static investment theory says that investment occurs until the marginal internal rate of return κ is equal to the discount rate i. In the case of the dynamic approach also the rate of change of the shadow price $\dot{q}_3/q_3$ and the term which includes the volume of traffic ϵ are to be taken into account. The concavity of the curves in Fig. 4 follows directly from the paths A,A′ and B shown in Fig. 3. The case of convexity is not reasonable, because path B′ in Fig. 3 would then never cross the horizontal line $q_3 = \gamma$ and, therefore, total investments would be infinite. Although this case is very unlikely in the real world, it could happen within the framework of this model in case the data specifications are not adequate.

Next, the paths B and C are considered. Both paths start from an initial value of the capacity shadow price q_3 which is less than the marginal cost to invest γ. According to the bang–bang principle of eqn (31) investments do not occur: either never, as in the case of path C, or until the time path has reached the horizontal line $q_3 = \gamma$, as in the cases of paths B and B′. When path B has crossed the horizontal line, investment at the maximum rate u_2^m will occur according to the bang–bang principle. The solution is now the same as described for paths A and A′. Again, we exclude the possible path B′ as unrealistic, but it could occur within the framework of this model if the data specifications are not adequate. The diagrammatic solutions for paths A, B and C are shown in Fig. 5. In the following we will assume that the time horizon t_1 is greater than t_A or t_B^e. We will discuss the solutions for the time paths A and B, only. Path A is characterized by the condition $q_3(t_0) > \gamma$, while path B by $q_3(t_0) < \gamma$.

3.1 $q_3(t_0) > \gamma$

Investments occur from the beginning. We have for the investments u_2 and the capacity c the following expressions:

$$u_2(t) = \begin{cases} u_2^m, \text{ if } t_0 \le t \le t_A \\ 0, \text{ if } t_A < t \le t_1 \end{cases} \tag{37}$$

$$c(t) = \begin{cases} c_0 + u_2^m(t - t_0), \text{ if } t_0 \le t \le t_A \\ \min(c^m, c(t_A)), \text{ if } t_A \le t \le t_1 \end{cases} \tag{38}$$

Either the maximum capacity c^m or the value $c(t_A)$ holds. The time t_A when investment stops is determined from the condition that either the maximum capacity c^m or the value $q_3 = \gamma$ is reached. In the former case, time t_A is simply

$$t_A = \frac{c^m - c_0}{u_2^m} + t_0. \tag{39}$$

In the unconstraint case where $c(t_A) < c^m$, the Lagrange multiplier μ in eqn (33c) is zero. With eqn (38) the system (33a)–(33c) is determined and can be solved for the boundary conditions† below:

$$v(t_0) = v_0; \quad q_2(t_1) = 0; \quad q_3(t_1) = 0. \tag{40}$$

†With an infinite time horizon the boundary condition $y_2(\infty) \equiv e^{-i\infty} q_2(\infty) = e^{-i\infty} q_2^* = 0$ could be used, although q_2 will possibly never reach q_2^*, but $v(\infty) > 0$.

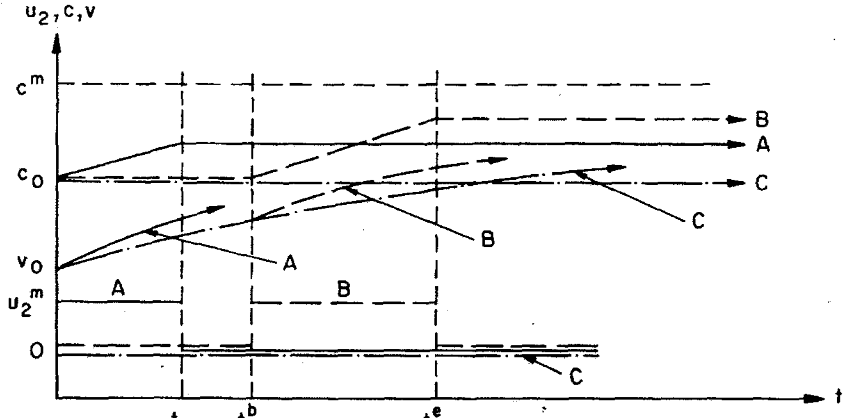

Fig. 5. Solutions of capacity problem.

Therefore, time t_A is determined from

$$q_3(t_A) = \gamma. \tag{41}$$

Equations (39) and (40) together give the condition to determine time t_A

$$t_A = \min\left(\frac{c^m - c_0}{u_2{}^m} + t_0, \quad q_3(t_A) = \gamma\right). \tag{42}$$

The volume of traffic is given by the differential equation (33b) when using eqn (38).

3.2 $q_3(t_0) < \gamma$

Investments occur within the time interval $[t_B{}^b, t_B{}^e]$. Therefore, investments u_2 and capacity c are given by

$$u_2(t) = \begin{cases} 0, & \text{if } t_0 \le t < t_B{}^b \\ u_2{}^m, & \text{if } t_B{}^b \le t \le t_B{}^e \\ 0, & \text{if } t_B{}^e < t \le t_1 \end{cases} \tag{43}$$

$$c(t) = \begin{cases} c_0, & \text{if } t_0 \le t \le t_B{}^b \\ c_0 + u_2{}^m(t - t_B{}^b), & \text{if } t_B{}^b \le t \le t_B{}^e \\ \min(c^m, c(t_B{}^e)), & \text{if } t_B{}^e \le t \le t_1. \end{cases} \tag{44}$$

Again, eqn (44) and (40) determine now the system (33). The time points $t_B{}^b$ and $t_B{}^e$ are obtained from

$$q_3(t_B{}^b) = q_3(t_B{}^e) = \gamma \tag{45}$$

and

$$t_B{}^e = \min\left(\frac{c^m - c_0}{u_2{}^m} + t_B{}^b, \quad q_3(t_B{}^e) = \gamma\right). \tag{46}$$

The conditions (45) and (46) are obvious from Figs. 3 and 5.

4. INVESTMENT IN SMOOTHNESS AND CAPACITY

Now, the objective functional of the joint problem follows from eqns (6) and (28):

$$\max_{\{u_1, u_2\}} J = \int_{t_0}^{t_1} \{B(v, c, s) - \Phi(c, s) - K(u_1, u_2, s)\}\, e^{-i(t-t_0)}\, dt + F(s) \tag{47}$$

subject to

$$\dot{s} = u_1 - \delta_1 s$$

$$\dot{v} = \delta_2 v - \delta_3 \left(\frac{v^2}{c}\right)$$

$$\dot{c} = u_2$$

$$s(t_0) = s_0, \quad v(t_0) = v_0,$$

$$0 \le u_2 \le u_2{}^m, \quad c \le c^m$$

with

$$B = \beta_0 \left(v - \frac{v^3}{c^2}\right) + \beta_1 c \ln s$$

$$\Phi = \phi_0 + \phi_1 c + \phi_2 \frac{c}{s}$$

$$K = (s + u_1)^2 - s^2 + \gamma u_2$$

$$F = s_1{}^2.$$

The necessary conditions for a maximum of J are given by eqns (18), (31) and (33). The differential equations for smoothness and shadow price of capacity are now interdependent. Again, we can distinguish between two cases according to whether $q_3(t_0) \gtrless \gamma$. The solutions follow directly from those derived in Section 3. The only difference being that now the points t_A, $t_B{}^b$ and $t_B{}^e$ in time must be determined from the system (33) together with (18), instead from the system (33) alone. For smoothness still a stable branch exists with a slightly different path due to the time-dependent capacity given in eqn (38) or (44), respectively. The equilibrium values for smoothness s^* and investment in smoothness $u_1{}^*$ are obtained when substituting $c(t_1)$ for c into eqns (14) and (15). Although the differential equations cannot be solved in closed form, numerical solutions may be obtained.

5. INVESTMENTS SUBJECT TO A BUDGET CONSTRAINT

Now, we assume that the expenditures in investment are subject to a constant amount of money per time interval (in real terms), but still the operating costs are not restricted. Since the operating costs are comparably small to the investment expenditures, this assumption might be realistic enough. Further, we assume that the budget Z is big enough to allow investment in capacity, if necessary. Therefore, the budget constraint is

$$K(u_1, u_2, s) \le Z, \qquad Z > u_2{}^m \gamma. \tag{48}$$

Now, the objective functional (47) must be maximized with respect to the control variables u_1, u_2 subject to the budget constraint (48). We introduce the Lagrangian

$$L \equiv H + \lambda^*(Z - K(u_1, u_2, s)) + \mu(c^m - c) \tag{49}$$

where H is the Hamiltonian of (47) and λ^* a Lagrange multiplier. The necessary conditions for a maximum of L are given by the Kuhn–Tucker equations. If the budget constraint is not binding, then the Lagrange multiplier λ^* is zero and the solution discussed in the previous sections holds. Here, we are interested in the case where the budget constraint is binding at any point in time, i.e. we disregard for the moment the case where the budget constraint is both binding and non-binding over the total time period $[t_0, t_1]$. With a binding budget constraint the necessary first-order conditions for a maximum of J are

$$-\frac{\partial K}{\partial u_1} + q_1 - \lambda \frac{\partial K}{\partial u_1} = 0 \tag{50a}$$

$$-\gamma + q_3 - \lambda\gamma \gtreqless 0 \tag{50b}$$

$$Z = K(u_1, u_2, s). \tag{50c}$$

Here, we used the following definition for the shadow price of budget λ^*:

$$\lambda^* \equiv \lambda\, e^{-i(t-t_0)}. \tag{51}$$

It follows from eqn (50b) that we have again a bang–bang solution for the investment in capacity:

$$u_2(t) = \begin{Bmatrix} 0 \\ u_2^m \end{Bmatrix}, \text{ if } q_3 \begin{Bmatrix} \leq \\ > \end{Bmatrix} \gamma(1+\lambda). \quad (52)$$

Investments occur when the shadow price of capacity q_3 is greater than the product of marginal cost to invest γ and one plus the shadow price of budget $(1+\lambda)$. Similarly to the previous sections, we distinguish between the cases where $q_3(t_0) \lesseqgtr \gamma(1+\lambda)$. First we turn to the case where

5.1 $q_3(t_0) > \gamma(1+\lambda)$

In this case we have a path as displayed as path A or A' in Fig. 3. Investment in capacity occur from the beginning so that u_2 and c are given by eqns (37) and (38). The canonical eqns (33) are still valid, but, the differential equation for smoothness (18) has to be replaced. Substituting eqns (37) and (50a) into (50c), we get for the shadow price of budget

$$(1+\lambda) = \begin{cases} \dfrac{q_1}{2(Z+s^2-\gamma u_2^m)^{1/2}}, \text{ if } t_0 \leq t \leq t_A \\ \\ \dfrac{q_1}{2(Z+s^2)^{1/2}}, \text{ if } t_A < t \leq t_1. \end{cases} \quad (53)$$

Substituting eqn (53) back into eqn (50a), we get an expression for the optimal investments in smoothness when $t_0 \leq t \leq t_A$:

$$u_1 = \frac{1}{2}[-2s \pm \sqrt{(4s^2 + 4(Z-\gamma u_2^m))}]. \quad (54)$$

If the budget Z is less than the investment cost for capacity widening γu_2^m, then neither investment in capacity nor in smoothness occur. This case was excluded by assumption (48). Investment in capacity u_2 but not in smoothness u_1 would occur, if $Z = \gamma u_2^m$. Also this case was excluded by assumption (48). Here, we concentrate on the case where both investment types are possible. With assumption (48) we have the optimal investment in smoothness to be

$$u_1 = \begin{cases} -s + \sqrt{(s^2 + Z - \gamma u_2^m)}, \text{ if } t_0 \leq t \leq t_A \\ \\ -s + \sqrt{(s^2 + Z)}, \text{ if } t_A \leq t \leq t_1. \end{cases} \quad (55)$$

Finally, the differential equation for smoothness is obtained when substituting (55) into the equation of motion (1):

$$\dot{s} + (1+\delta_1)s = \begin{cases} \sqrt{(s^2 + Z - \gamma u_2^m)}, \text{ if } t_0 \leq t \leq t_A \\ \\ \sqrt{(s^2 + Z)}, \text{ if } t_A \leq t \leq t_1. \end{cases} \quad (56)$$

Note that we do not need a differential equation for the shadow price of smoothness, for the binding budget determines directly the optimal investment programme. Similarly to eqn (42), the point in time t_A when investment stops, is obtained from

$$t_A = \min\left(\frac{c^m - c_0}{u_2^m} + t_0, \quad q_3(t_A) = \gamma(1+\lambda)\right), \quad (57)$$

where q_3 is determined through system (33) together with the differential equation for smoothness (56) and the equation for the shadow price of budget (53). Note that we need now only one boundary condition for the differential equation for smoothness, i.e. $s(t_0) = s_0$. The equilibrium value of smoothness s^* is obtained from eqn (56) when setting $\dot{s} = 0$:

$$s^* = \sqrt{\left(\frac{Z}{(1+\delta_1)^2 - 1}\right)}, \quad \text{if } t_1 = \infty. \quad (58)$$

And the equilibrium value of investment in smoothness u_1^* is still given by (15).

5.2 $q_3(t_0) < \gamma(1+\lambda)$

The path discussed here, is that of B in Fig. 3. The solutions follow straightforward from those given in Section 5.1. We get for the shadow price of budget, investment in smoothness, and smoothness:

$$(1+\lambda) = \begin{cases} \dfrac{q_1}{2(Z+s^2)^{1/2}}, \text{ if } t_0 \leq t < t_B^b, t_B^e < t \leq t_1 \\ \\ \dfrac{q_1}{2(Z+s^2-\gamma u_2^m)^{1/2}}, \text{ if } t_B^b \leq t \leq t_B^e \end{cases} \quad (59)$$

$$u_1 = \begin{cases} -s + \sqrt{(s^2 + Z)}, \text{ if } t_0 \leq t \leq t_B^b, t_B^e \leq t \leq t_1 \\ \\ -s + \sqrt{(s^2 + Z - \gamma u_2^m)}, \text{ if } t_B^b \leq t \leq t_B^e \end{cases} \quad (60)$$

$$\dot{s} + (1+\delta_1)s = \begin{cases} \sqrt{(s^2 + Z)}, \text{ if } t_0 \leq t \leq t_B^b, t_B^e \leq t \leq t_1 \\ \\ \sqrt{(s^2 + Z - \gamma u_2^m)}, \text{ if } t_B^b \leq t \leq t_B^e. \end{cases} \quad (61)$$

Investment in capacity and capacity are given by (43) and (44), respectively. The shadow price of capacity is determined through system (33) together with eqns (59) and (61). The time points t_B^b and t_B^e are obtained from

$$q_3(t_B^b) = q_3(t_B^e) = \gamma(1+\lambda) \quad (62)$$

and

$$t_B^e = \min\left(\frac{c^m - c_0}{u_2^m} + t_B^b, \quad q_3(t_B^e) = \gamma(1+\lambda)\right). \quad (63)$$

The diagrammatic solution of the paths A and B is shown in Fig. 6. Clearly, the solution of the case when the budget constraint is both binding and non-binding over the total time interval $[t_0, t_1]$ is a combination of those discussed in Sections 4 and 5. The possibility of a non-binding budget constraint can easily be found when comparing budget and investment costs due to smoothness at equilibrium. For $t_1 = \infty$, total costs per time interval are

$$K(u_1^*, s^*, u_2 = 0) = \begin{cases} s^{*2}[(1+\delta_1)^2 - 1], \text{ non-binding} \\ \\ Z, \text{ binding budget constraint} \end{cases} \quad (64)$$

s^* is given by eqn (14) for the case of a non-binding

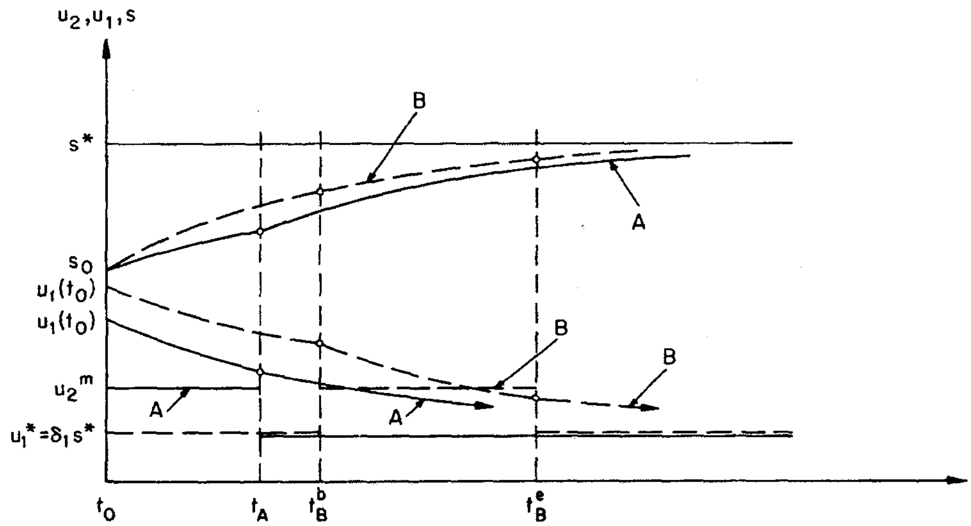

Fig. 6. Solution with binding budget constraint.

budget constraint. If Z is greater than the upper expression in (64), then the possibility of a non-binding budget constraint can certainly occur during the time interval $[t_0, \infty)$. In practice, for each point in time total costs would have to be compared to the budget Z.

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REFERENCES

Arrow K. J. and Kurz M. (1970) *Public Investment, the Rate of Return, and Optimal Fiscal Policy.* Hopkins, Baltimore and London.

Bergendahl G. (1969) *Models for Investments in a Road Network,* Department of Business Administration, Stockholm University, Bonniers.

Hutchinson B. G. (1975) *Programming of Highway Systems Capital Improvements,* Transport Group, Department of Civil Engineering, University of Waterloo, September.

Intriligator M. D. (1971) *Mathematical Optimization and Economic Theory.* Prentice Hall, Englewood Cliffs, New Jersey.

Pavement Management Guide (1977) Roads and Transportation Association of Canada, Ottawa, p. 3.9.