Evaluation of Callable Bonds

Finite Difference Methods, Stability and Accuracy

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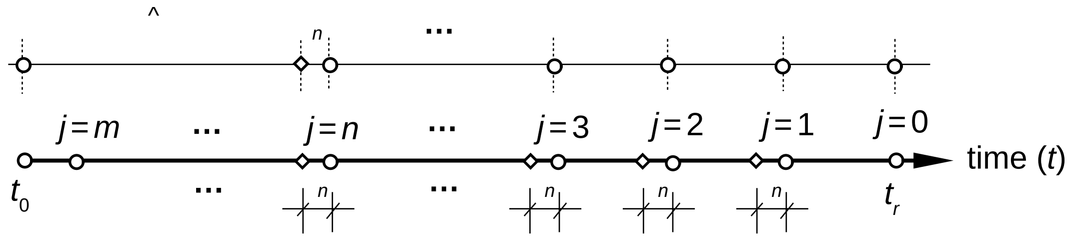
1. INTRODUCTION

• Callable Bond is a straight bond (coupon bond) with the provision that the bond can repurchased by the debtor before the final redemption date at a prespecified price (call price).

• Semi-American callable bond has discrete call dates (1 – 10). Notice period: 2 months.



• Compound security: straight bond long + embedded call option short ⇒ Price Callable Bond = Price Straight Bond – Price of Embedded Call Option.

• **Motivation**: our experience with finite difference methods as well as wrong numerical results published in GIBSON [1990] and LEITHNER [1992].

Three Examples:

1. GIBSON [1990, Table 6]): computed prices of two almost identical embedded call options:

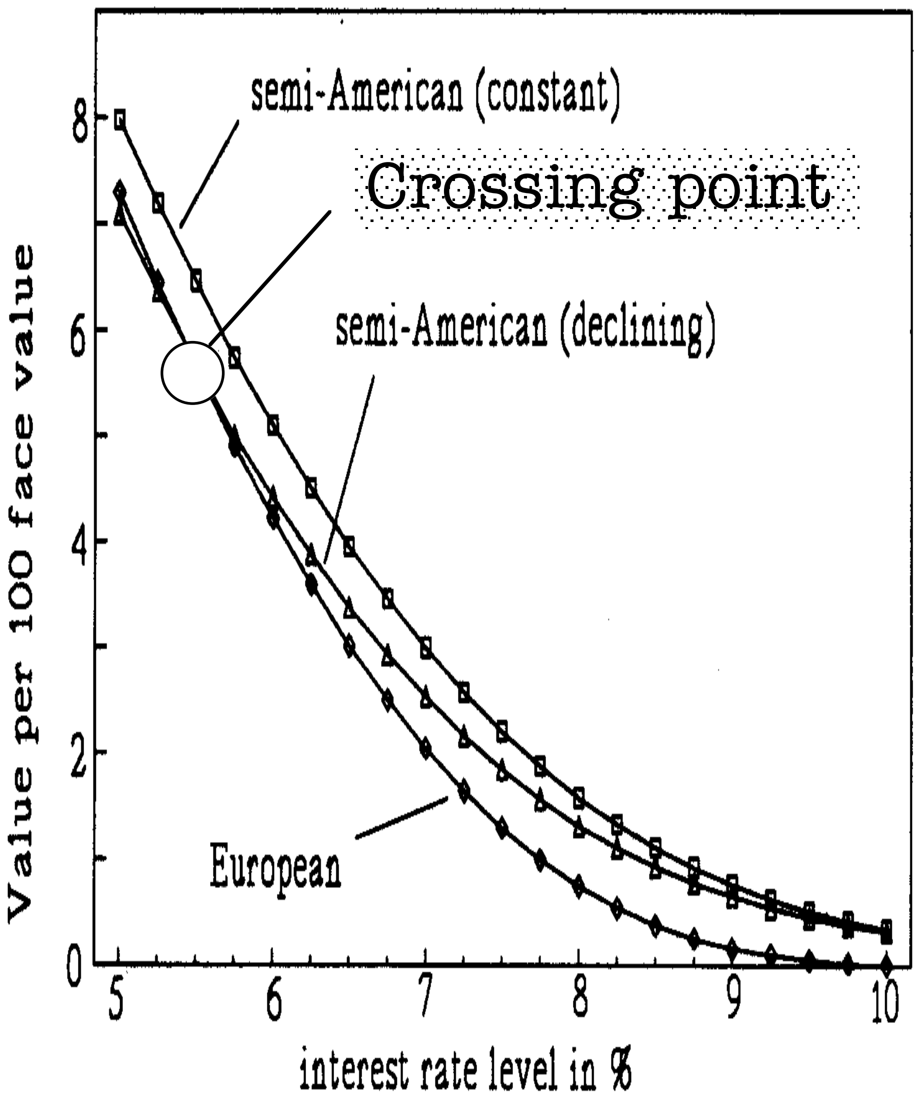
Rem. time	Call #1:	Call #2:
≈ 11 years	1.1303	2.6164
≈ 10 years	0.0878	0.2994
≈ 9 years	0.0482	0.1471
≈ 8 years	15.3192	5.3789

   (a) Time profile is wrong. (b) price variation is too large. (c) our computation: difference is ≈ 0.2 (see #15 745 & #15 227 in Table A.4).

2. GIBSON [1990, p. 670]: "Surprisingly, we observed —out of 26 price comparisons— that 14 times the price of the callable bond exceeded the straight bond price substantially thereby including the absurd conclusion that the market priced the call feature *negatively*."

3. LEITHNER [1992, Figure 5.5]: the computed price of the Semi-American call option is less than that of the corresponding European call for small interest rates, but greater for large interest rates.

Source: LEITHNER [1992, Figure 5.5, p. 145]



• LONGSTAFF [1992, p. 571]: "I document that callable U. S. Treasury-bond prices frequently imply *negative* values for the implicit call option. For example, I find that nearly two-thirds of the call values implied by a sample of recent callable bond prices are negative."

• **Purpose**: Numerical evaluation by means of finite difference methods. Benchmark: analytical solution (BÜTTLER & WALDVOGEL). As an example, we consider VASICEK's model of the *real* term structure of interest rates.

3. The Model

• Example: Vasicek's bond price model.

• Driving factor of the real term structure: instantaneous interest rate.

• Instantaneous interest rate follows the Ornstein-Uhlenbeck process.

• Vasicek derives a PDE for the price of a discount bond by means of no-arbitrage arguments.

• Callable bond satisfies PDE of discount bond except for coupon dates and notice dates.

• **Early redemption condition** (3.7): minimum {time value of call price on notice day, price of callable bond an instant before notice day}.

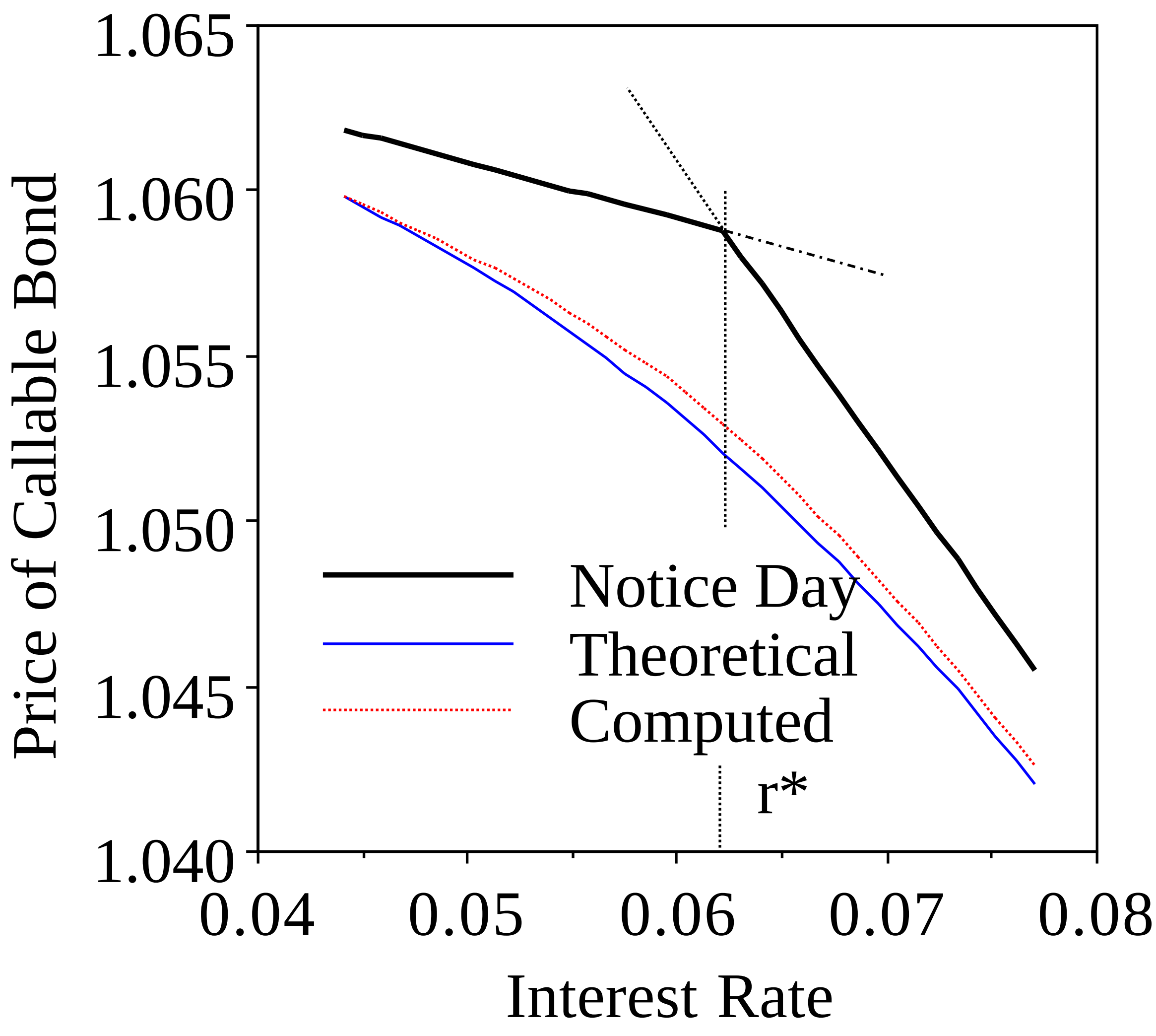


Figure 3.

4. The Estimation of Model Parameters

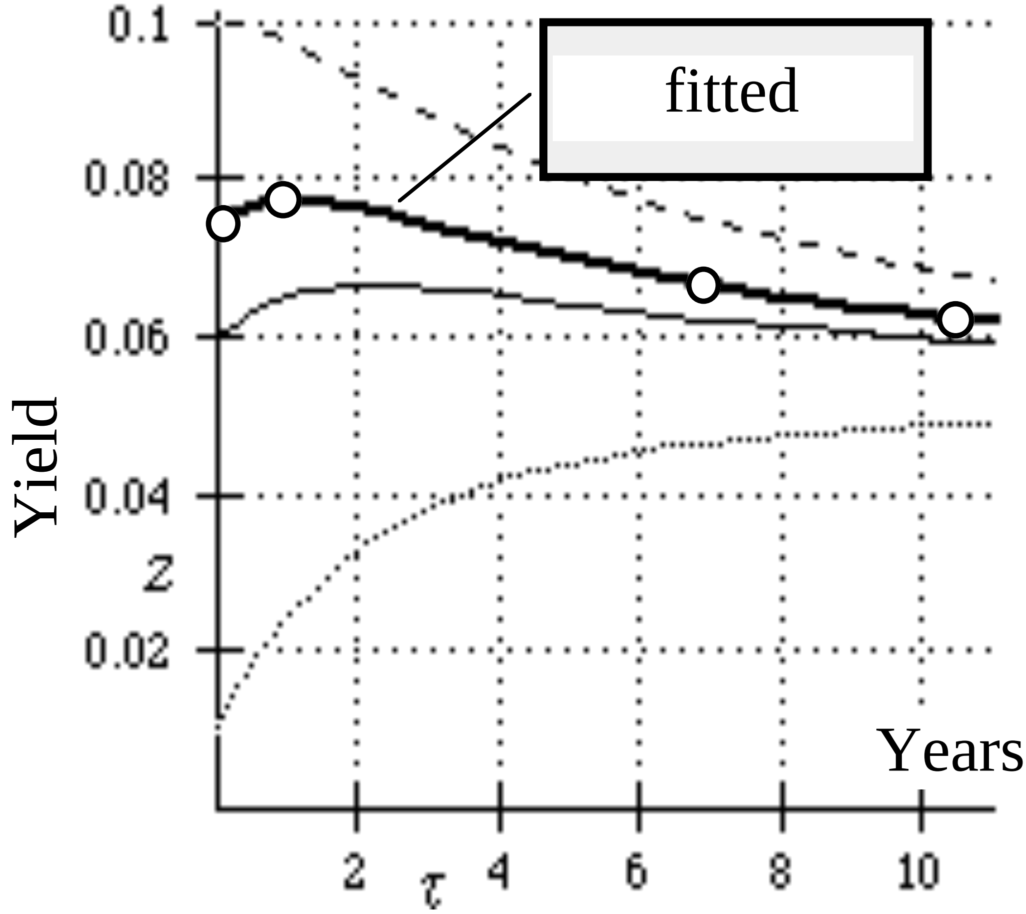


Figure 2.

• Result: α = 0.4418, ρ = 0.1326 and q = 0.2117.

• How well does Vasicek's model explain observed bond prices? → Table A.1.

5. Finite Difference Methods

• Explicit, implicit, Crank-Nicolson, Lawson-Morris.

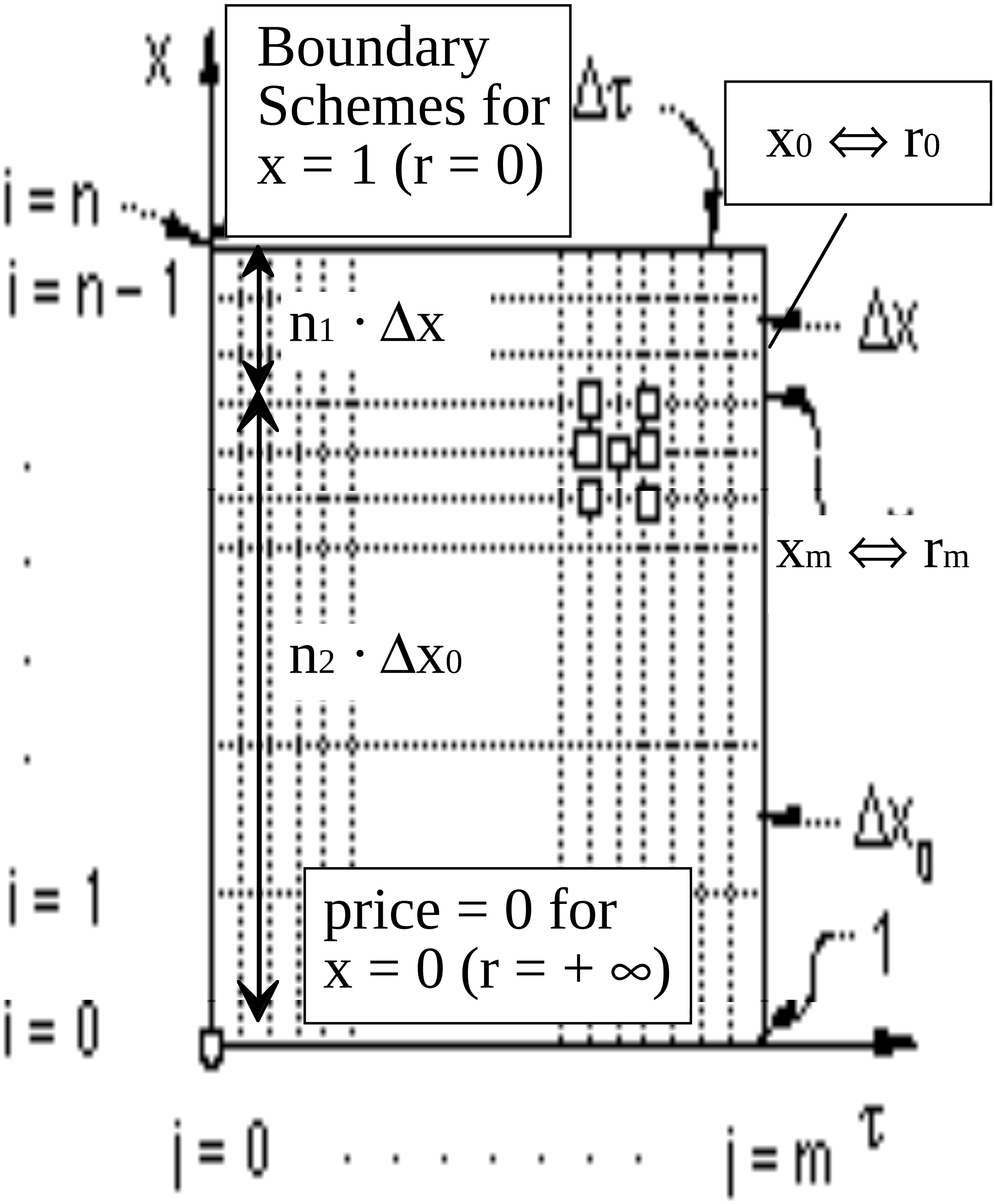
• Explicit method $\Leftrightarrow$ binomial model, implicit method $\Leftrightarrow$ multinomial model with jumps (Brennan & Schwartz [1978]).

• Consider only positive interest rates in Vasicek's PDE.

• Transformation of variables (Brennan & Schwartz [1979]):

$r \in [0, +\infty] \rightarrow x \in [0, 1]$,

$t \in [0, +\infty] \rightarrow \tau \in [0, 1]$.



Meshes.

• Standard second-order finite differences for internal mesh points.

Table 5.1: Five Boundary Schemes.

#	Der.	Den.	F_n	F_{n-1}	F_{n-2}	F_{n-3}	F_{n-4}
1	F_x	Δx	1	– 1	—	—	—
	F_{xx}	$(\Delta x)^2$	1	– 2	1	—	—
2	F_x	$2\ \Delta x$	3	– 4	1	—	—
	F_{xx}	$(\Delta x)^2$	1	– 2	1	—	—
3	F_x	$2\ \Delta x$	3	– 4	1	—	—
	F_{xx}	$(\Delta x)^2$	2	– 5	4	– 1	—
4	F_x	$6\ \Delta x$	11	– 18	9	– 2	—
	F_{xx}	$(\Delta x)^2$	2	– 5	4	– 1	—
5	F_x	$12\ \Delta x$	25	– 48	36	– 16	3
	F_{xx}	$12\ (\Delta x)^2$	35	– 104	114	– 56	11

Comment: Read the second row as $F_x = [1 \cdot F_n - 1 \cdot F_{n-1}] / \Delta x$ and similarly the other rows.

• Boundary schemes #1 & #2 impose the “not-a-knot” condition on the second partial derivatives.

6. Accuracy of Finite Difference Methods

- Machine precision: 19 decimal digits.
- Calibration with ***discount bond***:
  1. Boundary schemes #1 – #4 are stable, #5 is unstable.
  2. Explicit method: unstable for "big" mesh ratios, computationally not efficient.
  3. Slowly decaying finite oscillations:
     (a) The larger the number of interest-rate intervals, the larger is the range of eigenvalues of $\mathbf{A}^{-1}\mathbf{B} \rightarrow$ stiff PDE.
     (b) Discontinuities in the initial values or between initial values and boundary values.

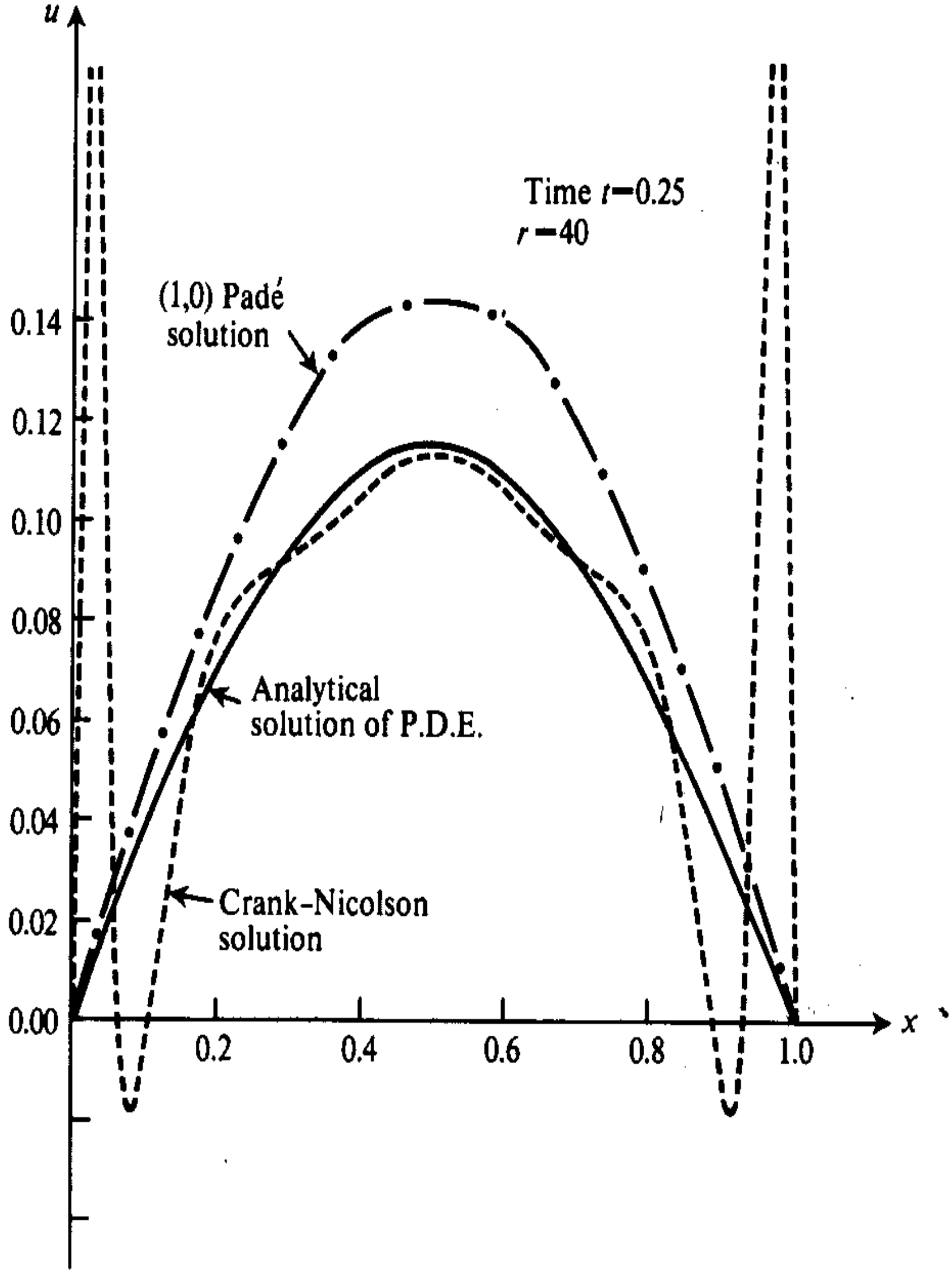


Fig. 3.2

Source: SMITH [1985].

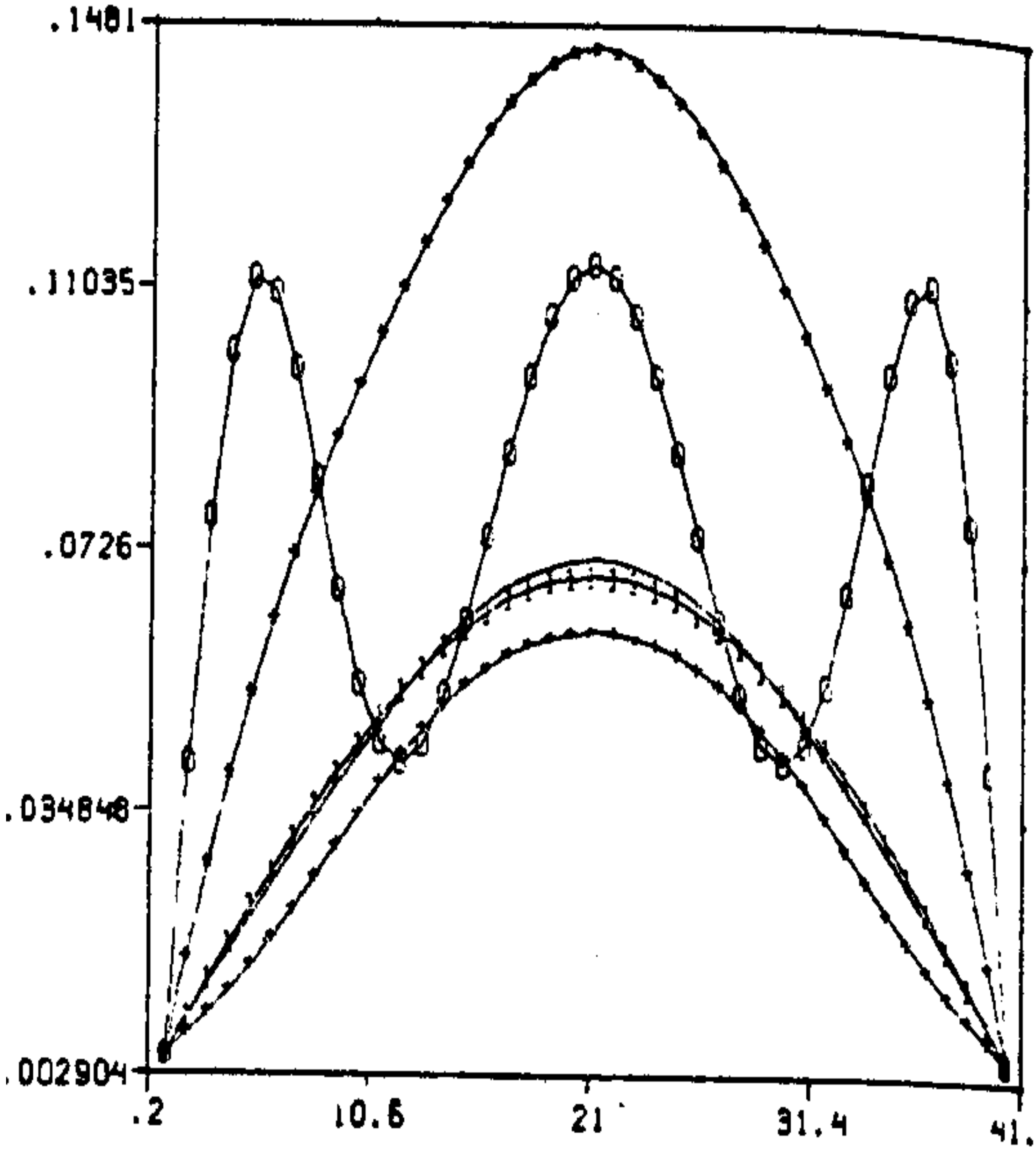


FIG. 20. *Solution of problem* (2.11) *at* $t = 1.2$, $h = 0.05$, $\tau = 0.1$ ($r = 40$). $\theta = \frac{1}{2}$:

Source: GOURLAY AND MORRIS [1980].

4. LAWSON-MORRIS method performs slightly best.

- Numerical accuracy of finite difference methods when applied to ***callable bonds*** is rather poor (comparison with similar problem).

  Numerical accuracy is *one* digit; some computed prices of embedded call option are *negative* $\rightarrow$ Table A.3.

- What is the reason for this poor accuracy?
- Look at numerical error for European callable bond: maximum life of 6.8 years and annual coupon of 7%.

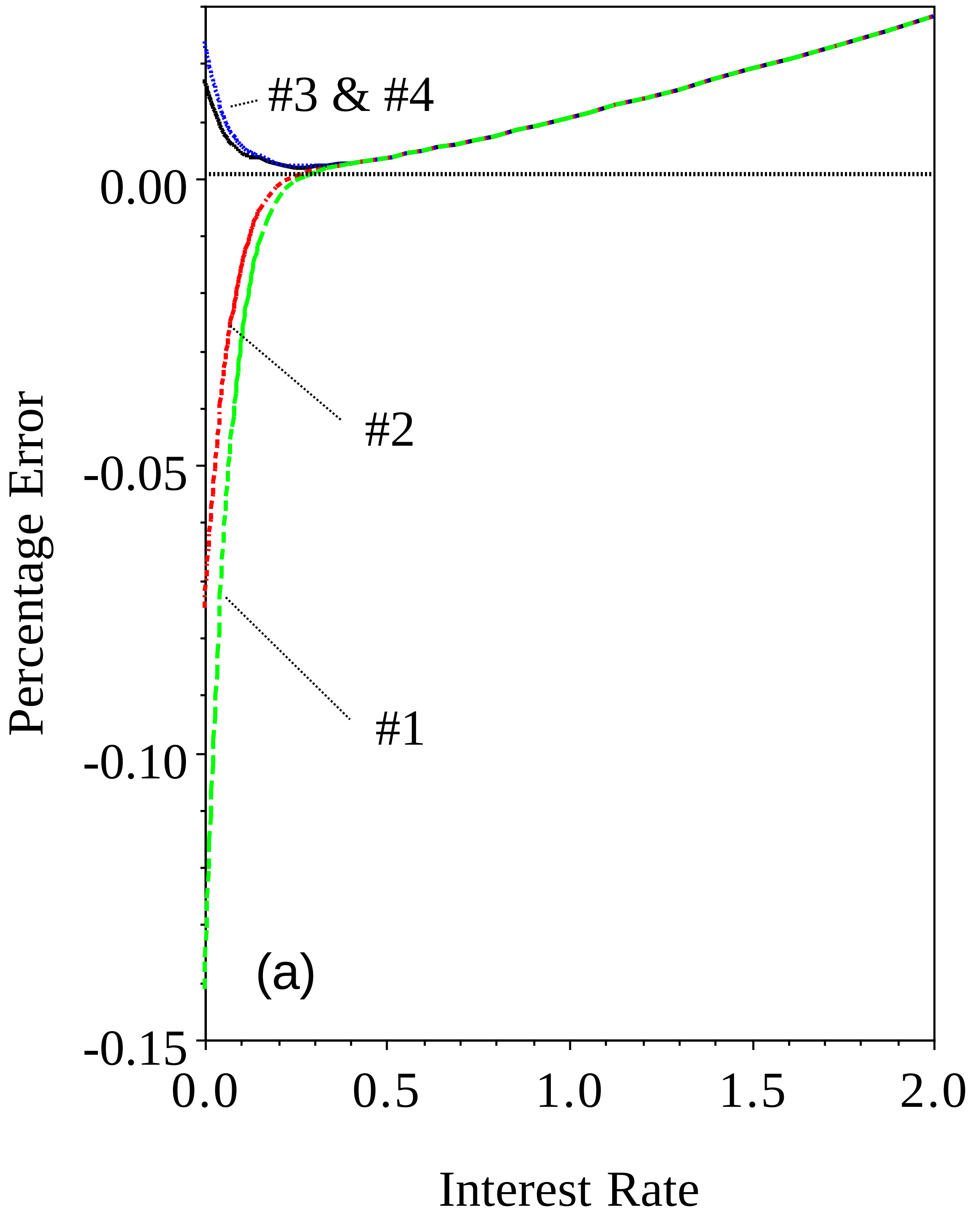


Figure A.1a: Relative Error on the Notice Day.

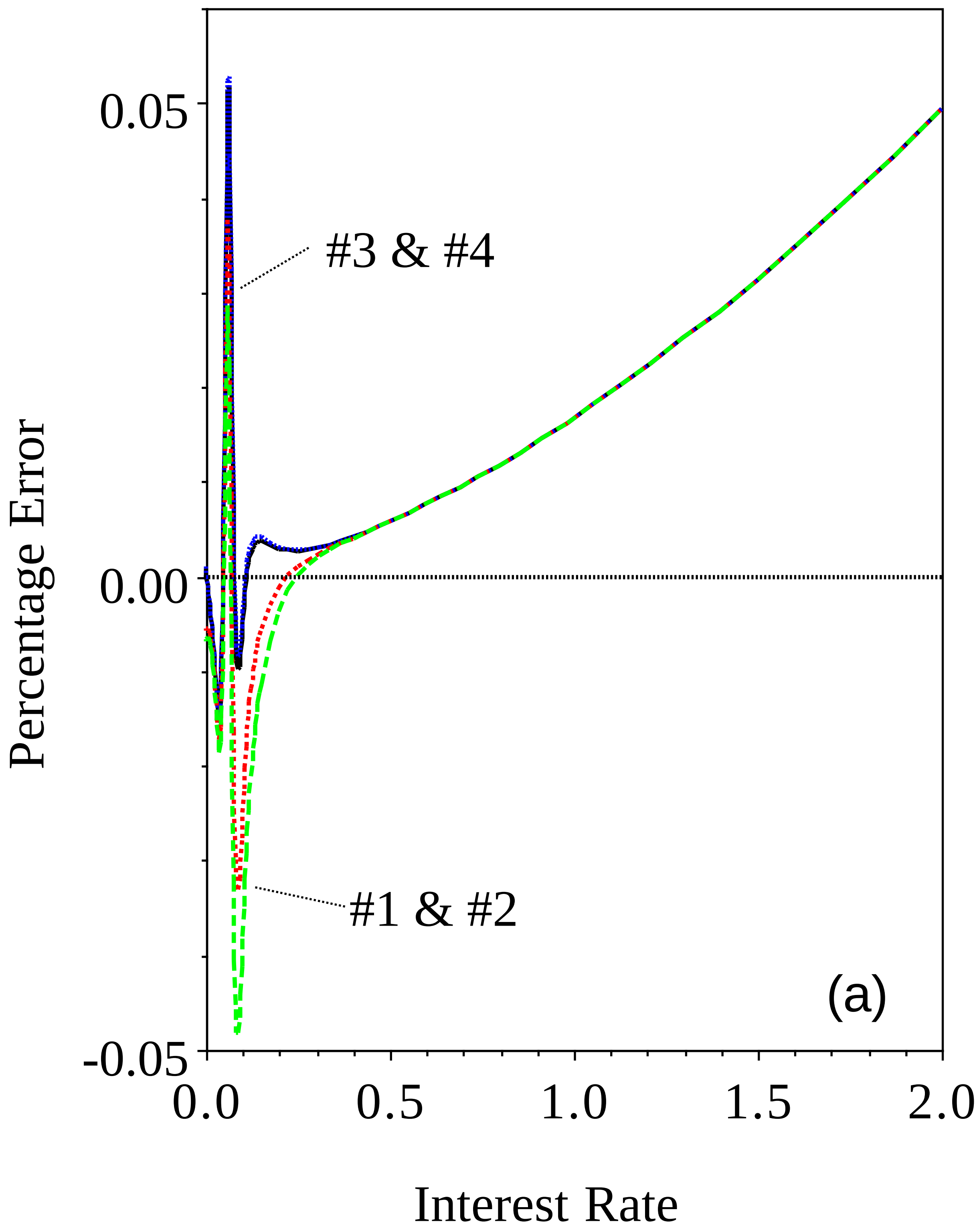


Figure A.2a: Relative Error One Time Step after the Notice Day.

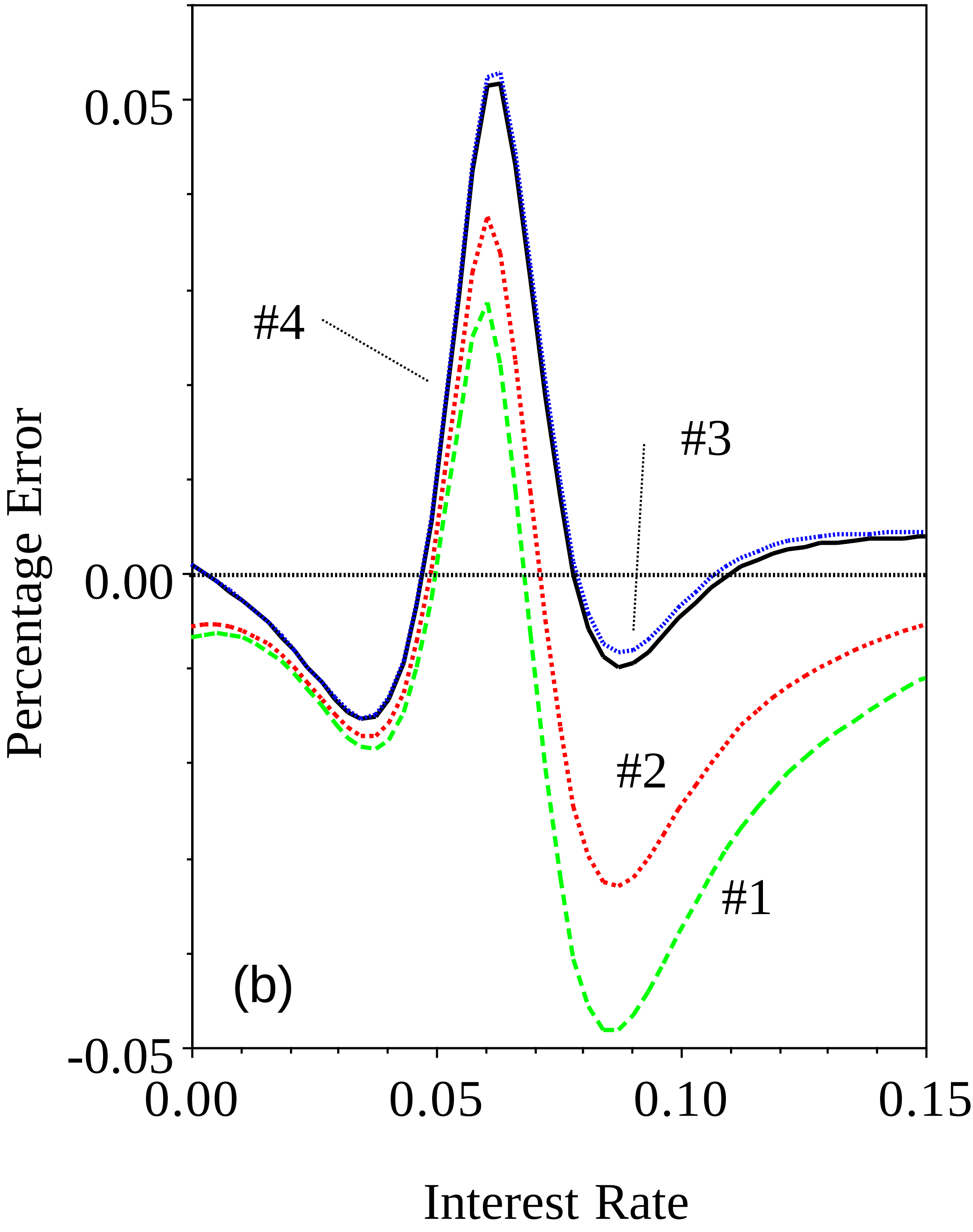


Figure A.2b: Relative Error One Time Step after the Notice Day.

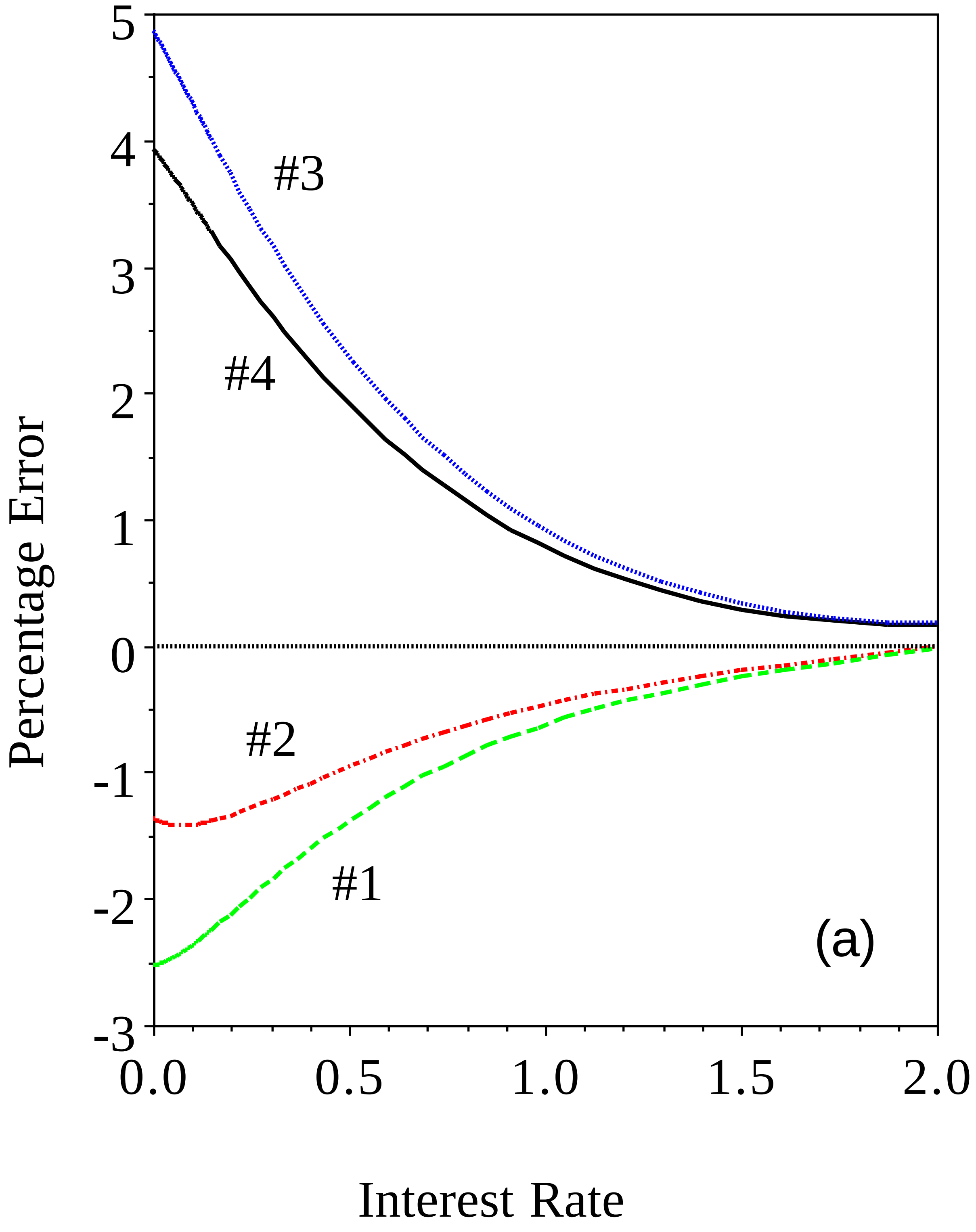


Figure A.4a: Relative Error after 6.811 Years.

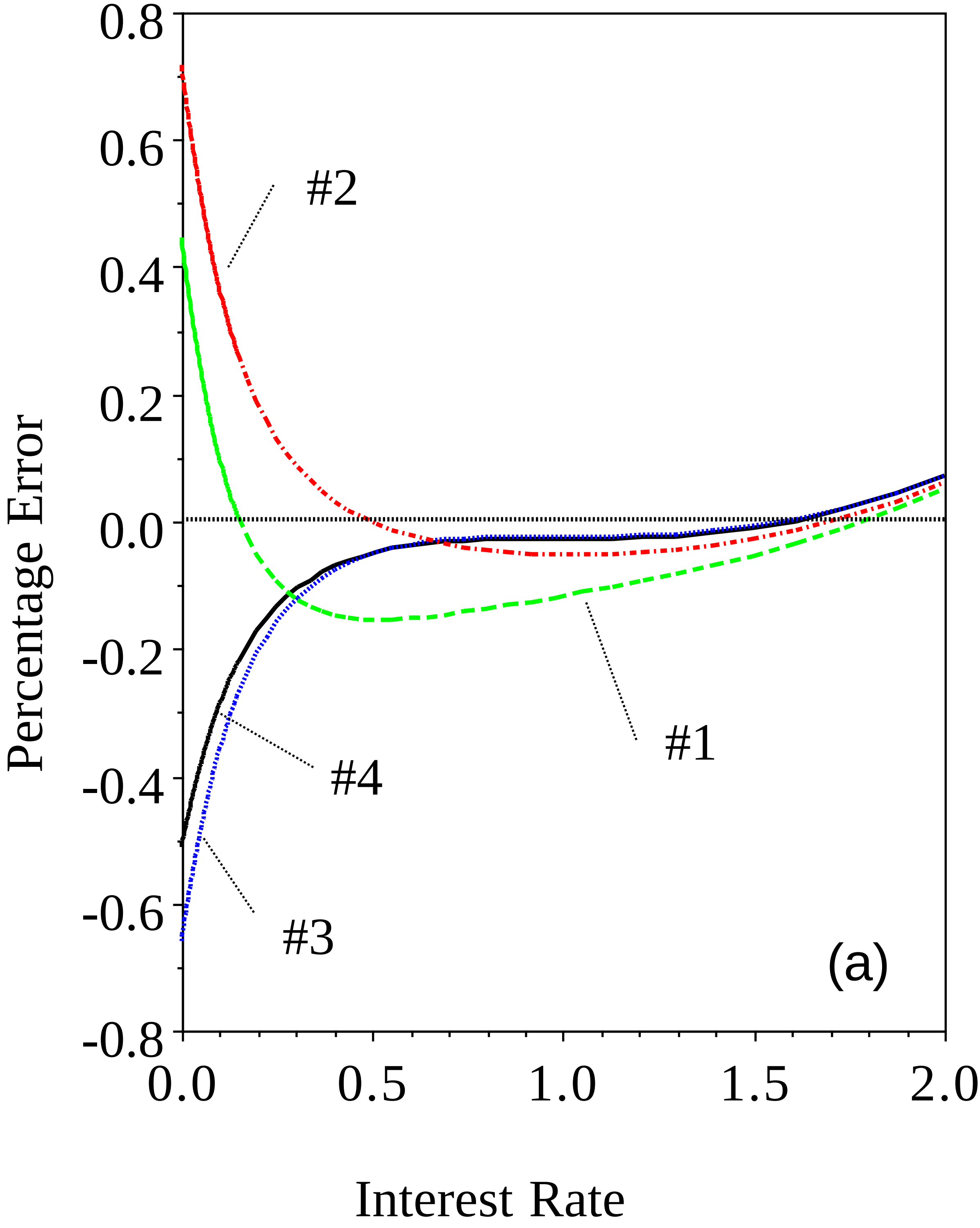


Figure A.5a: Relative Error of the Underlying Straight Bond.

• The accuracy in the relevant range of interest rates is determined by the boundary scheme under consideration.

• Which boundary scheme should be chosen?

Table A.2: RMSE of #4 is smallest.

Table A.3: RMSE of #2 is smallest.

Why? The “not-a-knot” condition of #1 and #2 helps to stabilize oscillations.

7. Are Negative Option Prices Possible?

Table A.4: Comparison of implied price of embedded call option with analytical price.

8. Conclusions

(1) Discontinuity in the values of the early redemption condition ⇒ the numerical error is greater for the callable bond than for the straight bond.
For a comparable number of intervals: the accuracy is one digit only ⇒ not useful in real applications.
Many computed prices turn out to be negative, particularly for the boundary schemes #3 & #4.
The phenomenon of negative computed prices has been observed, but has not been explained in Gibson [1990].

(2) The numerical accuracy in the relevant range of interest rates is entirely determined by the boundary scheme.

None of the four boundary schemes considered is fully compatible with the finite difference scheme for internal mesh points.

(3) Choosing the boundary scheme that gives you the smallest error for the straight bond is misleading:

#4 has smallest RMSE for straight bond.

#2 has smallest RMSE for callable bond.

Explanantion: “not-a-knot” condition.

(4) All but two call option prices implied by the sample of Table A.4 are negative. → LONGSTAFF [1992].

I argue that the negative computed or implied prices of the embedded call option might be due to the numerical error of the finite difference methods.

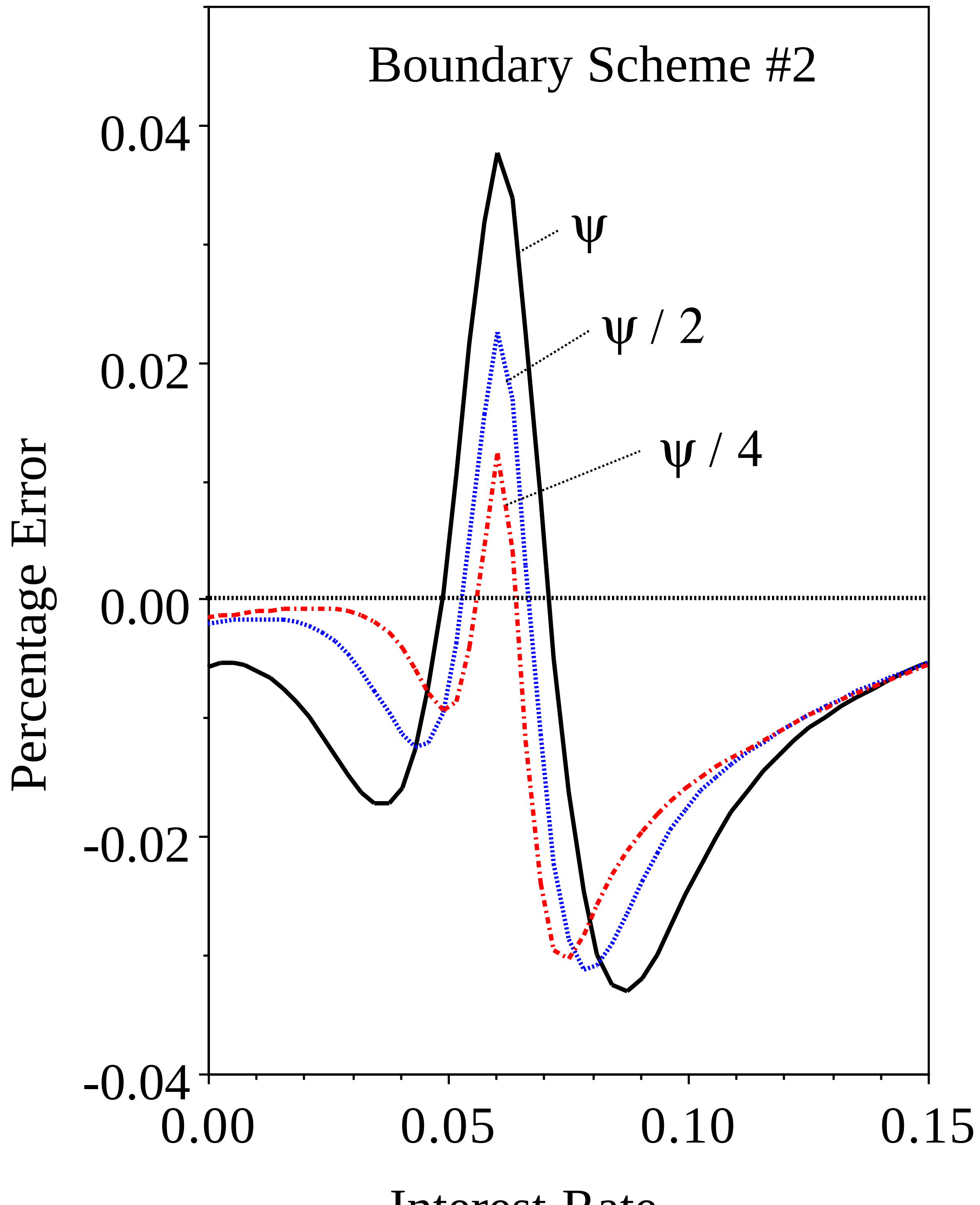


Figure A.6b: Relative Error One Time Step after the Notice Day.

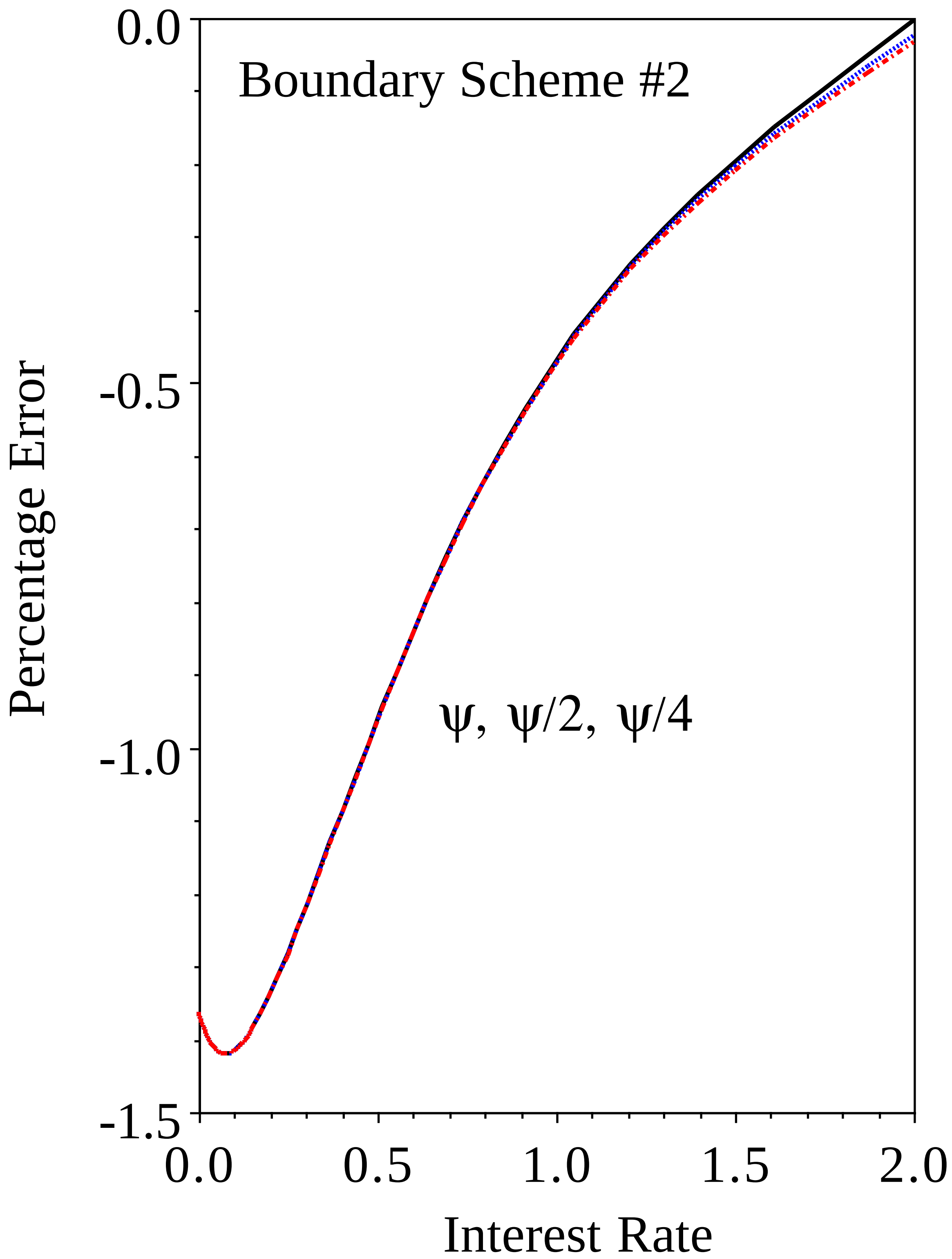


Figure A.7a: Relative Error after 6.811 Years.

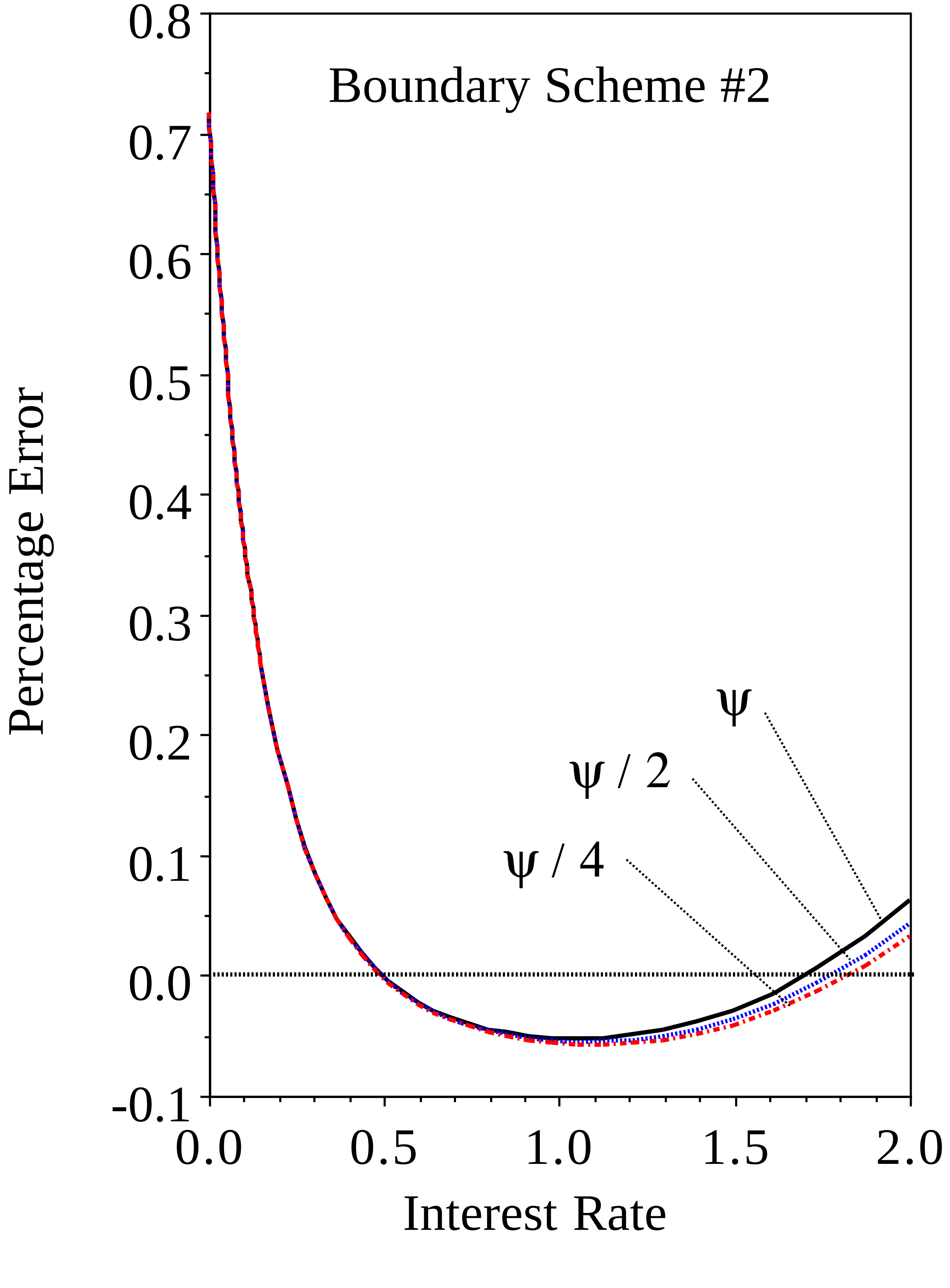


Figure A.8a: Relative Error of the Underlying Straight Bond.

• Finite Differences for Internal Mesh Points.

	Der.	Denomin.	F_{i+1}	F_i	F_{i-1}
Equal	F_x	$2\,\Delta x$	1	—	– 1
Interval	F_{xx}	$(\Delta x)^2$	1	– 2	1
Unequal	F_x	$\Delta x + \Delta x_0$	1	—	– 1
Interval	F_{xx}	$\Delta x\, \Delta x_0 \cdot$	$2\,\Delta x_0$	– 2 ·	$2\,\Delta x$
		$[\Delta x + \Delta x_0]$		$[\Delta x + \Delta x_0]$	

Comment: Read the second row as $F_x = [1 \cdot F_{i+1} - 1 \cdot F_{i-1}] / [2\,\Delta x]$ and similarly the other rows. Δx_0 denotes the interval length to the left of the internal mesh point in question and Δx the interval length to the right of this mesh point.