Pricing Callable Bonds by Means of Green's Function

by

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1. What is a Callable Bond?
2. How to Value a Callable Bond?
3. The Numerical Methods Generally Used
4. Green's Function Approach
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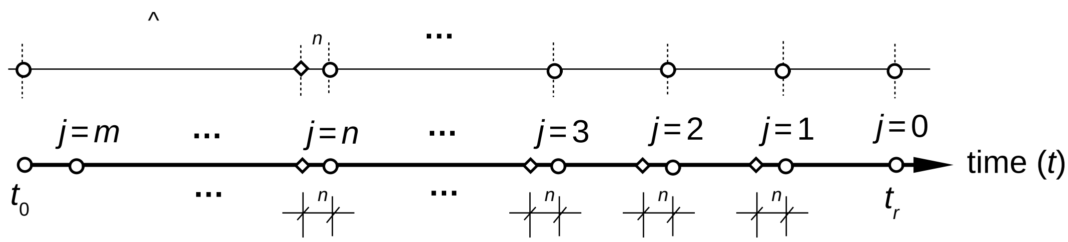
1. WHAT IS A CALLABLE BOND?

• A ***Callable Bond*** is a straight bond (coupon bond) with the provision that the bond can be redeemed by the debtor before the final redemption date at a prespecified price (call price).

• A ***Semi-American*** callable bond has discrete call dates (1 – 10). Notice period: 2 months.



• ***Compound security***: straight bond long + embedded call option short ⇒ Price Callable Bond = Price Straight Bond – Price of Embedded Call Option.

• ***Motivation***: our experience with finite difference methods as well as wrong numerical results published in GIBSON [1990] and LEITHNER [1992].

2. How to value a callable bond?

• Basic security: discount bond.

• Coupon bond: add coupons at discrete points in time.

• Callable bond: add early redemption condition.

• One-factor model ⇒ ***discount bond***: $P(r, t, T)$.

r: instantaneous rate of interest.

t: present date.

T: maturity date.

• ***Modelling the interest rate***: Brennan and Schwartz, Vasicek, Longstaff, Dothan, etc. Cox, Ingersoll and Ross (CIR):

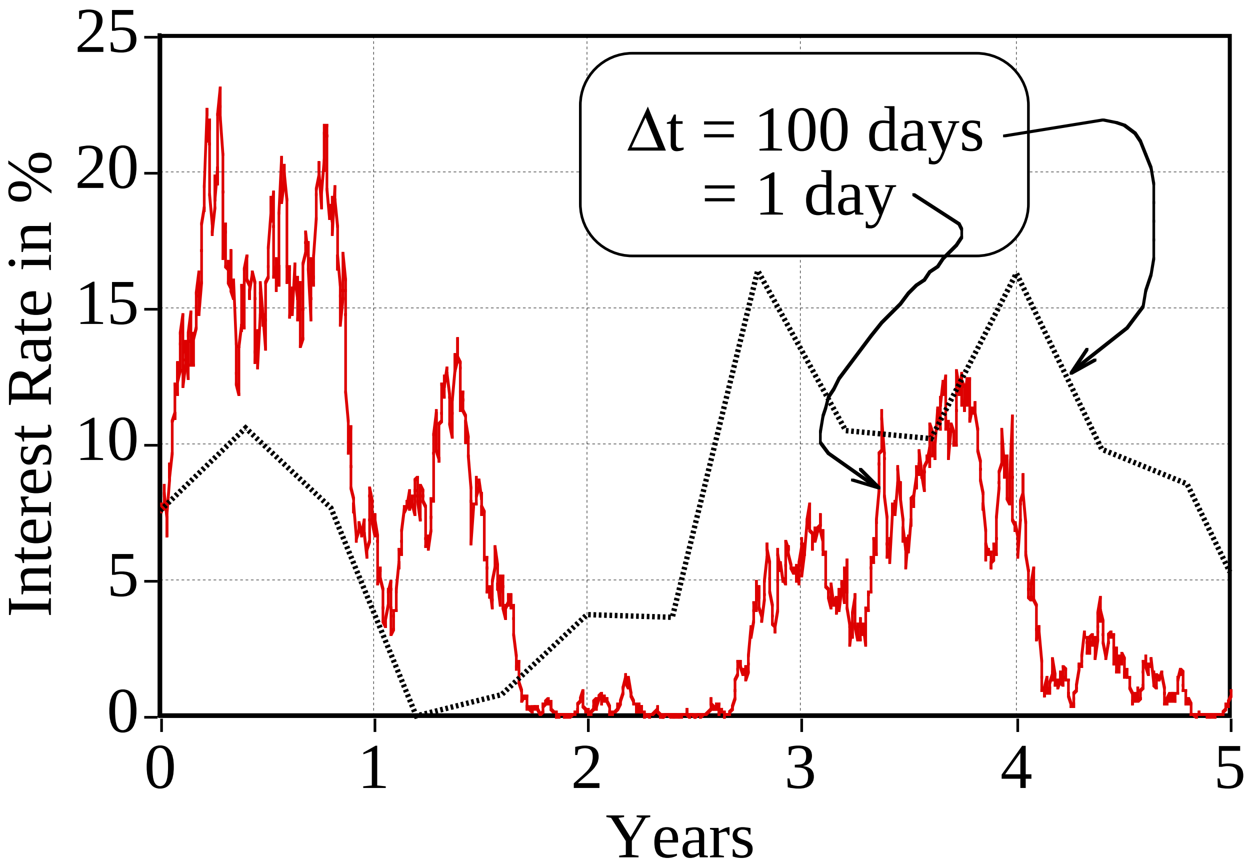
$$dr = \kappa\,(\theta - r)\,dt + \sigma\sqrt{r}\;dz, \quad dz \sim \mathcal{N}\left(0, \sqrt{dt}\right)$$

κ: speed of adjustment (0.55).

θ:long-run mean value (3.5% p. a.).

σ: instantaneous volatility (38.8% p. a.).

z: Gauß-Wiener process.



- CHAN, KAROLYI, LONGSTAFF AND SANDERS [1992]: „An Empirical Comparison of Alternative Models of the Short-Term Interest Rate“, *The Journal of Finance*, XLVII: 1209 – 1227:
“*The models that best describe the dynamics of interest rates over time are those that allow the conditional volatility of interest rate changes to be highly dependent on the level of the interest rate.*”
- Deriving the ***differential equation for the discount bond*** $P(r, t, T)$.

First step: ITO’s lemma.

$$\mathrm{d}P = P_r\,\mathrm{d}r + P_t\,\mathrm{d}t + \frac{1}{2}P_{rr}\underbrace{(\mathrm{d}r)^2}_{\sim\,\mathrm{d}t}$$

$$\mathcal{E}\left\{(\mathrm{d}z)^2\right\} = \mathrm{d}t, \quad \mathcal{V}\left\{(\mathrm{d}z)^2\right\} = (\mathrm{d}t)^2$$

$$\mathrm{d}P = \mu(r, t, T)\,\mathrm{d}t - \sigma(r, t, T)\,\mathrm{d}z \qquad (1)$$

$$\mu(r, t, T) = P_t + \kappa\left[\theta - r\right]P_r\,\frac{1}{2}\,\sigma^2\,r\,P_{rr}$$

$$\sigma(r, t, T) = -\,\sigma\,\sqrt{r}\,P_r$$

Second step: Constructing a riskless portfolio. sell short w_1 of $P(r, t, T_1)$ and buy w_2 of $P(r, t, T_2)$. Choose w_1 and w_2 in such a way that the total worth of the Portfolio $W = w_2 - w_1$ changes deterministically. Then this portfolio earns the instantaneous rate of return, r, over the next instant of time (no arbitrage). Putting all together:

$$q(r, t) = \frac{\mu(r, t, T_1) - r}{\sigma(r, t, T_1)} = \frac{\mu(r, t, T_2) - r}{\sigma(r, t, T_2)} \qquad (2)$$

q: the *market price of* (yield) *risk*.

• Equ. (1) and (2) give ***the partial differential equation (PDE) to value a discount bond***:

$$P_\tau = \frac{1}{2}\sigma^2 r P_{rr} + [\kappa\theta - \zeta r]P_r - rP$$

$\zeta = \kappa - \sigma\sqrt{r}\,q$ (= $\kappa + \lambda$ [CIR]), $q = -\lambda\sqrt{r}/\sigma$.

$\tau = T - t$ (remaining time period until maturity).

• ***Boundary conditions***:

$P(r, 0) = 1$ (Initial condition)

$P(r, \tau) \to 0$ as $r \to \infty$

$P_r(r, \tau)$ finite as $r \to 0$

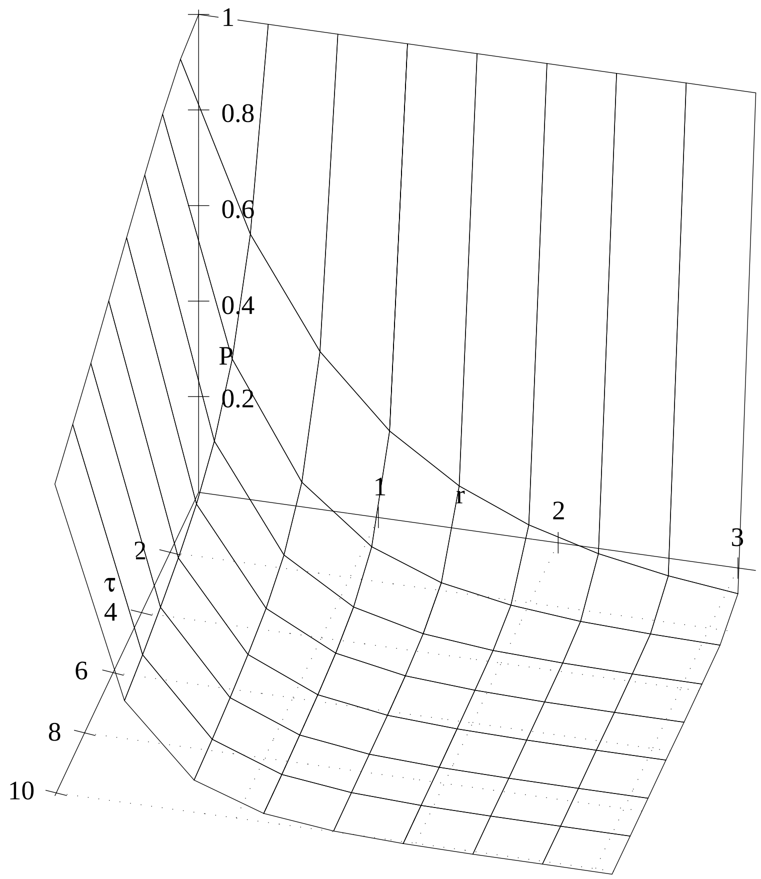
• Solution of the ***price of a discount bond***:

$$P(r, \tau) = f(\tau)\exp\big(g(\tau)\,r\big), \text{ where}$$

$$f(\tau) = \left[\frac{2\gamma\, e^{\frac{\zeta+\gamma}{2}\tau}}{2\gamma + [\zeta+\gamma][e^{\gamma\tau}-1]}\right]^{\frac{2\kappa\theta}{\sigma^2}}$$

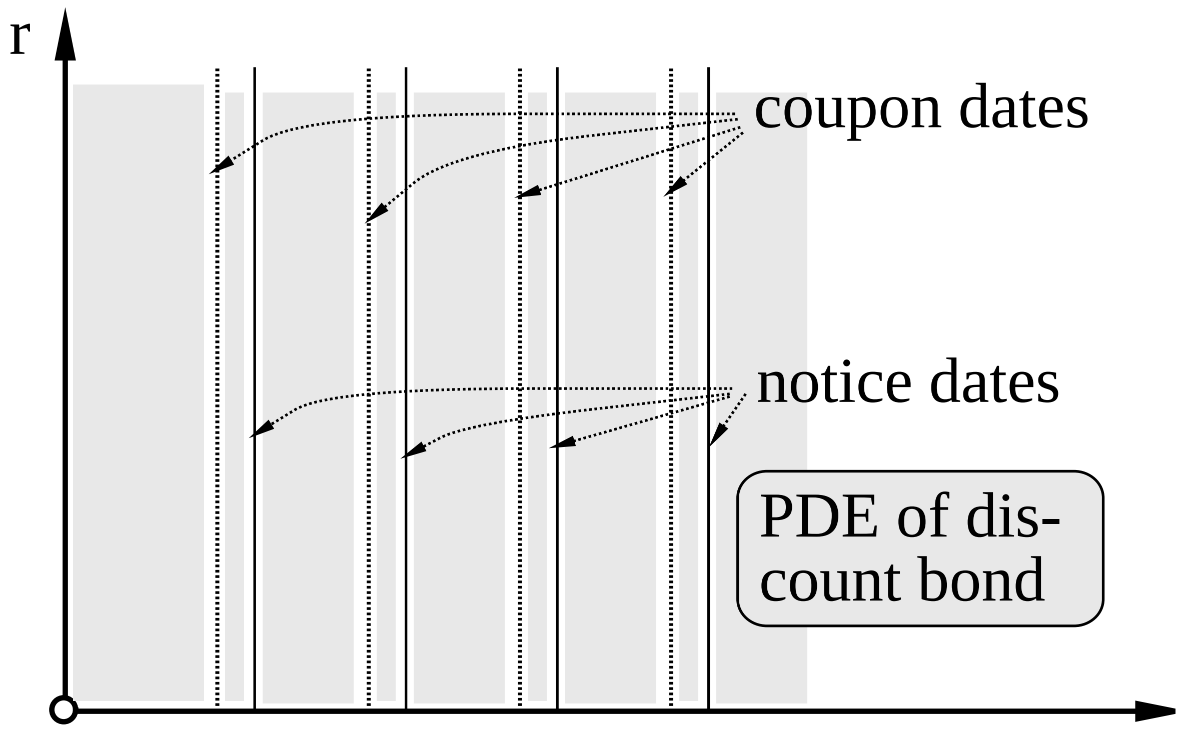
$$g(\tau) = -\frac{2\,[e^{\gamma\tau}-1]}{2\gamma + [\zeta+\gamma][e^{\gamma\tau}-1]}$$

$$\gamma = +\sqrt{\zeta^2 + 2\sigma^2}$$

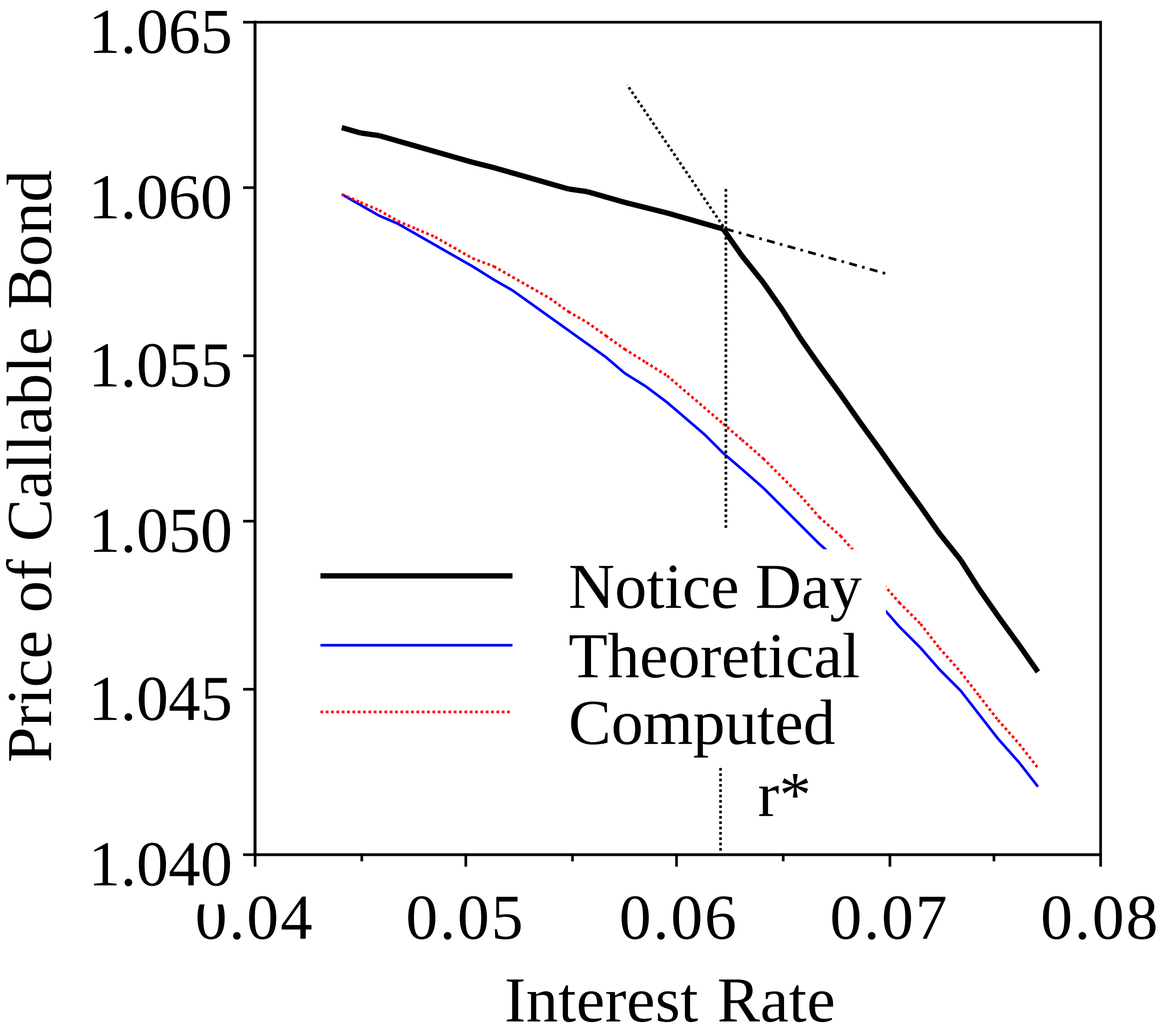


The Price of a Discount Bond.

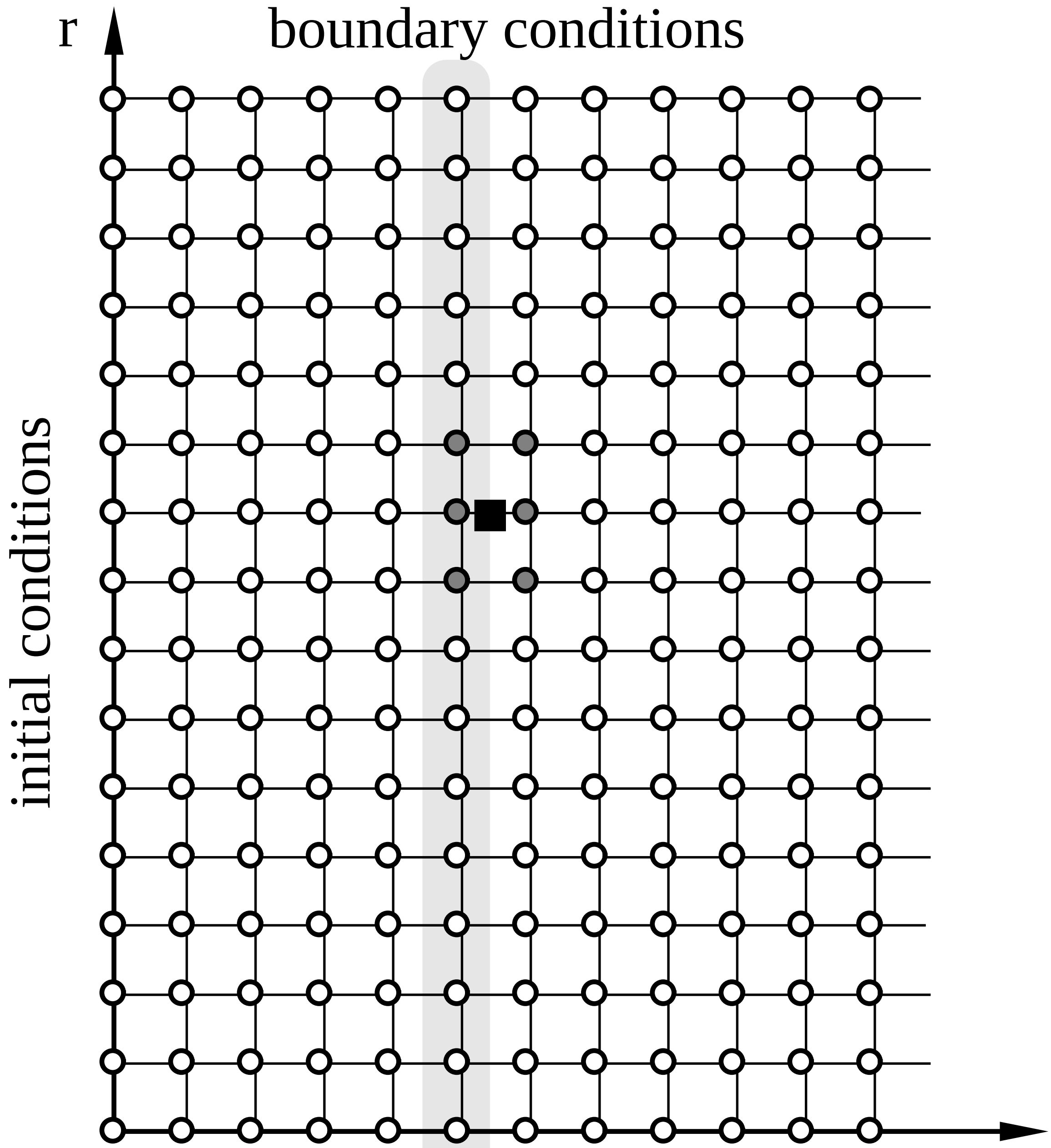
- Solving for the ***callable bond***.



• ***Early redemption condition***: minimum {time value of call price on notice day, price of callable bond an instant before notice day}.



3. The Numerical Methods Generally Used



• ***Finite difference methods***: explicit ⇔ binomial model, implicit ⇔ multinomial model with jumps (Brennan and Schwartz).

• Three Examples:

1. GIBSON [1990, Table 6]) uses the SHAEFER-SCHWARTZ model. Computed prices of two almost identical embedded call options:

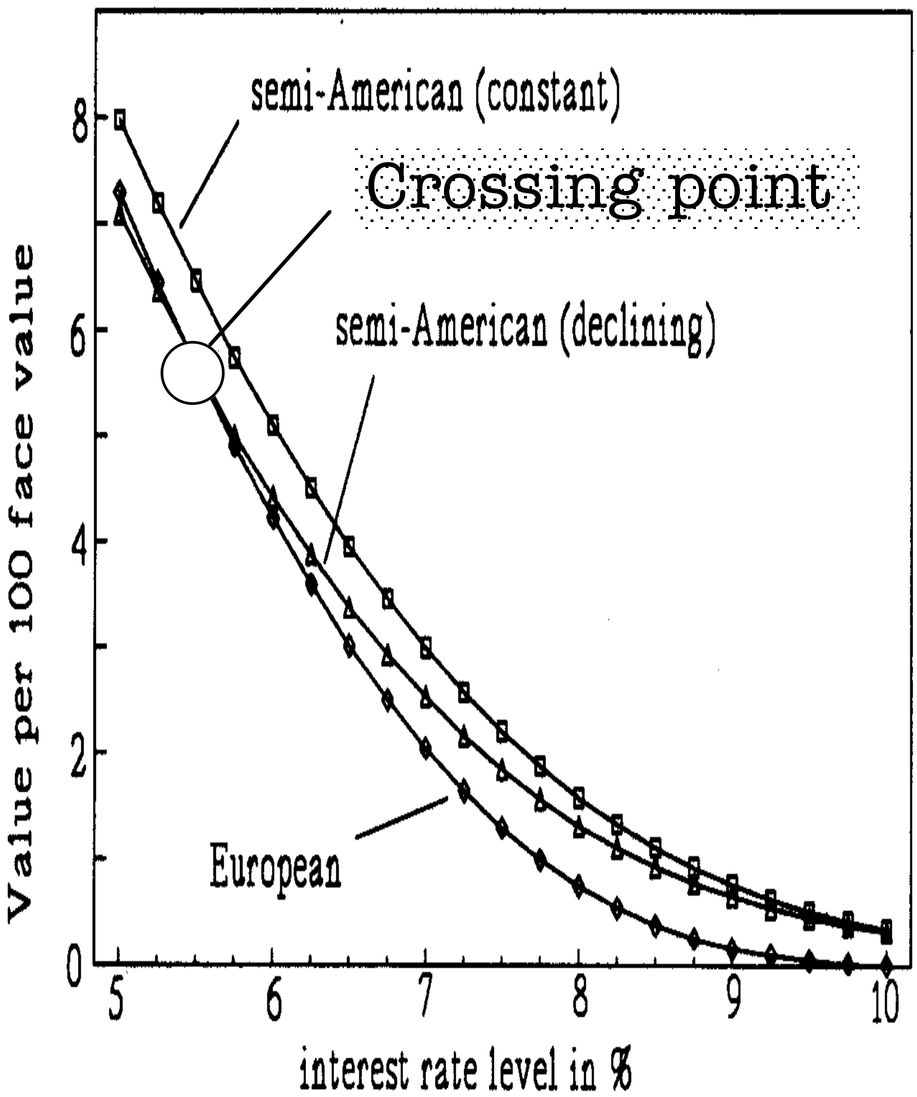
Rem. time	Call #1:	Call #2:
≈ 11 years	1.1303	2.6164
≈ 10 years	0.0878	0.2994
≈ 9 years	0.0482	0.1471
≈ 8 years	15.3192	5.3789

(a) Time profile is wrong. (b) price variation is too large. (c) our computation: difference is ≈ 0.2.

2. GIBSON [1990, p. 670]: *"Surprisingly, we observed —out of 26 price comparisons— that 14 times the price of the callable bond exceeded the straight bond price substantially thereby including the absurd conclusion that the market priced the call feature* ***negatively****."*

3. LEITHNER [1992, Figure 5.5] uses the CIR model. The computed price of the Semi-American call option is less than that of the corresponding European call for small interest rates!

Source: LEITHNER [1992, Figure 5.5, p. 145]



• Why do we get wrong results frequently?

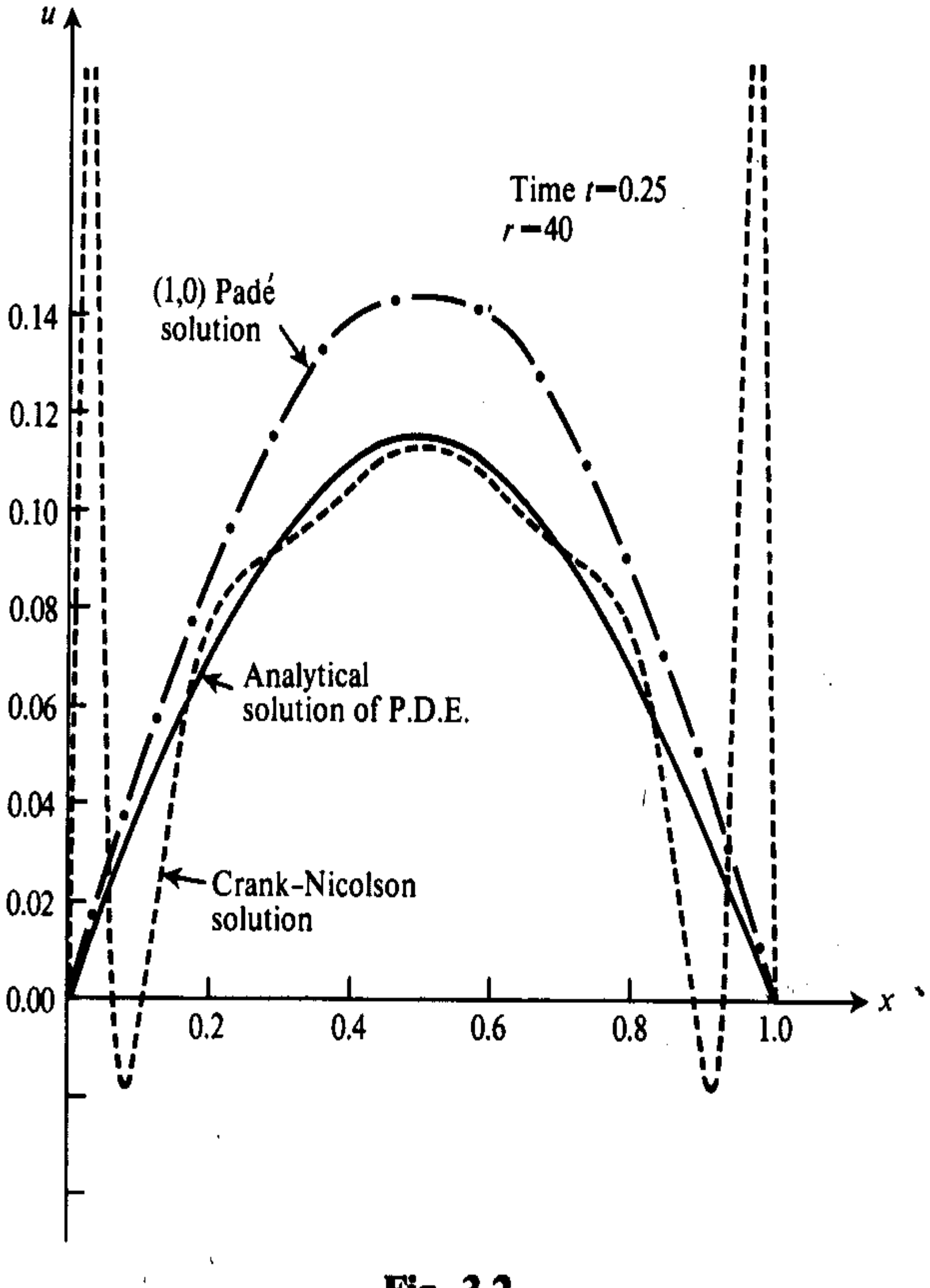
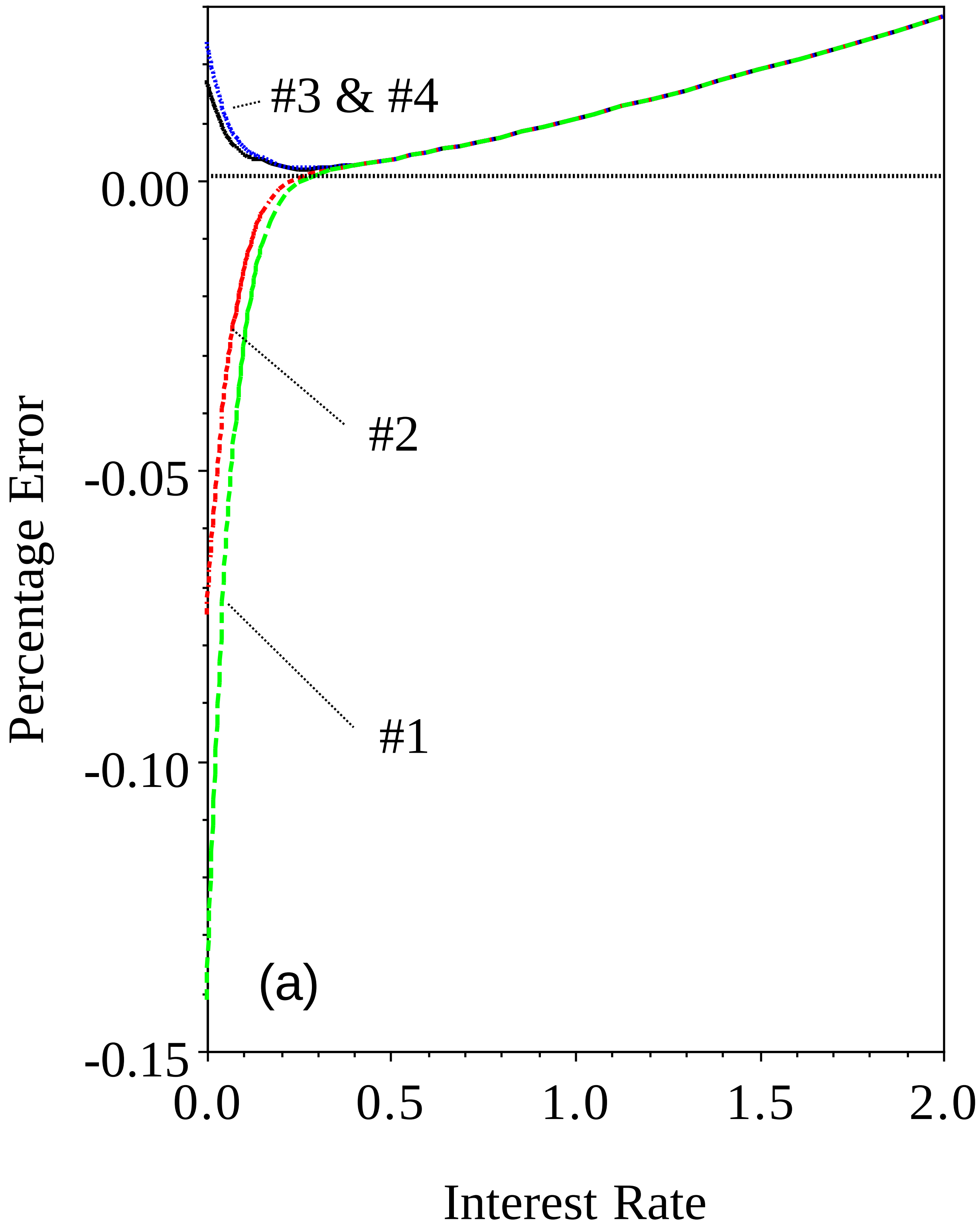
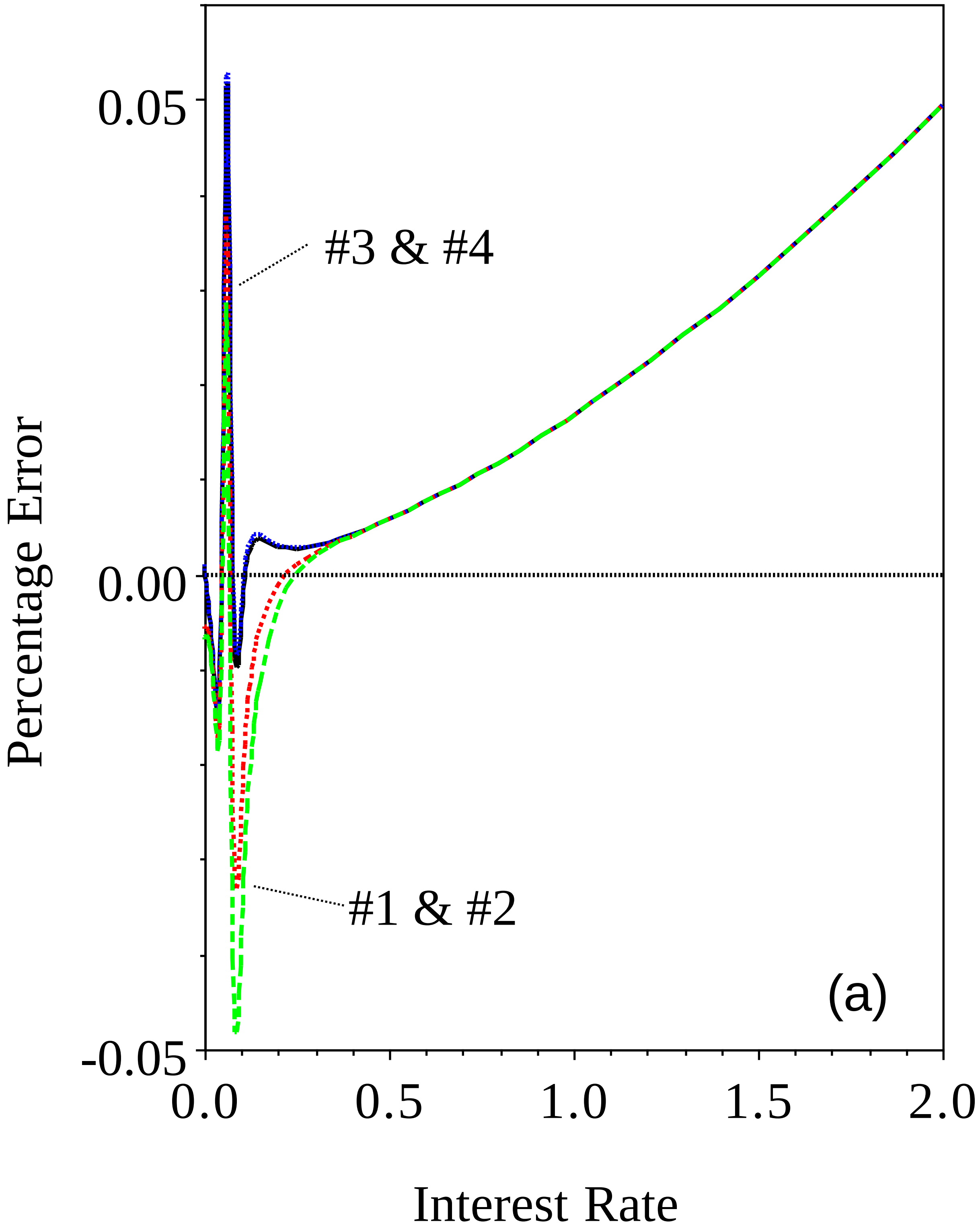


Fig. 3.2

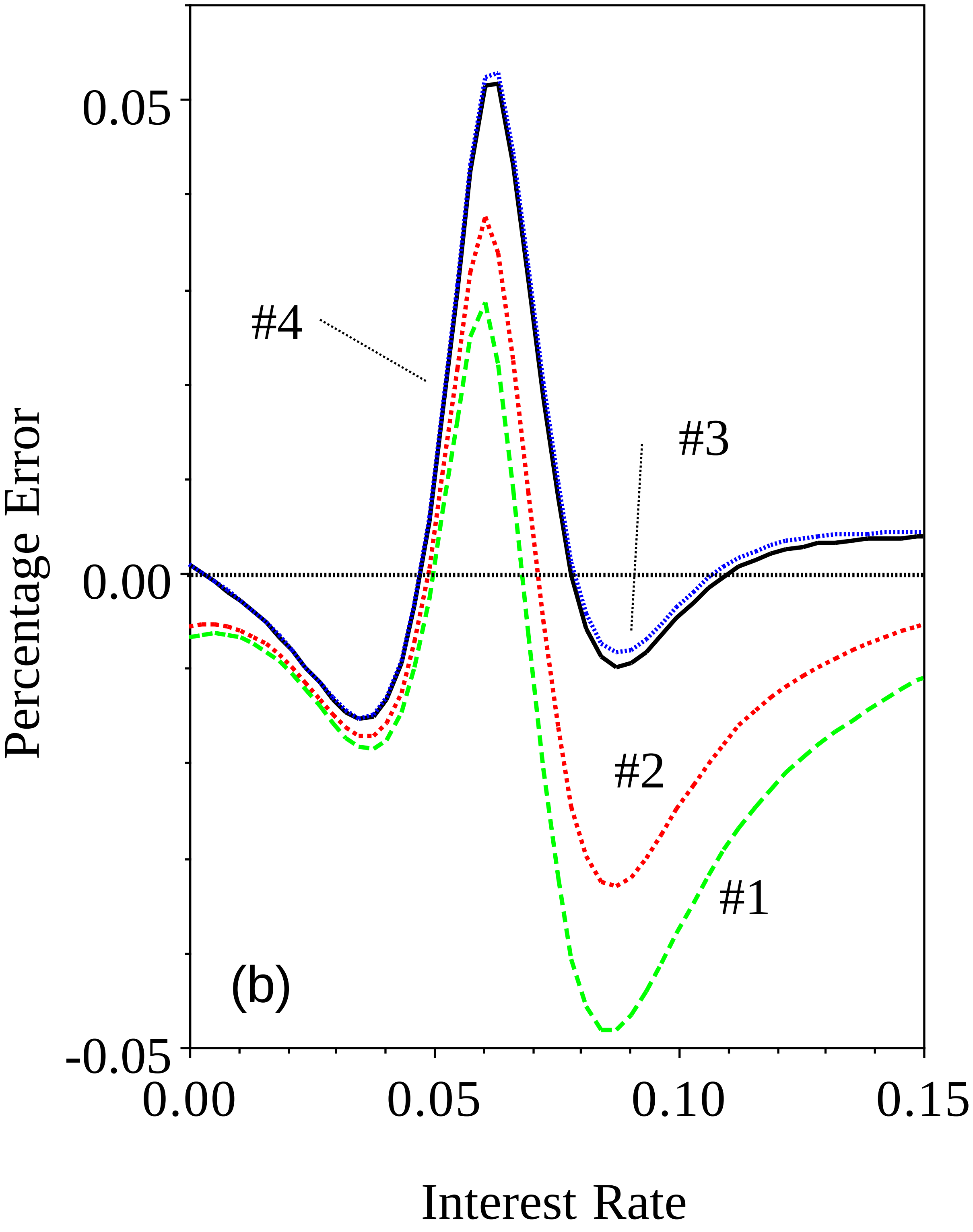
Source: SMITH [1985].



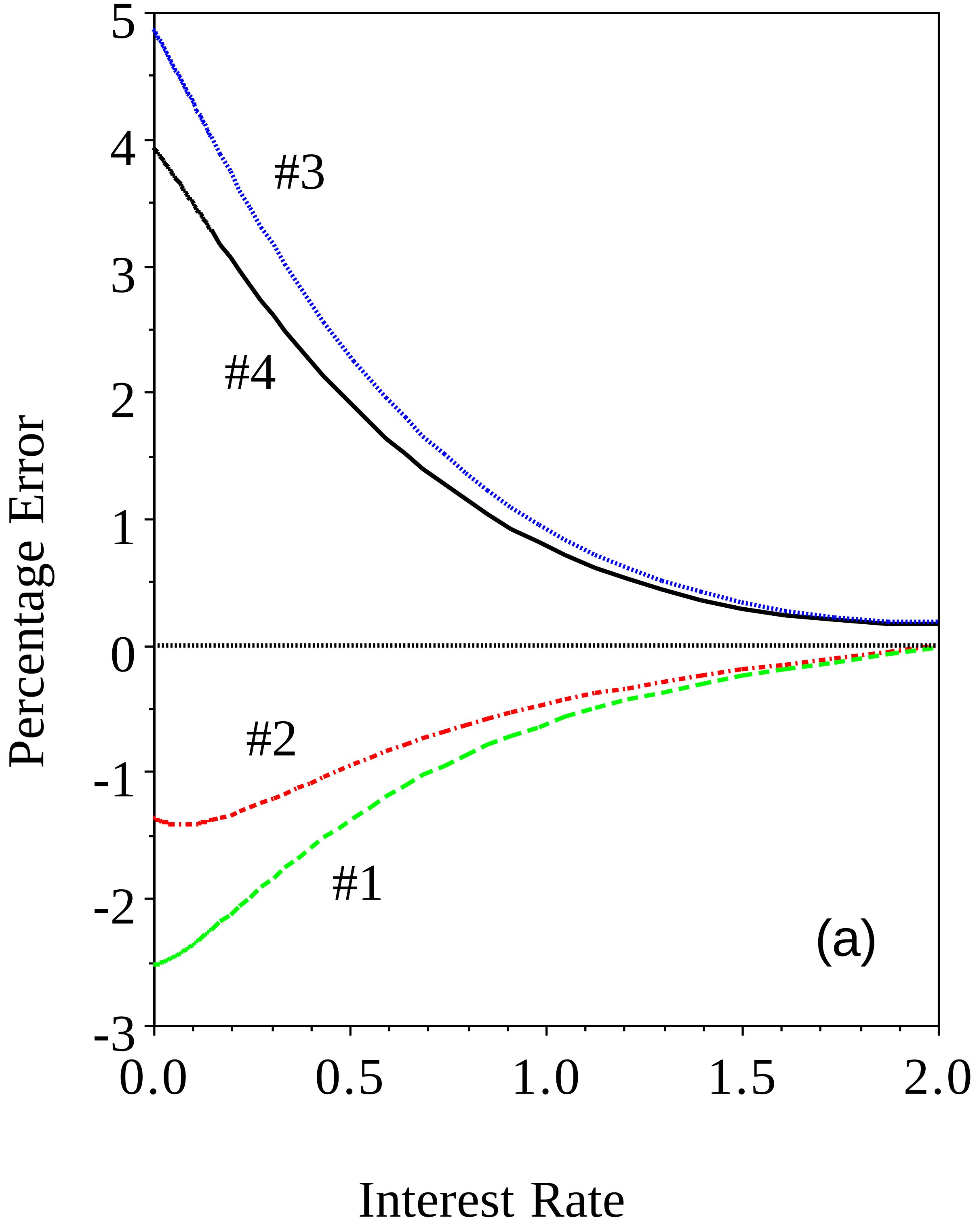
Relative Error of the European Callable Bond on the Notice Day (BÜTTLER [1995]).



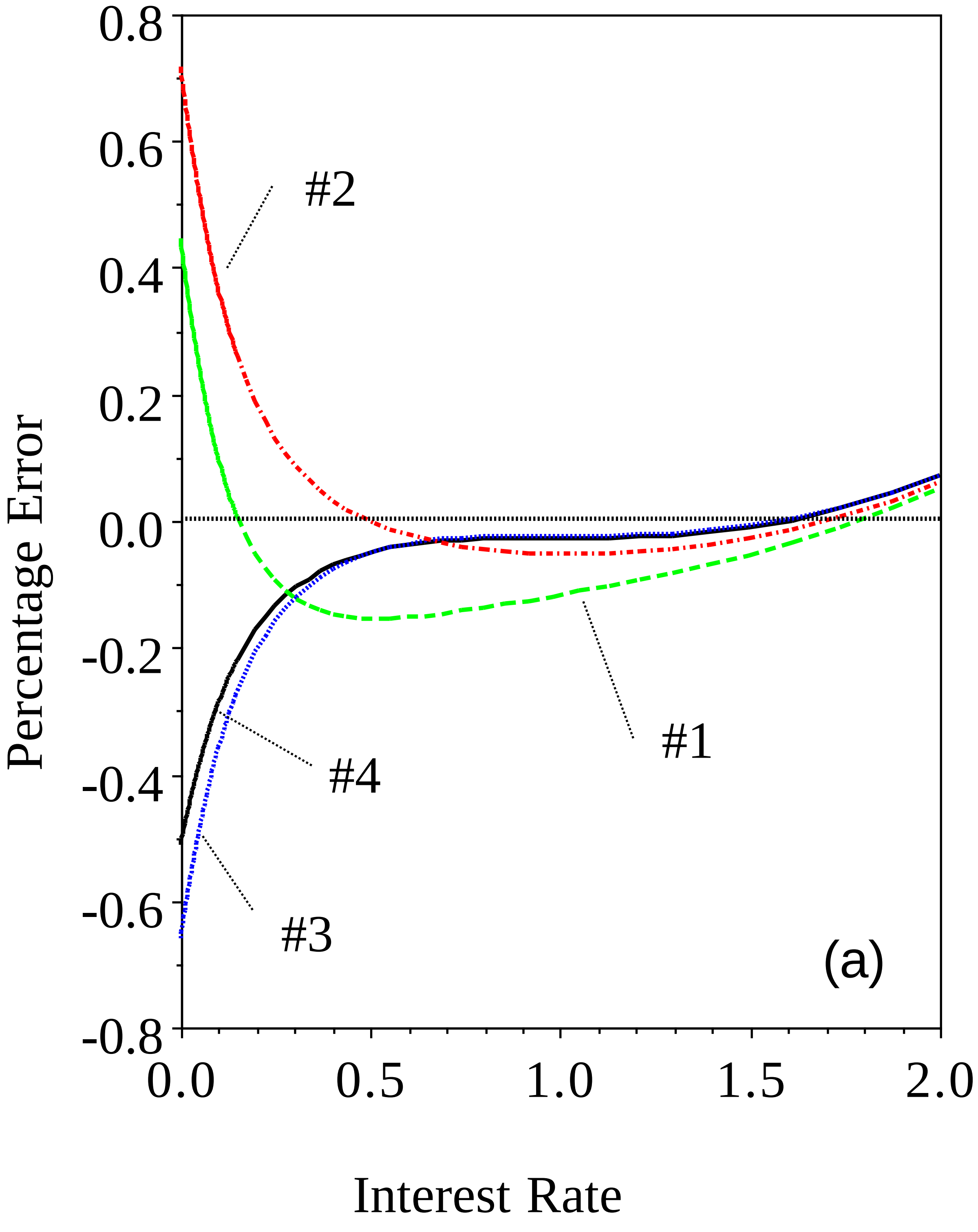
Relative Error of the European Callable Bond One Time Step after the Notice Day (BÜTTLER [1995]).



Relative Error of the European Callable Bond One Time Step after the Notice Day (BÜTTLER [1995]).



Relative Error of the European Callable Bond after 6.811 Years (BÜTTLER [1995]).



Relative Error of the Underlying Straight Bond after 6.811 Years (BÜTTLER [1995]).

• LONGSTAFF [1992, p. 571]: *"I document that callable U. S. Treasury-bond prices frequently imply* ***negative*** *values for the implicit call option. For example, I find that nearly two-thirds of the call values implied by a sample of recent callable bond prices are* ***negative****."*

• ***Multinomial models***:

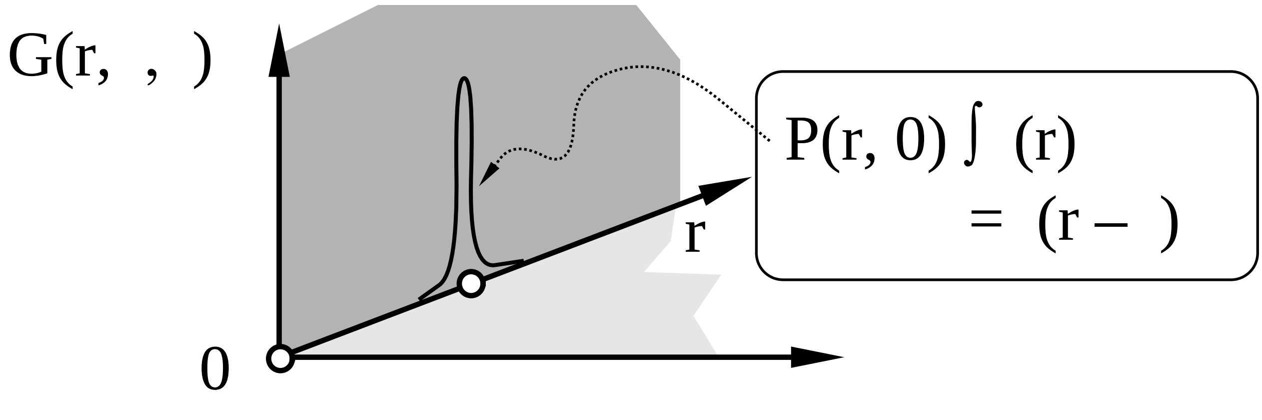
a) A *binomial* model: HO AND LEE (applicability to be proved), HEATH, JARROW AND MORTON (large computing time).

b) A *generalized* binomial model: NELSON AND RAMASWAMY (poor numerical accuracy; e. g. with 200 time steps a 13% error for a five-year discount bond).

c) A *trinomial* model: HULL AND WHITE (accuracy?).

4. GREEN'S FUNCTION APPROACH

• ***Green's function***, G, or the fundamental solution of the partial differential equation to value a discount bond represents the price of a ***primitive security*** at time t_0, given an instantaneous interest rate $r(t_0)$, and which promises to pay one unit of money at the maturity date t_1 if the instantaneous interest rate $r(t_1)$ will be equal to the particular value $\rho(t_1)$, but nothing in all other events. Obviously, the initial condition is the DIRAC delta function, $\delta(r - \rho)$.



Payoff of Primitive Security in the State ρ.

$$G(r, \tau, \rho) = P(r, \tau) \cdot 2\,p(\tau)\, H\big(2p(\tau)\rho \,|\, 2\alpha + 2, \lambda(r, \tau)\big)$$

$$p(\tau) := \frac{\zeta + \gamma - [\zeta - \gamma]\,\mathrm{e}^{-\gamma\tau}}{\sigma^2 \left[1 - \mathrm{e}^{-\gamma\tau}\right]} > 0, \quad \alpha = \frac{2\,\kappa\,\theta}{\sigma^2} - 1$$

$$\lambda(r, \tau) := \frac{8\gamma^2\,\mathrm{e}^{-\gamma\tau}\,r}{\sigma^2\left\{2\gamma + [\zeta - \gamma]\left[1 - \mathrm{e}^{-\gamma\tau}\right]\right\}\left[1 - \mathrm{e}^{-\gamma\tau}\right]} \geqq 0$$

$$H(x|\nu, \lambda) := \frac{1}{2}\left[\frac{x}{\lambda}\right]^{(\nu-2)/4} \mathrm{e}^{-(\lambda+x)/2}\, I_{(\nu-2)/2}\left(\sqrt{\lambda x}\right) = \sum_{n=0}^{\infty} \frac{\left[\lambda/2\right]^n \mathrm{e}^{-(\lambda+x)/2}\, x^{\nu/2+n-1}}{2^{\nu/2+n}\, n!\, \Gamma(\nu/2+n)},$$

$I_\alpha(\cdot)$: Modified BESSEL function of the first kind of order α.

$\Gamma(\cdot)$: EULER's gamma function.

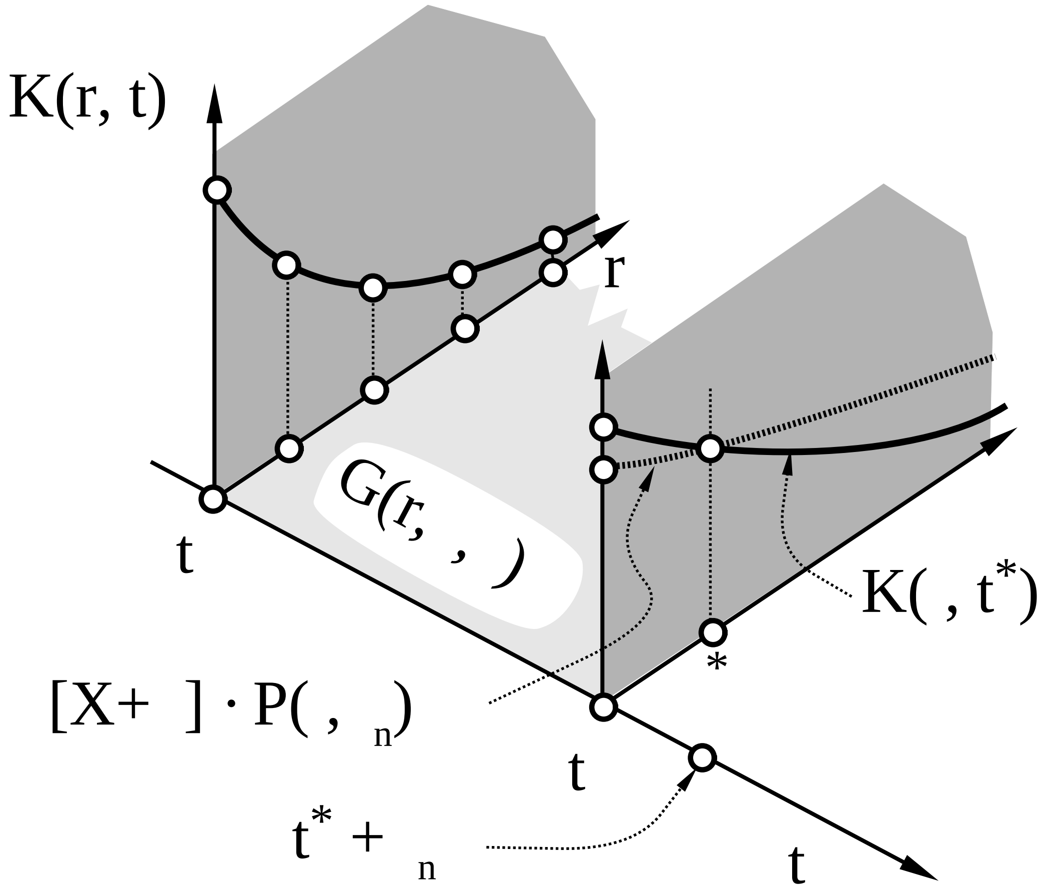
$H(\cdot)$: "Forward-risk-adjusted" non-central chi-square distribution with $\nu = 2\,\alpha + 2$ degrees of freedom and noncentrality parameter $\lambda = \lambda(r, \tau)$.

• Following JAMSHIDIAN [1991, theorem 1, p. 143, and corollary 2, p. 145], the price of a primitive security is equal to the price of a discount bond times the "forward-risk-adjusted" expectation of the DIRAC delta function. Hence, GREEN's function is equal to the price of a discount bond, $P(r, \tau)$, multiplied by the ***"forward-risk-adjusted" probability*** density function of the instantaneous interest rate prevailing at the terminal date (when looking forwards), conditional on the current value of the instantaneous interest rate.

• "forward-risk" adjustment:

$$\mathrm{d}r = \left[\kappa\,(\theta - r) - (\zeta - \kappa)\,r + \sigma^2\, r\, g(t)\right]\mathrm{d}t + \sigma\sqrt{r}\;\mathrm{d}z$$

• ***Price of callable bond***, *K*:



$$K(r,t) = \left[X + \eta\right] \int_0^{\rho^*} G(r,\ \tau,\ \rho)\, P(\rho,\ \tau_n)\, \mathrm{d}\rho$$

$$+ \int_{\rho^*}^{+\infty} G(r,\ \tau,\ \rho)\, K\left(\rho, t^*\right) \mathrm{d}\rho + \eta\, P(r,\ \tau_n),$$

• ***An expample***:

Table C.2: The Characteristics of ….

Name of security [1]	4 1/4% Swiss Confederation 1987–2012
Maximum life of callable bond	20.172 years
Notice period	0.1666 years (2 months)
Interpolation method	Exponential spline with equidistant knots
Year [2]	Call Price
1. – 5.	1.000
6.	1.005
7.	1.010
8.	1.015
9.	1.020
10.	1.025

[1] Security number 15 718. The bond bears an annual coupon of 4 1/4%. Trading day December 23, 1991.

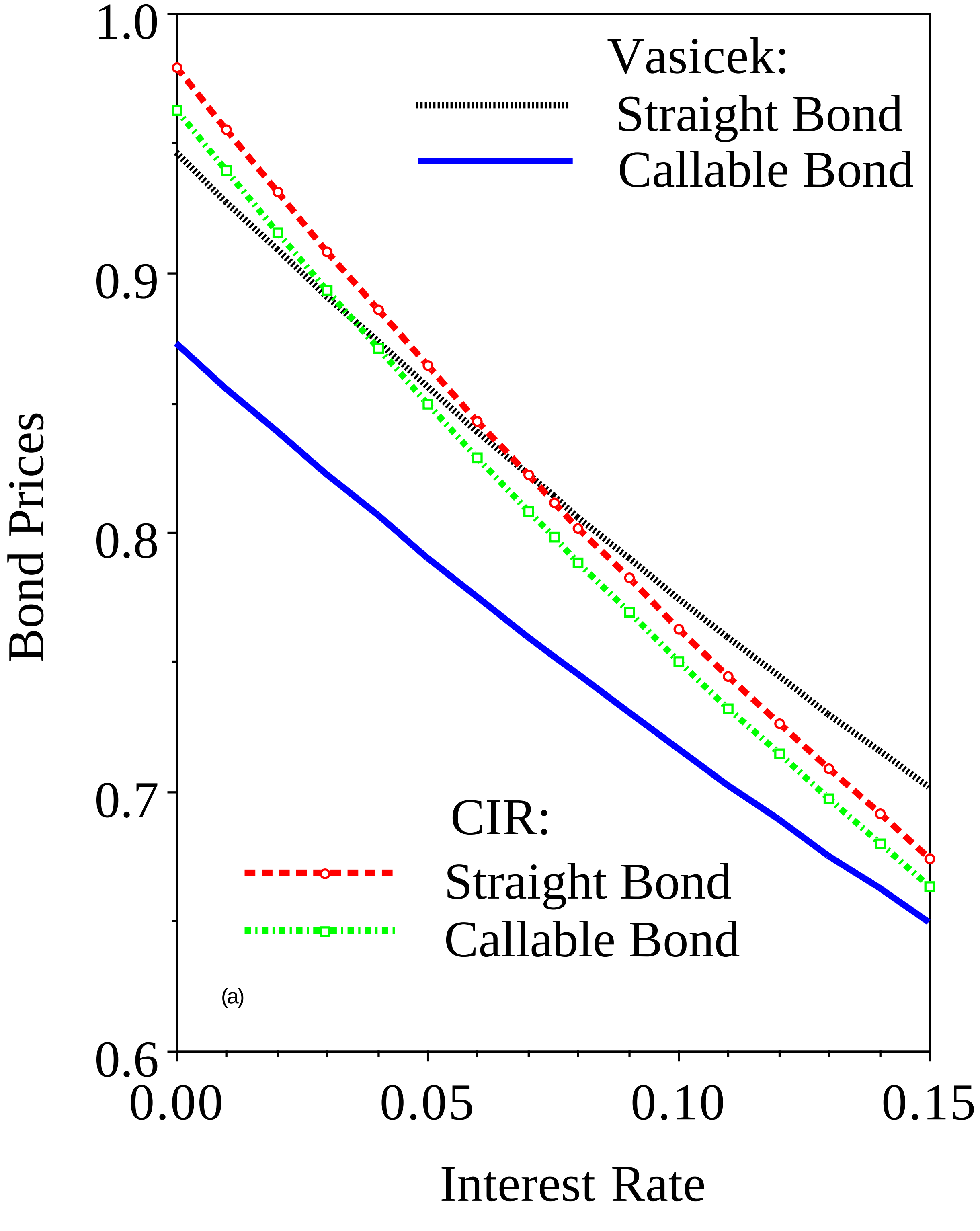
[2] Ordinal number of the call date when counted backwards in time. The ten call dates are equal to the ten coupon dates prior to the last possible redemption day. The computations are based on the assumption that the debtor makes his call decision on the notice days (notional call dates) which lie two months ahead of the effective call dates.

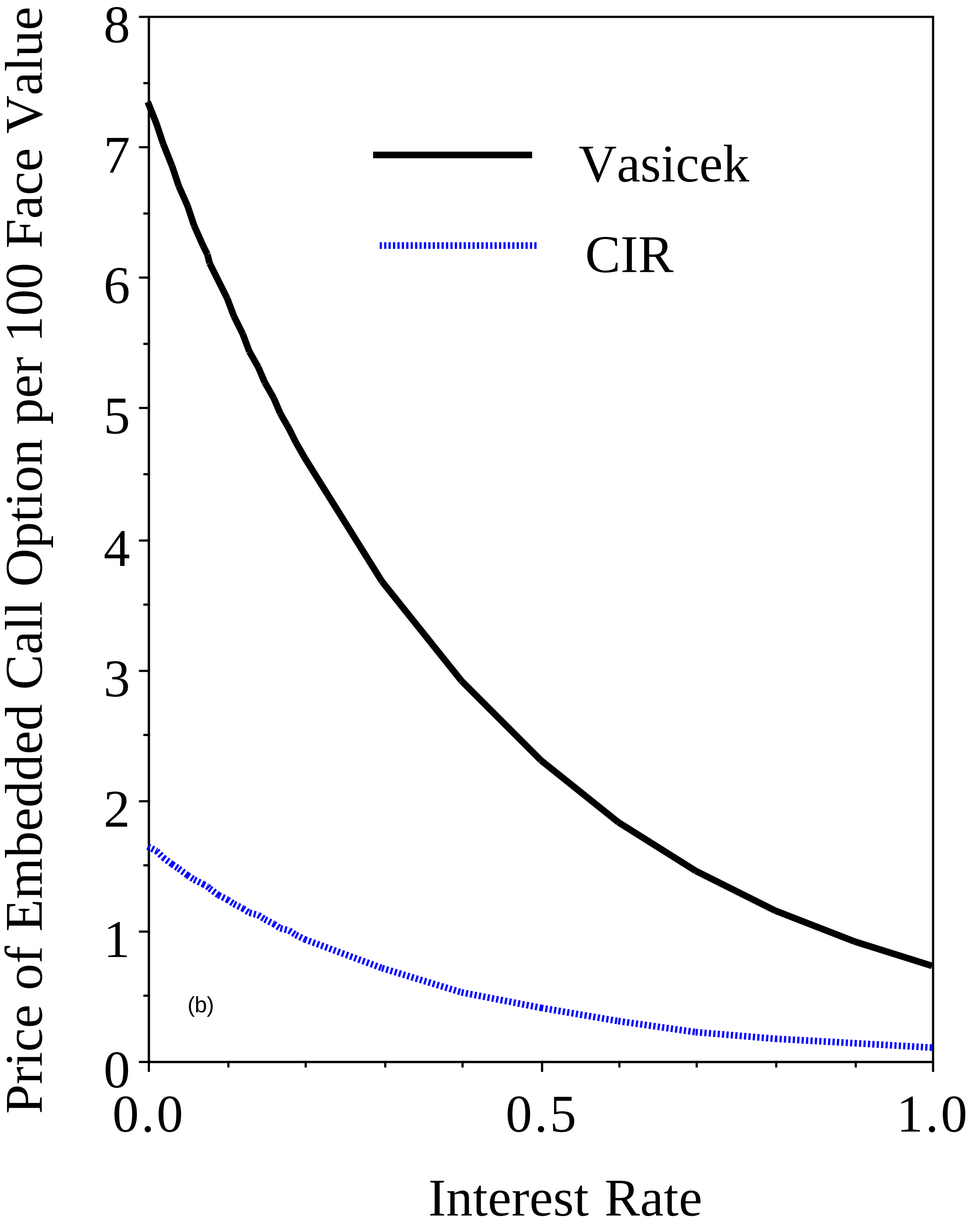
Table C.4: The Break-even Interest Rates. [1]

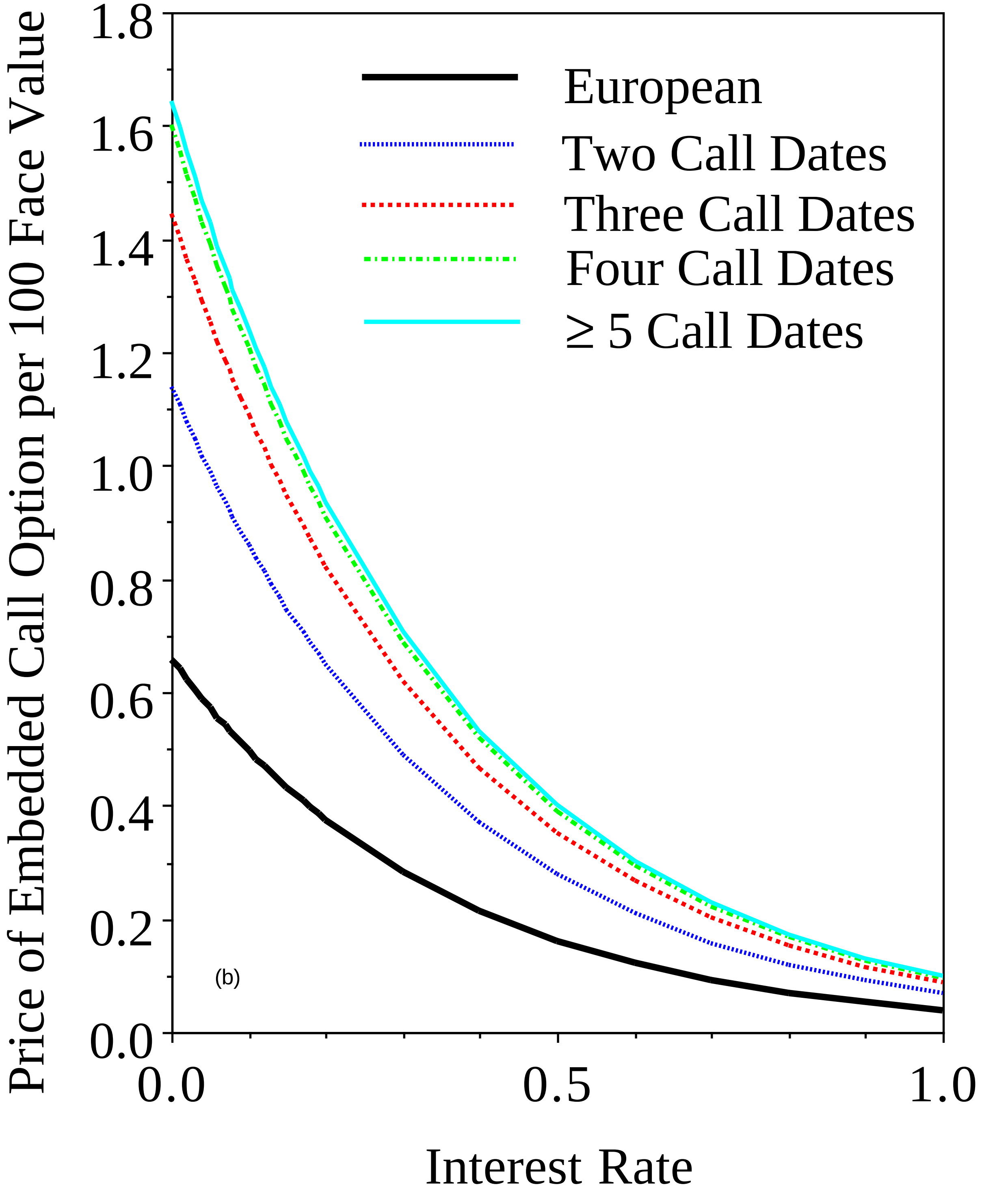
Year [2]	VASICEK	CIR
1.	0.0270644976	0.0338871564
2.	- 0.0101266321	0.0179273733
3.	- 0.0333187039	0.0097892562
4.	- 0.0557203925	0.0048817260
5.	- 0.0743127490	0.0015784739
6.	- 0.0935658091	0
7.	- 0.1088988626	0
8.	- 0.1214749074	0
9.	- 0.1314698228	0
10.	- 0.1391392481	0

1 Break-even interest rates obtained for 650 knots.

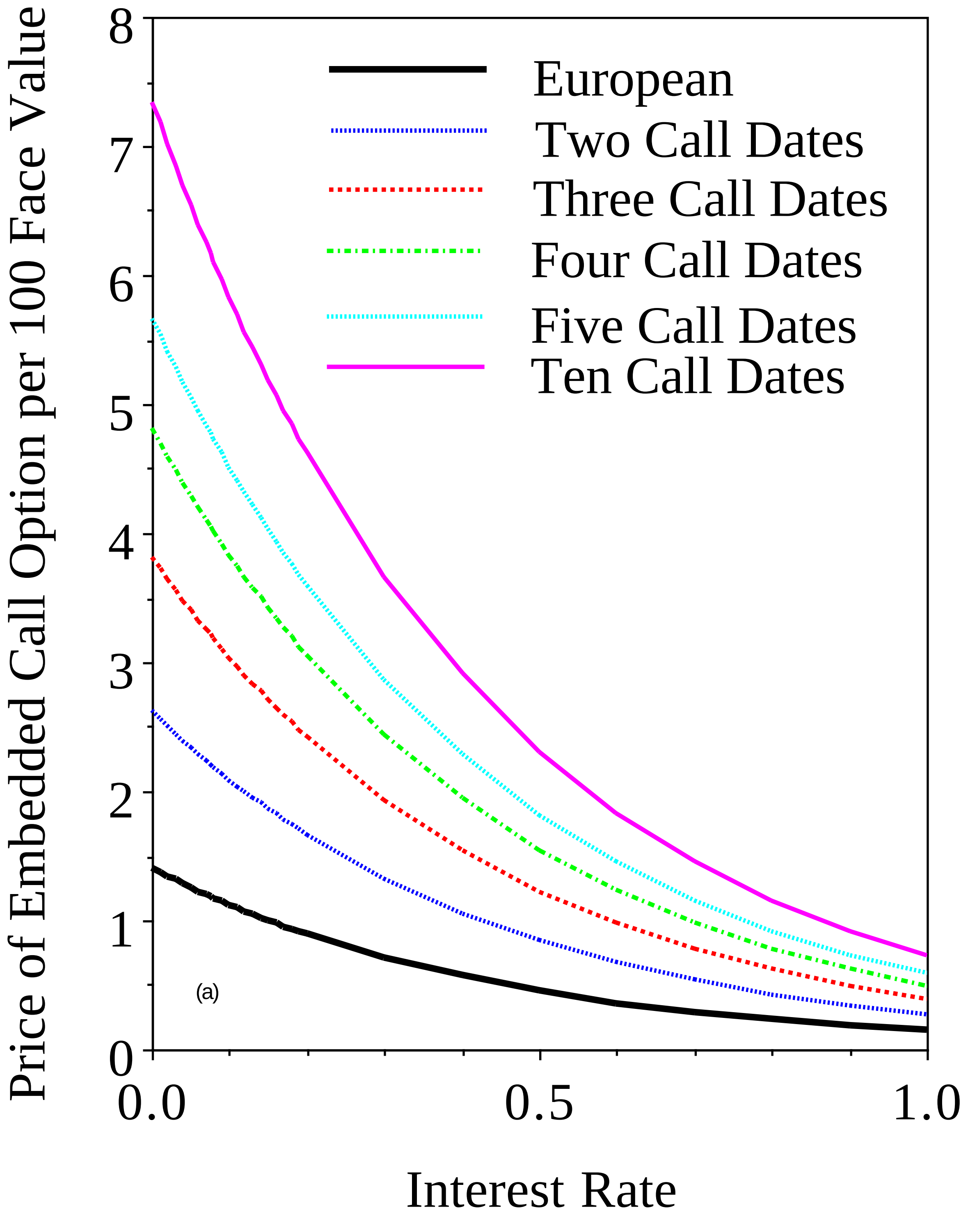
2 Ordinal number of notional call date (i. e., notice date) counted backwards in time.


Vasicek:
Straight Bond
Callable Bond
CIR:
Straight Bond
Callable Bond
(a)
Bond Prices
Interest Rate
1.0
0.9
0.8
0.7
0.6
0.00
0.05
0.10
0.15


Price of Embedded Call Option per 100 Face Value
8
7
6
5
4
3
2
1
0
Vasicek
CIR
(b)
0.0
0.5
1.0
Interest Rate


Price of Embedded Call Option per 100 Face Value
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
European
Two Call Dates
Three Call Dates
Four Call Dates
≥ 5 Call Dates
(b)
0.0
0.5
1.0
Interest Rate

CIR Model.



Vasicek's Model.

5. Conclusions

(1) ***Green's function*** is the discounted value of the "forward-risk-adjusted" probability density function of the underlying instantaneous interest-rate process, or, alternatively,
Green's function is the price of a primitive security.
(2) By means of Green's function, the price of the ***Semi-American callable bond*** can be expressed in terms of a multiple integral of the adjusted probability density function of the instantaneous interest-rate process.
(3) Since the numerical quadrature of a multiple integral with inter-dependent limits of integration is prohibitively expensive in terms of computing time, the ***proposed algorithm*** increases the speed of computation by calculating a sequence of one-dimensional integrals involving Green's function.

(4) Since the boundary conditions are built into the algorithm, ***no numerical instabilities*** occur, in contrast to finite difference methods.

(5) The ***accuracy of the algorithm*** depends on both the numerical quadrature and the interpolation.

(a) The ***numerical quadrature*** procedure proposed in our paper can, in principle, achieve machine precision.

(b) The ***interpolation*** of the price function relies on the series solution we derived for the price of a discount bond. In principle, the interpolation does not achieve machine precision, but the accuracy is much higher than for a finite difference method.

(6) The proposed algorithm offers both ***higher accuracy and higher speed of computation*** than finite difference methods in the example considered in this paper.