Pricing Callable Bonds by means of Green's Function

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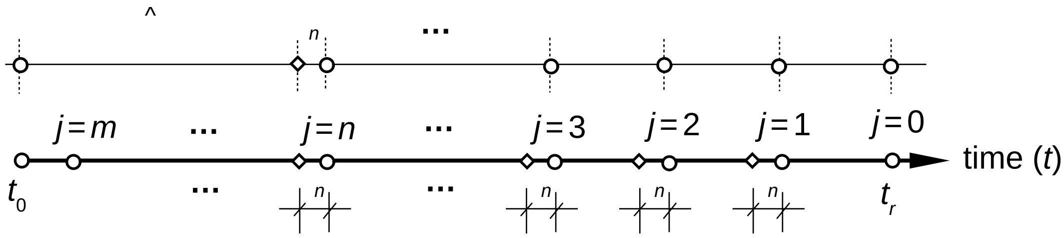
1. INTRODUCTION

• Callable Bond is a straight bond (coupon bond) with the provision that the bond can be redeemed by the debtor before the final redemption date at a prespecified price (call price).

• Semi-American callable bond has discrete call dates (1 – 10). Notice period: 2 months.



• Compound security: straight bond long + embedded call option short ⇒ Price Callable Bond = Price Straight Bond – Price of Embedded Call Option.

• **Motivation**: our experience with finite difference methods as well as wrong numerical results published in GIBSON [1990] and LEITHNER [1992].

Three Examples:

1. GIBSON [1990, Table 6]): computed prices of two almost identical embedded call options:

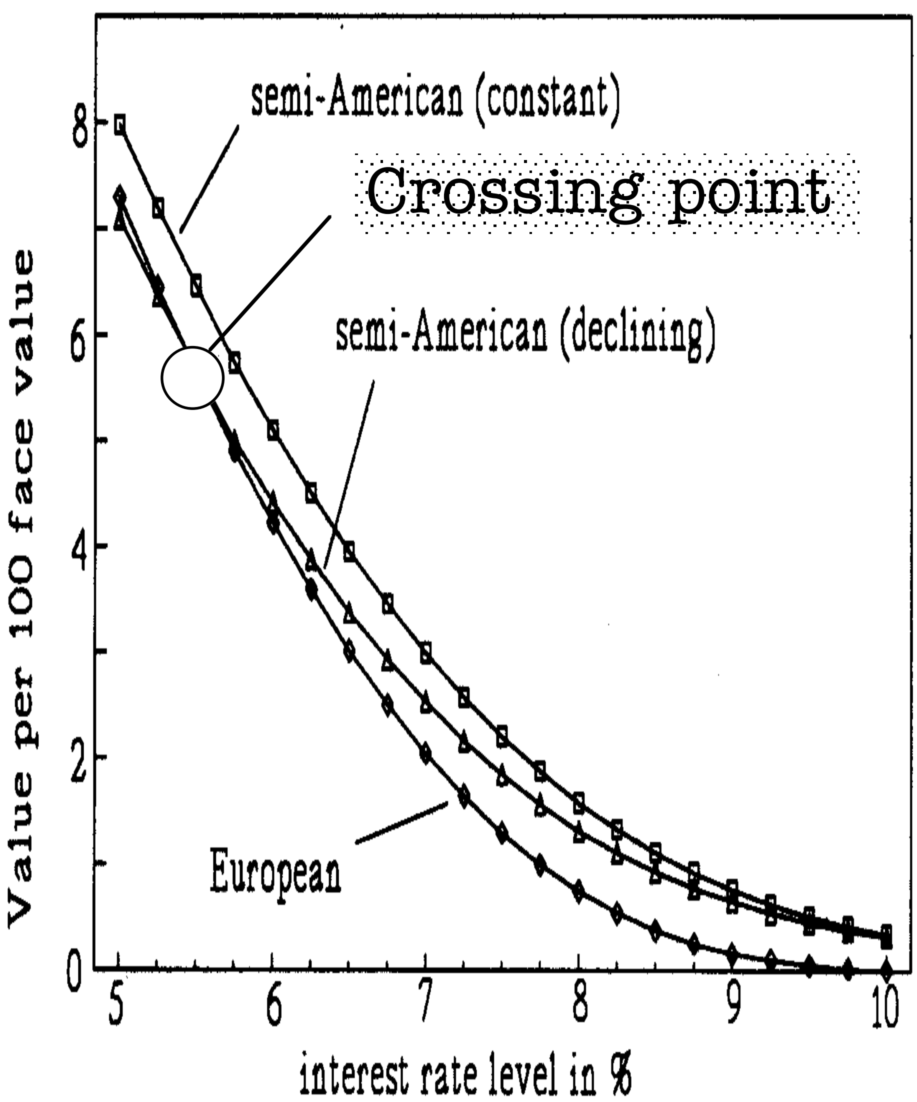
Rem. time	Call #1:	Call #2:
≈ 11 years	1.1303	2.6164
≈ 10 years	0.0878	0.2994
≈ 9 years	0.0482	0.1471
≈ 8 years	15.3192	5.3789

   (a) Time profile is wrong. (b) price variation is too large. (c) our computation: difference is ≈ 0.2 (see #15 745 & #15 227 in Table A.4).

2. GIBSON [1990, p. 670]: “Surprisingly, we observed —out of 26 price comparisons— that 14 times the price of the callable bond exceeded the straight bond price substantially thereby including the absurd conclusion that the market priced the call feature *negatively*.”

3. LEITHNER [1992, Figure 5.5]: the computed price of the Semi-American call option is less than that of the corresponding European call for small interest rates, but greater for large interest rates.

Source: LEITHNER [1992, Figure 5.5, p. 145]



• LONGSTAFF [1992, p. 571]: "I document that callable U. S. Treasury-bond prices frequently imply *negative* values for the implicit call option. For example, I find that nearly two-thirds of the call values implied by a sample of recent callable bond prices are negative."

• **Purpose**: Derive, and numerically implement, the analytical price of a callable bond.

(a) The algorithm should not suffer from instabilities.

(b) If possible, it should be computationally more efficient than a finite difference method.

• An analytical price of the Semi-American callable bond within the framework of the GAUSSian interest rate model has been derived by JAMSHIDIAN [1991].

2. THE SERIES SOLUTION OF GREEN'S FUNCTION

• Series Solution of discount bond:

Price of discount bond(*spot rate*, *time*) = exponential function(*spot rate*)

$\times$ infinite series{exponential function(*time*) $\times$ orthogonal polynomial(*spot rate*)}. (3.18)

• Green's function:

price of a primitive security which pays one unit of money if the future spot rate equals a certain value, but nothing in all other events.

Green's function(present spot rate, time, future spot rate) =

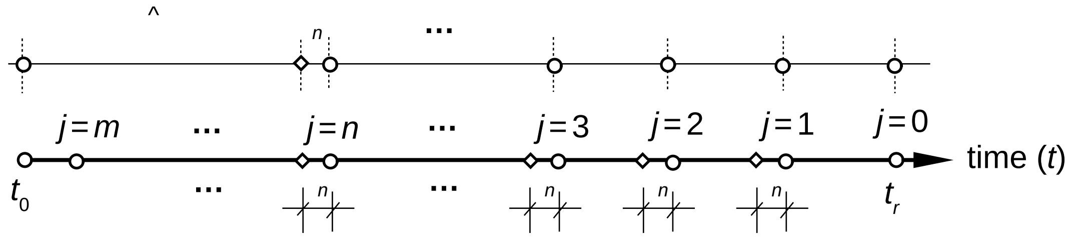
price of discount bond(present spot rate, time) $\times$ probability density function(future spot rate | certain value of the present spot rate). (5.6)

FELLER [1951], BEAGLEHOLE & TENNEY [1991], JAMSHIDIAN [1991].

3. The European Callable Bond

• Special case of the Semi-American.

4. THE SEMI-AMERICAN CALLABLE BOND



• **Price of callable bond**(present spot rate, remaining time period) =

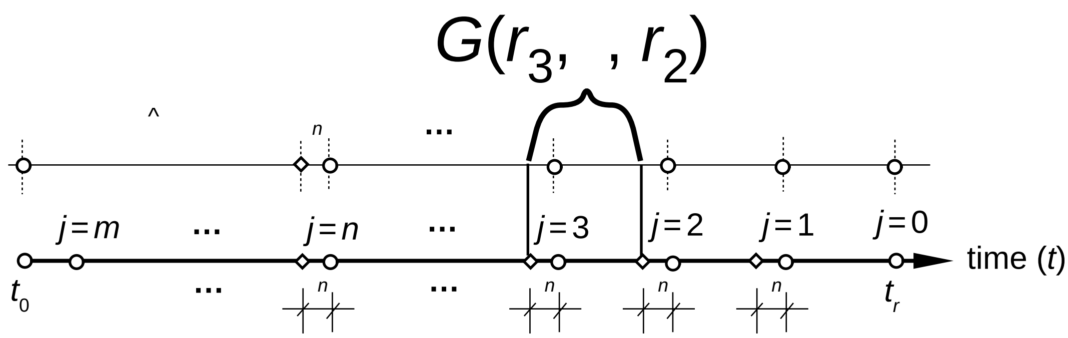
sum{call prices × price of discount bonds(present spot rate, time periods) × multiple probability integrals("break-even" interest rates)}

+ price of discount bond(present spot rate, remaining time period) × multiple probability integrals("break-even" interest rates)

+ coupon × sum{price of discount bonds(present spot rate, time periods) × multiple probability integrals("break-even" interest rates)}

+ coupon cash flow with certainty.

5. THE NUMERICAL ALGORITHM



• Three operations on each notice day:

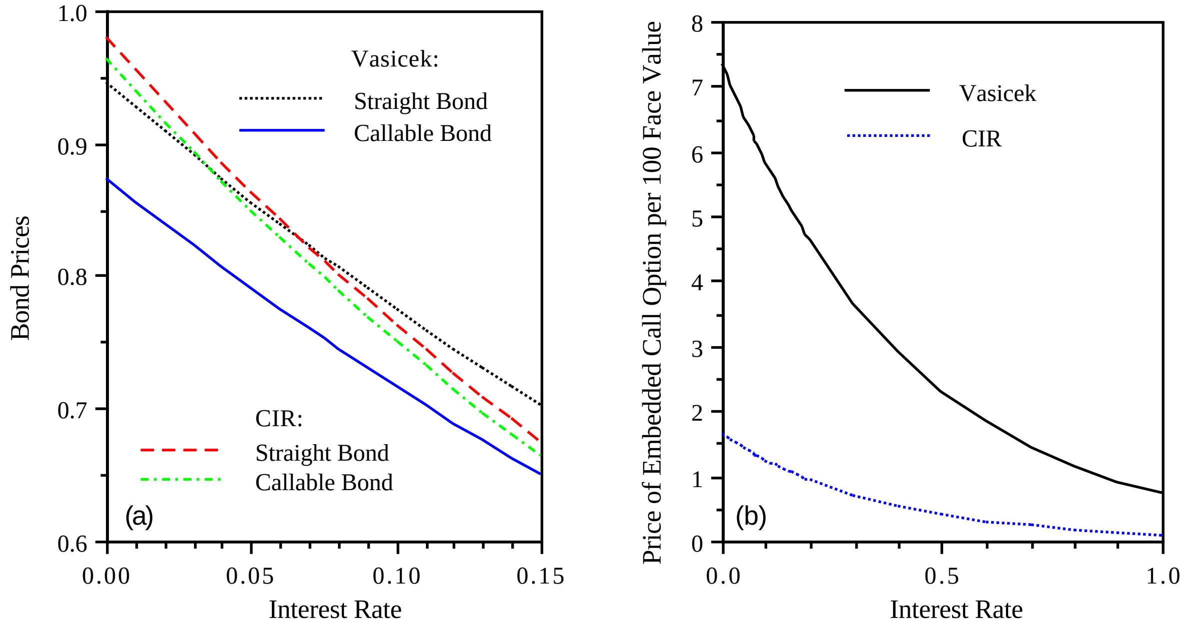
1. Interpolate the computed price of the callable bond.
   (a) Polynomial interpolation.
   (b) Exponential cubic spline.
   (c) Hybrid interpolation.
2. Determine “break-even” interest rate: Time value of call price is equal to the (interpolated) price of the callable bond.
3. Integrate GREEN’s function over the “initial data”:

(a) Time value of call price for spot rate less than "break-even" interest rate.
Can be done analytically.

(b) Otherwise, price of callable bond. Use numerical quadrature.

6. AN EXAMPLE

Figures 1 & 2, Table A.1 – Table A.5: Comparison of the VASICEK and CIR models.



Figures 1a & b: Ten call dates and twenty years remaining until expiration.

7. CONCLUSIONS

(1) The GREENian function is the discounted value of the conditional probability density function of the underlying instantaneous interest-rate process.
(2) By means of GREEN's function, the price of the Semi-American callable bond can be expressed in terms of a multiple integral of the probability density function of the instantaneous interest-rate process.
(3) Since the numerical quadrature of a multiple integral with inter-dependent limits of integration is prohibitively expensive in terms of computing time, the proposed algorithm increases the speed of computation by calculating a sequence of one-dimensional integrals involving GREEN's function.

(4) Since the boundary conditions are built into the algorithm, ***no*** numerical instabilities occur, in contrast to finite difference methods.

(5) The accuracy of the algorithm depends on both the numerical quadrature and the interpolation.

(a) The numerical quadrature procedure proposed in our paper can, in principle, achieve machine precision.

(b) The interpolation of the price function relies on the series solution we derived for the price of a discount bond. In principle, the interpolation does not achieve machine precision, but the accuracy is much higher than for a finite difference method.

(6) We conclude, therefore, that the proposed algorithm offers both higher accuracy and higher speed of computation than finite difference methods.