The Optimal Capital Structure of a Liquidity-insuring Bank

by

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1 What is a Bank?

2 A Simple Model of a Bank

3 A Numerical Sensitivity Analysis

4 Conclusions

1 What is a Bank?

- Why explaining the ***liabilities*** of a bank?

☞ Modigliani and Miller's [1958] proposition of the ***irrelevancy*** of the capital structure.

☞ Vast literature has evolved thereafter in more than three decades.

- In view of Modigliani and Miller's [1958] paper, the ***asset structure*** of a bank would be irrelevant, too. Viewed as a financial portfolio, it can be replicated by any investor.
- Why is there a need for ***capital regulation*** (BIS propositions)?
- ***Stylized facts*** about the historical evolution of banks:

☞ 19th century: a high equity-to-debt ratio (0.6 – 0.8) and a high interest-rate differential.

☞ 20th century: a very low equity-to-debt ratio (0.03 – 0.1) and a low interest-rate differential.

• ***Purpose*** of the paper: Optimal capital structure for a ***competitive*** bank.

• Why do banks exist?

☞ Banks offer a very special contract: a ***sight deposit*** contract which promises a payment on demand at par value and and a floating-rate interest thereon as long as ***confidence*** is maintained.

☞ Bank is an ***insurer of unpredictable liquidity*** demanded by depositors in the sense described by DIAMOND AND DYBVIG [1983].

☞ A competitive claim market would sell liquidity insurance less efficiently than banks for two reasons:

⇐ Difficult pricing of financial product (puttable bond with uncertain floating-rate coupon) which corresponds with sight deposit contract.

⇐ Transaction costs are higher.

- ***Sight deposits*** imply

☞ Assets and deposits are ***output***.

    ✍ Multi-dimensional cost function

☞ Doubts about solvency ⇒ loss of confidence ⇒ bank run ⇒ solvency and liquidity are interrelated.

    ✍ Financial states of the bank

    ✍ Reaction function of depositors

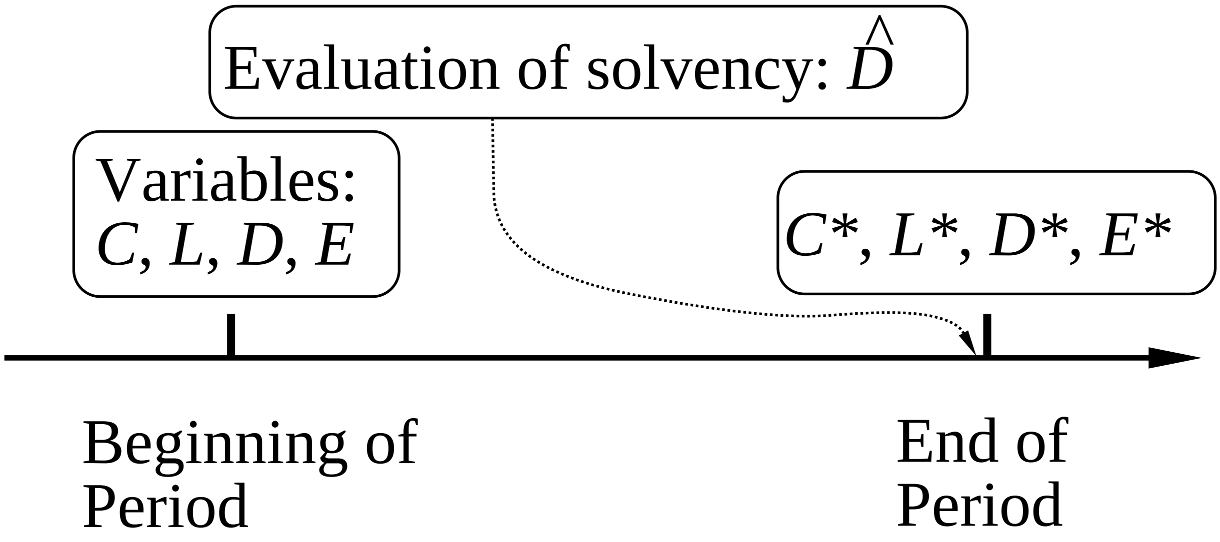
    ✍ Degree of informativeness

    ✍ Joint probability density function for deposits and loans

- ***Optimization*** from the point of view of equity owner (no agency problems considered).

✍ Yield-on-equity constraint

2 THE MODEL



Balance Sheet at the Beginning of the Period	
Assets	Liabilities
Cash C Loans L	Deposits D Equity E
$C+L$	$D+E$

- C: "Cash" balances bear no interest
- L: Market value of loans
- E: Market value of equity

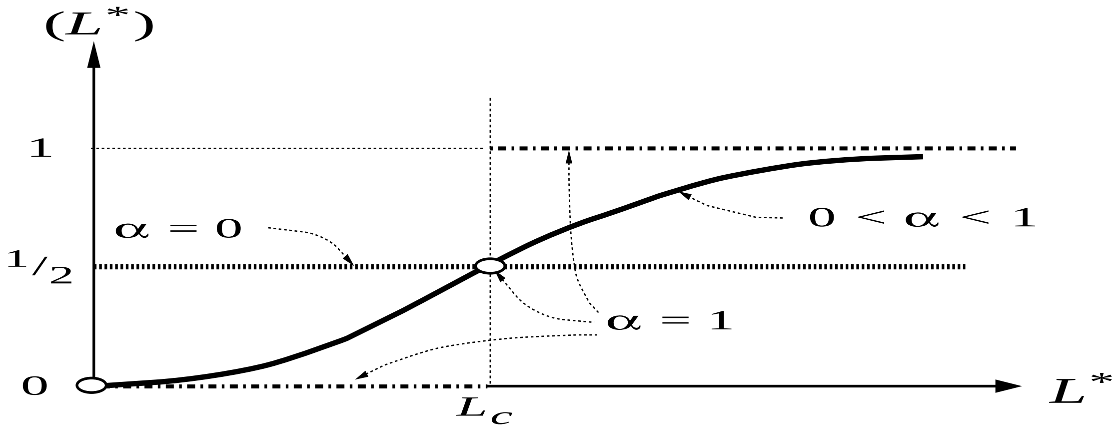
Profit and Loss Account for the Model Period	
Expenditures	Earnings
Interest on deposits $r_d\, D$ Costs $\varphi(L, D, C)$ Profit P	Interest on loans $r_\ell\, L$ Change in value ΔL
$r_d\, D + \varphi(L, D, C) + P$	$r_\ell\, L + \Delta L$

- $\varphi(L, D, C)$: $\partial\varphi/\partial L > \partial\varphi/\partial D > \partial\varphi/\partial C > 0$ for $L = D = C$
- Joint probability density function, $f(\hat{D}, L^*; D, L)$, where $\hat{D}$ denotes the ***intermediate*** deposit balances before depositors react to the observed financial status of the bank in a second round.
- Depositors' reaction function

$$D^* = \widehat{D}\, \boldsymbol{h}(L^*)$$

$$\text{with } \boldsymbol{h}(L^*) = \left[1 - \frac{1}{1 + \left(L^*/L_c\right)^{\operatorname{arctanh}(\alpha)}}\right]$$

$$\text{where } 0 \leqq \alpha \leqq 1$$



• Derived ***Probablity density function*** $f^*(D^*, L^*; D, L)$:

$$f^*\left(D^*,\ L^*\right) = \frac{1}{\hbar\left(L^*\right)}\, f\left(\frac{D^*}{\hbar\left(L^*\right)},\ L^*\right) \quad \text{if } 0 < \alpha < 1$$

Balance Sheet at the End of the Model Period	
Assets	Liabilities
$C^* = C + r_\ell\, L - r_d\, D$ $\quad - \varphi(L, D, C) + \Delta D$ $L^* = L + \Delta L$	$D^* = D + \Delta D$ $E^* = E + \Delta E = E + P$
$C^* + L^*$	$D^* + E^*$

• Maximize the owner's utility of final wealth:

$$E^* = C^* + L^* - D^* = L^* - L_c$$

$$\text{with } L_c \equiv \left(1+r_d\right)D + \varphi\left(L,\ D,\ C\right) - r_\ell L - C$$

• L_c denotes the ***critical*** level of the value of ***loans*** the bank needs at least to keep its solvency

• ***Insolvency*** if $E^* < 0 \Rightarrow L^* < L_c$

• ***Illiquidity*** if $C^* < 0 \Rightarrow D^* < L_c$

• ***Pay-offs*** in the four financial states considered

	solvent and ...		insolvent and ...	
	liquid state 1	illiquid state 2	liquid state 3	illiquid s. 4
$[A_s]_{\alpha=1}$	E^*	max $[E^* - S, 0]$	0	0
$[A_s]_{\alpha=0}$	E^*	0	max $[r_\ell L - r_d D - \varphi(L, D, C), 0]$	0
A_s	E^*	max $[\alpha(E^* - S), 0]$	max $[(1-\alpha)\{r_\ell L - r_d D - \varphi(L, D, C)\}, 0]$	0

• ***Goal***

$$\max_{\{C,\ L, D,\ E\}} \mathcal{E}\,\mathcal{U}(A) = \max_{\{C,\ L, D,\ E\}} \sum_{s=1}^{4} \mathcal{E}_s\, \mathcal{U}\left(A_s\left(L^*\right)\right)$$

or explicitly

$$\mathcal{E}\,\mathcal{U}\,(A) =$$

$$\int_{L^*=L_c}^{L^*=\infty}\int_{D^*=L_c}^{D^*=\infty} \mathcal{U}\,(A_1)\, f^*\left(D^*, L^*\right) \mathrm{d}D^*\mathrm{d}L^* \text{ (state 1)}$$

$$+\int_{L^*=L_c}^{L^*=\infty}\int_{D^*=0}^{D^*=L_c} \mathcal{U}\,(A_2)\, f^*\left(D^*, L^*\right) \mathrm{d}D^*\mathrm{d}L^* \text{ (state 2)}$$

$$+\,\mathcal{U}\,(A_3)\int_{L^*=0}^{L^*=L_c}\int_{D^*=L_c}^{D^*=\infty} f^*\left(D^*, L^*\right) \mathrm{d}D^*\mathrm{d}L^* \text{ (state 3)}$$

$$+\,\mathcal{U}\,(A_4=0)\int_{L^*=0}^{L^*=L_c}\int_{D^*=0}^{D^*=L_c} f^*\left(D^*, L^*\right) \mathrm{d}D^*\mathrm{d}L^* \text{ (state 4)}$$

- ***Constraints***

$$(1)\ C, L, D, E \geqq 0$$

$$(2)\ C + L = D + E = 1$$

$$(3)\ L_c \geqq 0$$

$$(4)\ \mathcal{E}\left\{\frac{A_s(E^*)}{E} - 1\right\} \geqq r_f > r_d \quad \text{yield on equity}$$

3 A Numerical Sensitivity Analysis

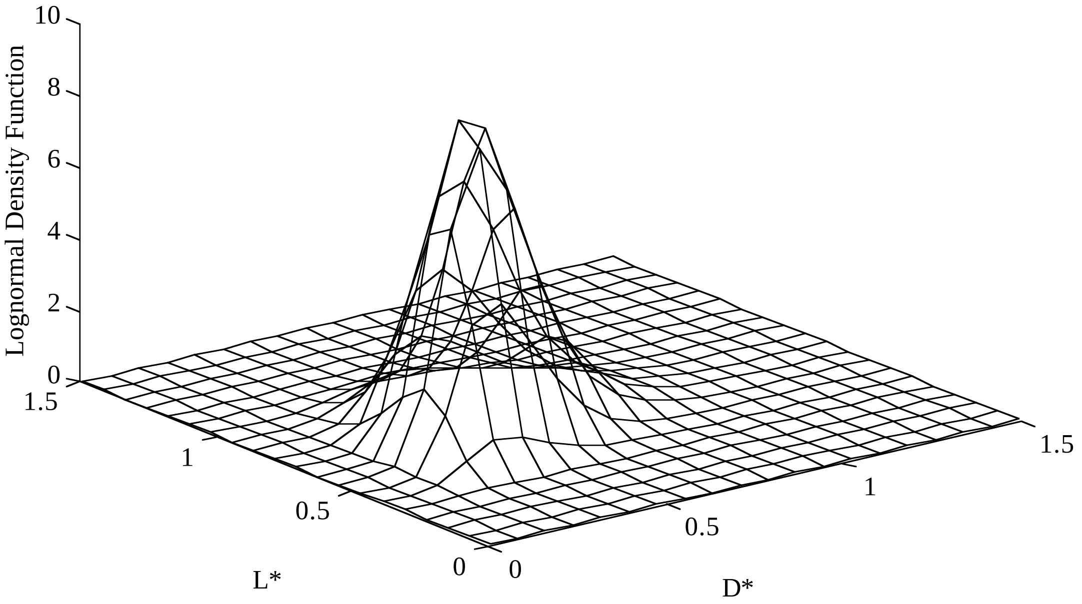
• The Probability density function for $(D;^{\wedge}, L^*)$ is the ***bivariate log-normal density function***

$$f\left(\widehat{D}, L^*\right) = \frac{1}{\widehat{D}\, L^*\, \sigma_d \sigma_\ell}$$

$$\times\, \boldsymbol{g}\left(\frac{\ln\left(\widehat{D}\right) - \left[\ln(D) + \mu_d\right]}{\sigma_d}, \frac{\ln\left(L^*\right) - \left[\ln(L) + \mu_\ell\right]}{\sigma_\ell}, \rho\right)$$

$$\boldsymbol{g}(x, y, \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}}\ \exp\left(-\frac{1}{2}\,\frac{x^2 - 2\rho x y + y^2}{1-\rho^2}\right)$$

• $f^*(D^*, L^*; D, L)$ for Basic Set of Parameter Values and $\{D = 0.66, L = 0.66\}$:



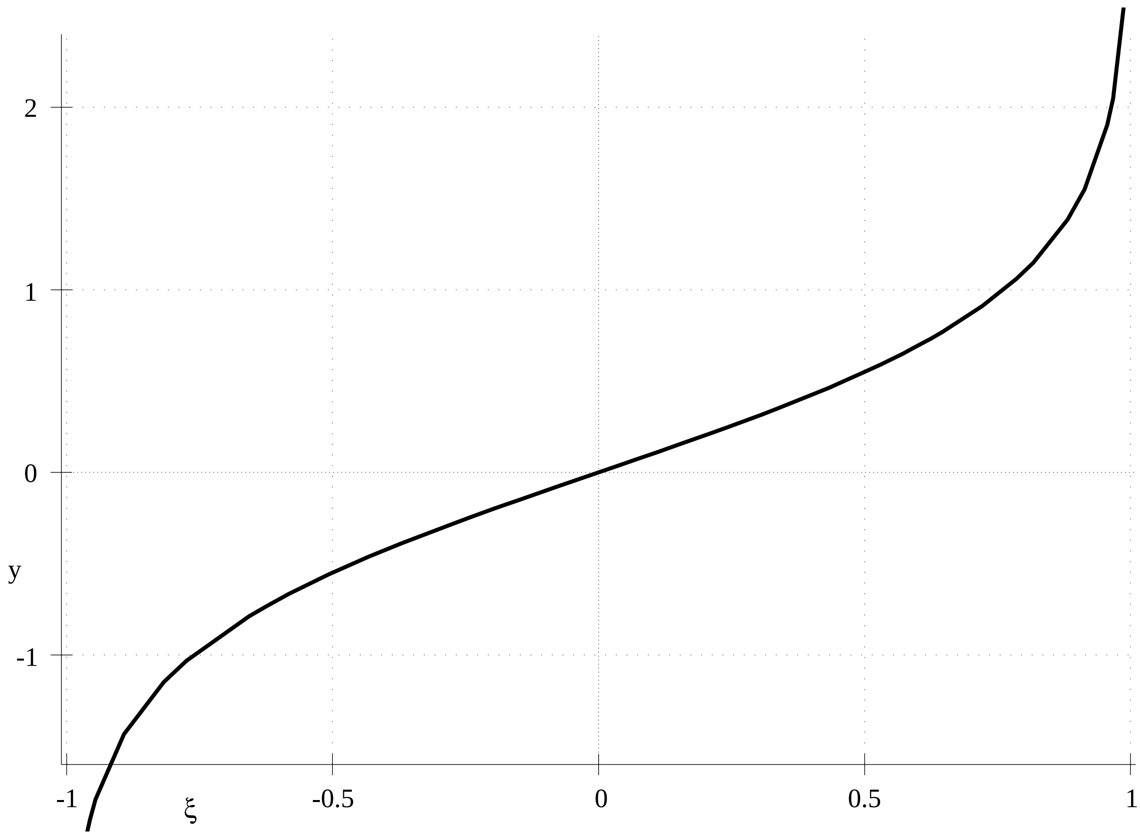
• ***Proposed Cost function***

$$\varphi(\mathbf{x}) = \delta + \kappa \operatorname{arctanh}(\xi), \quad \xi \equiv \theta_1 \|\mathbf{x}\| + \theta_2$$

$$\mathbf{x} = (x_1, x_2, \ldots, x_n), \quad 0 \leqq a \leqq x_i \leqq b < \infty$$

$$\|\mathbf{x}\| = \left[\beta_1 x_1^m + \beta_2 x_2^m + \ldots + \beta_n x_n^m\right]^{1/m}$$

$$0 < \beta_i \ (\forall i); \quad 0 < m$$



• ***Marginal costs***

$$\frac{\partial \varphi(\mathbf{x})}{\partial x_i} = \frac{\kappa\, \theta_1\, \beta_i\, x_i^{m-1}\, \|\mathbf{x}\|^{1-m}}{1 - \left[\theta_1 \|\mathbf{x}\| + \theta_2\right]^2}\ .$$

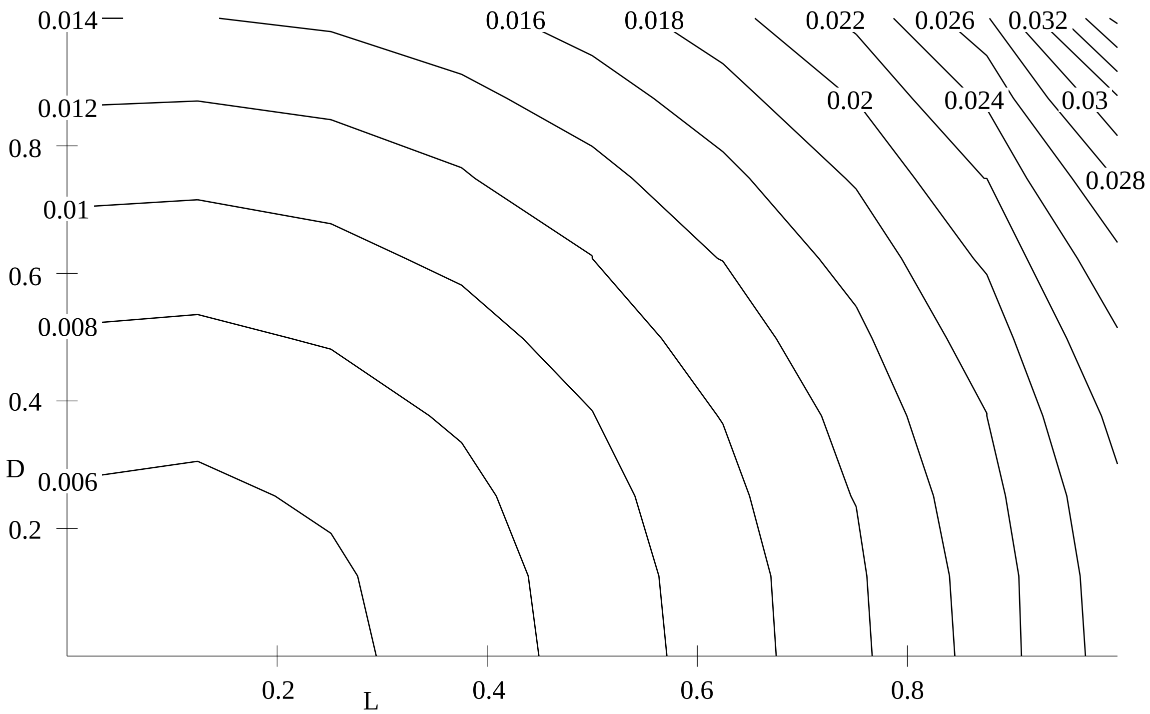
• Cost function for ***bank considered*** underlies ***decreasing economies of scale*** over the whole range of outputs (L, D, C):

$$\varphi(L,\ D,\ C) =$$
$$\delta + \kappa \operatorname{arctanh}\left(\theta_1\left[\beta_1 L^m + \beta_2 D^m + \beta_3 C^m\right]^{1/m} + \theta_2\right)$$

and

$$\theta_1 = \frac{1-\varepsilon}{\left[\beta_1 + \beta_2 + \beta_3\right]^{1/m}}, \quad \theta_2 = 0\ , \quad \delta = 0\ .$$

• Observe that δ has been obtained from the assumption that $\varphi(0, 0, 0) = 0$.



• ***HARA utility function*** (MERTON [1971])

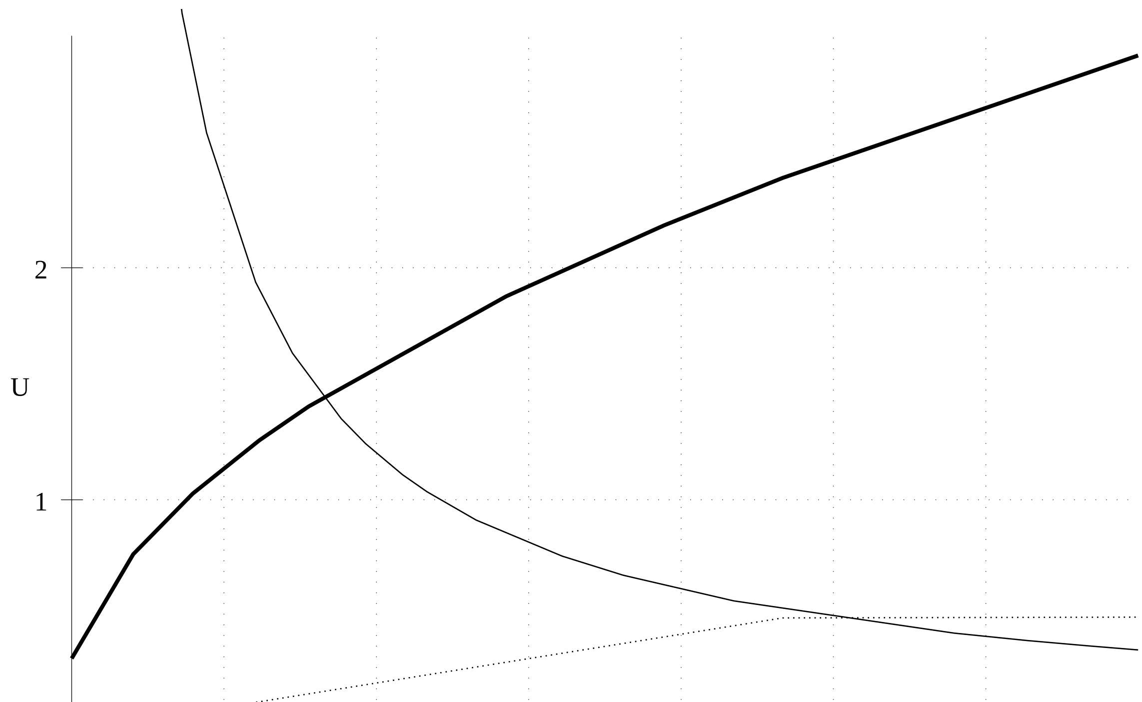
$$\mathcal{U}(A) = \frac{1-\gamma}{\gamma}\left[\frac{\lambda A}{1-\gamma} + \eta\right]^{\gamma}$$

with $\gamma \neq 1,\ \lambda > 0,\ \dfrac{\lambda A}{1-\gamma} + \eta > 0,\ \eta = 1 \text{ if } \gamma = -\infty$

• ***Assumptions*** (ARROW [1971, chapter 3])

☞ The absolute risk aversion ($-\mathcal{U}''(A) / \mathcal{U}'(A)$) decreases with increasing wealth.

☞ The relative risk aversion ($-\mathcal{U}''(A)\, A / \mathcal{U}'(A)$) increases with increasing wealth.



- ***Basic Set of Parameter Values***

m Degree of norm used for the cost function $\varphi(\cdot, \cdot, \cdot)$. [2].

r_d Rate of interest on sight deposits. [0.04].

r_f Risk-free interest rate. [0.06].

r_ℓ Rate of interest on loans. [0.08].

S Penalty cost for illiquid but solvent bank. [0.16].

α Level of information of depositors. [0.9].

β_i Weights of norm used for the cost function $\varphi(\cdot, \cdot, \cdot)$, $i = 1, \ldots, n$. The weights determine the marginal costs with respect to each of several outputs. [1, 0.5, 0.1].

ε Parameter of the cost function $\varphi(\cdot, \cdot, \cdot)$. [0.01].

γ Parameter of the HARA utility function. [0.5].

λ Parameter of the HARA utility function. [3].

η Parameter of the HARA utility function. [0.1].

κ Stretching parameter of the cost function $\varphi(\cdot,\ \cdot,\ \cdot)$. [0.02].

μ_d Mean of the logarithm of the deposit rate, $\ln(D;^{\wedge} / D)$. Given the geometric Brownian motion $\mathrm{d}D;^{\wedge} / D;^{\wedge} = \mu;^{\wedge}_d\ \mathrm{d}t + \sigma_d\ \mathrm{dz}$ with time t and Wiener process z, no growth implies that $\mu;^{\wedge}_d = 0$ and $\mu_d \equiv \mu;^{\wedge}_d - \sigma\ _{2;d} / 2 = -\ \sigma\ _{2;d} / 2$.

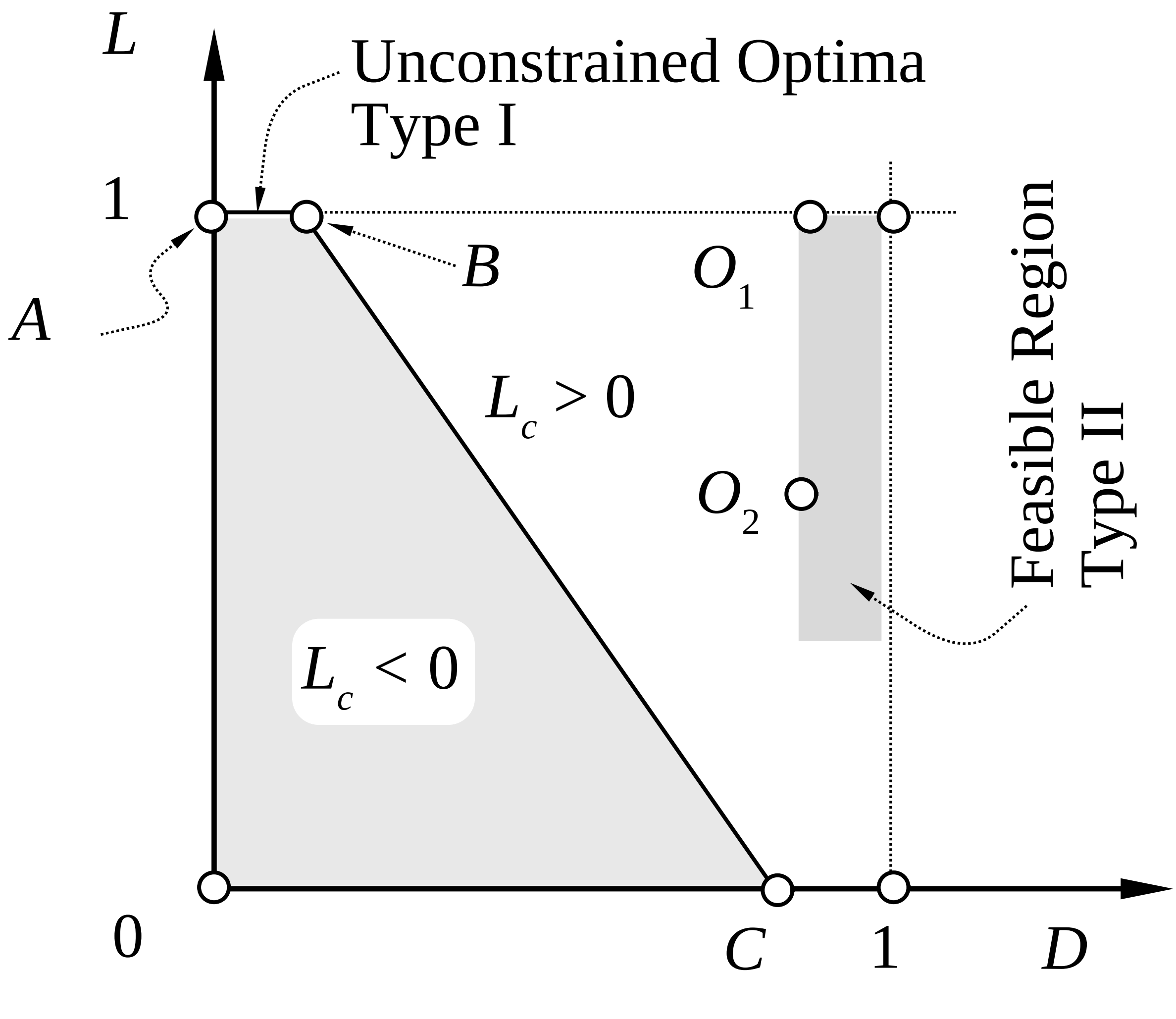
μ_ℓ Mean of the logarithm of the loan rate, $\ln(L^* / L)$. Given the geometric Brownian motion $\mathrm{d}L^* / L^* = \mu;^{\wedge}_\ell\ \mathrm{d}t + \sigma_\ell\ \mathrm{d}\zeta$ with time t and Wiener process ζ, no growth implies that $\mu;^{\wedge}_\ell = 0$ and $\mu_\ell \equiv \mu;^{\wedge}_\ell - \sigma\ _{2;\ell} / 2 = -\ \sigma\ _{2;\ell} / 2$. The two Wiener processes, z and ζ, are correlated with coefficient ρ.

ρ Coefficient of correlation between rates of return on deposits and rates of return on loans. [0].

σ_d Variance of rates of return on deposits. [0.3].

σ_ℓ Variance of rates of return on loans. [0.2].

• ***Two types of optimum***



• ***Unconstrained*** optimum (***Type I***) as considered in the existing literature ***cannot*** explain a low equity-to-deposit ratio.

- A small set of ***constrained*** optima (***Type II***) yield a low equity-to-deposit ratio.

FOUR CONCLUSIONS

1. There are ***two types*** of the constrained optimum.

☞ ***"Weak" competition*** (large interest-rate differentials, low volatility, low risk aversion) implies a ***type-I optimum***.

☞ A type-I optimum is equivalent to the ***unconstrained optimum*** as considered in the exisitng literature.

2. ***Type-I optimum*** ⇒

☞ High equity-to-deposit ratio (0.88 – 1)

☞ Loan-to-cash ratio = 1. (⇐ Cash balances do not bear interest.)

☞ Bank is ***not*** exposed to any risk at all. It cannot become insolvent nor illiquid.

☞ Banks founded in the 19th century started with a high equity-to-deposit ratio. Building up of confidence or reputation, respectively.

3. ***Type-II optimum*** $\Rightarrow$

- ☞ Low equity-to-deposit ratio (≈ 0.04)
- ☞ Loan-to-cash ratio = 1. ($\Leftarrow$ Cash balances do not bear interest.)
- ☞ Bank is now ***exposed*** to the risk of an insolvency or an illiquidity.
- ☞ Nowadays, banks show a low equity-to-deposit ratio. Building up confidence allows them to undergo the exposure of risk.

4. The model considered here is ***able to explain*** the ***stylized facts*** about the history of the banks.

APPENDIX D: NUMERICAL QUADRATURE

- Three standard types of ranges of integration:

$\mathscr{I}_1 \equiv (-\infty, +\infty) \times (-\infty, +\infty)$

$\mathscr{I}_2 \equiv (0, +\infty) \times (0, +\infty)$

$\mathscr{I}_3 \equiv (-1, +1) \times (-1, +1)$

- Reverse strategy as usual

$$\mathfrak{I} := \iint_{x_{n+2},\, y_{n+2} \in \mathscr{I}} f\left(x_{n+2}, y_{n+2}\right) \mathrm{d}y_{n+2}\, \mathrm{d}x_{n+2}$$

$$= \int_{x_0=-\infty}^{x_0=+\infty} \mathrm{d}x_0 \int_{y_0=-\infty}^{y_0=+\infty} f\left(x_0, y_0\right) p_x\, p_y\, \mathrm{d}y_0$$

- Transformation of variables

$$\left\{\begin{matrix} x_j = \sinh(x_{j-1}) \\ y_j = \sinh(y_{j-1}) \\ (j = 1, \ldots, n) \end{matrix}\right\}, \quad \left\{\begin{matrix} x_{n+1} = \cos(\varphi)\, x_n - \sin(\varphi)\, y_n \\ y_{n+1} = \sin(\varphi)\, x_n + \cos(\varphi)\, y_n \end{matrix}\right\},$$

$$\left\{\begin{matrix} x_{n+2} \\ y_{n+2} \end{matrix}\right\} = \begin{cases} \left\{\begin{matrix} x_{n+1} \\ y_{n+1} \end{matrix}\right\} & \text{if } \mathscr{I} = \mathscr{I}_1, \\ \left\{\begin{matrix} \exp\left(x_{n+1}\right) \\ \exp\left(y_{n+1}\right) \end{matrix}\right\} & \text{if } \mathscr{I} = \mathscr{I}_2, \\ \left\{\begin{matrix} \tanh\left(x_{n+1}\right) \\ \tanh\left(y_{n+1}\right) \end{matrix}\right\} & \text{if } \mathscr{I} = \mathscr{I}_3. \end{cases}$$

• Absolute value of Wronsky determinant: $p_x\, p_y$.

$$p_x = \begin{cases} \prod_{j=0}^{n-1} \cosh(x_j) & \text{if } \mathscr{I} = \mathscr{I}_1, \\ e^{x_{n+1}} \prod_{j=0}^{n-1} \cosh(x_j) & \text{if } \mathscr{I} = \mathscr{I}_2, \\ \dfrac{1}{\cosh^2(x_{n+1})} \prod_{j=0}^{n-1} \cosh(x_j) & \text{if } \mathscr{I} = \mathscr{I}_3. \end{cases}$$

• Trapezoidal rule (ε is machine tolerance)

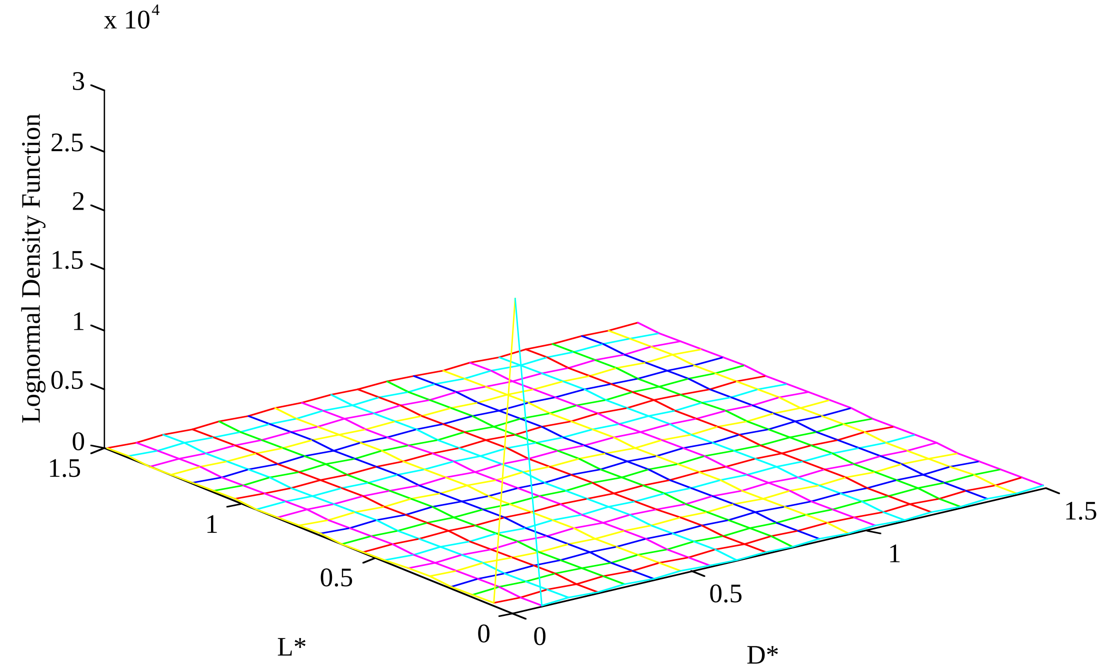
$$T(h) = h_x\, h_y\, S(h),$$

$$S(h) = \sum_{x_i = c_x \pm i\, h_x}^{|g(\cdot)| < \varepsilon} \; \sum_{y_j = c_y \pm j\, h_y}^{|g(\cdot)| < \varepsilon} g(x_i, y_j)\,,$$

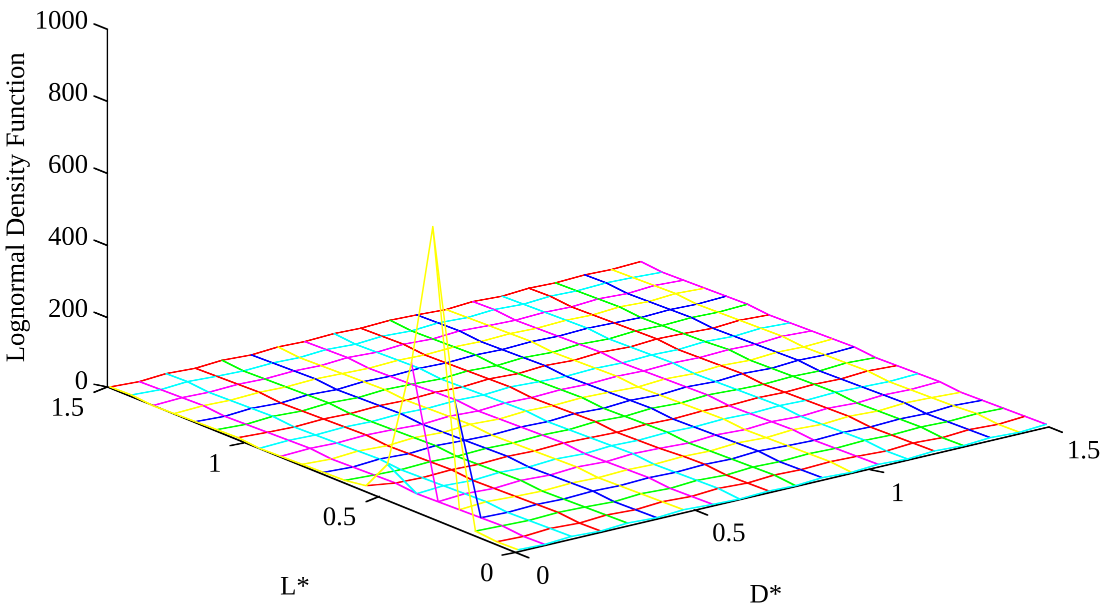
$$g(\cdot) = f(\cdot)\, p_x(\cdot)\, p_y(\cdot)\,.$$

• Repeated reduction of the step sizes is stopped when $|\, T(h) - T(h/2) \,| < \sqrt{\varepsilon}$. Then, $T(h/2)$ has accuracy ε. The ***convergence*** of the trapezoidal value to the integral is given by $T(h) - \mathfrak{I} = \mathbb{O}(e^{-\gamma/h})$ with γ a positive constant, given an analytic function $g(\cdot)$.

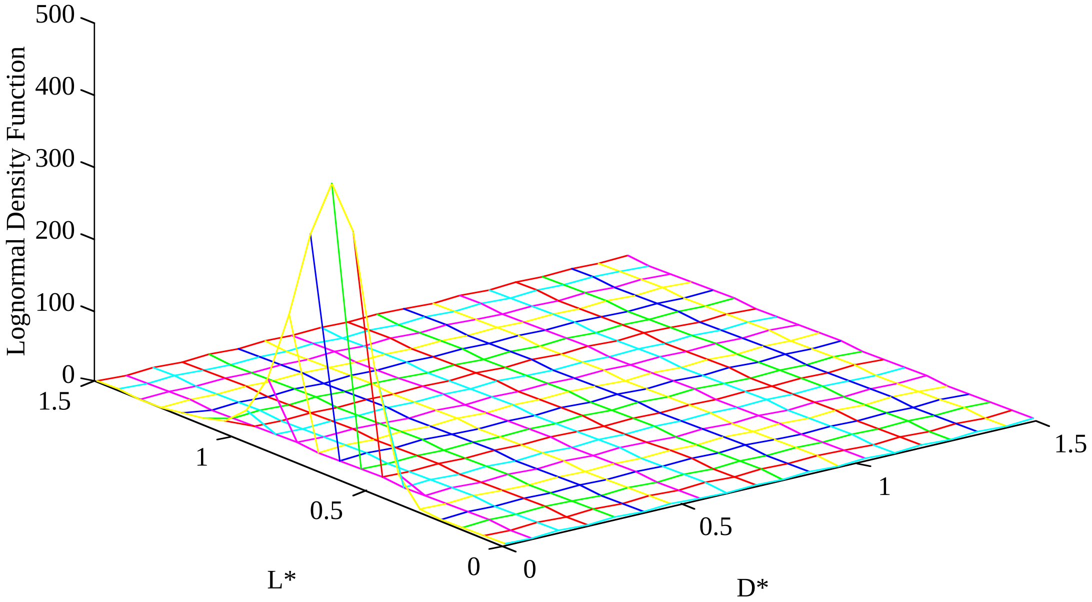
APPENDIX E: JOINT PROBABILITY DENSITY FUNCTION $f^*(D^*, L^*; D, L)$ FOR BASIC SET AND …



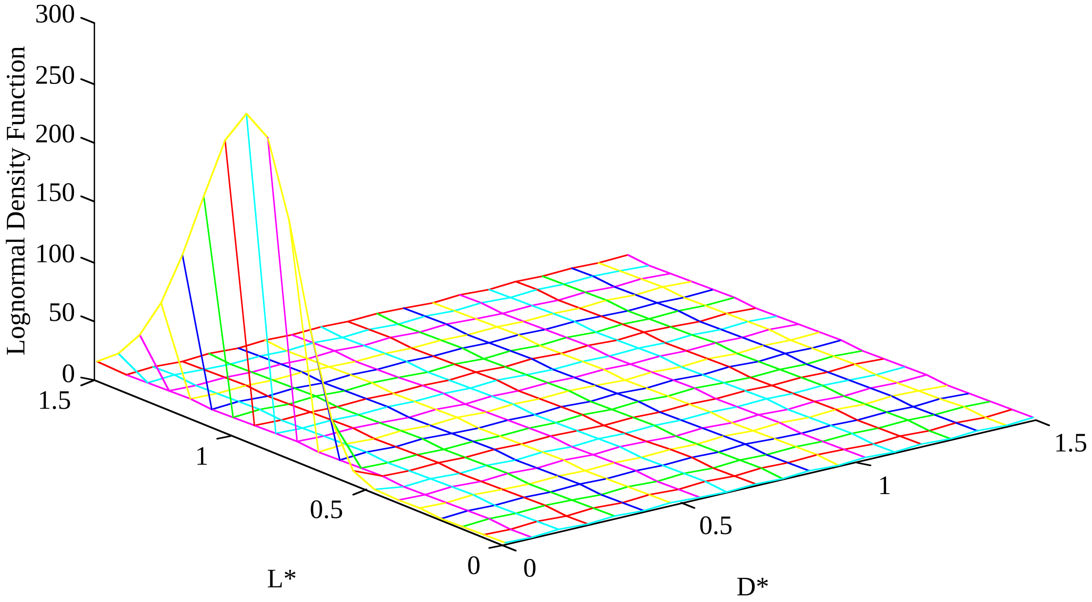
… D = 0.01 and L = 0.01.



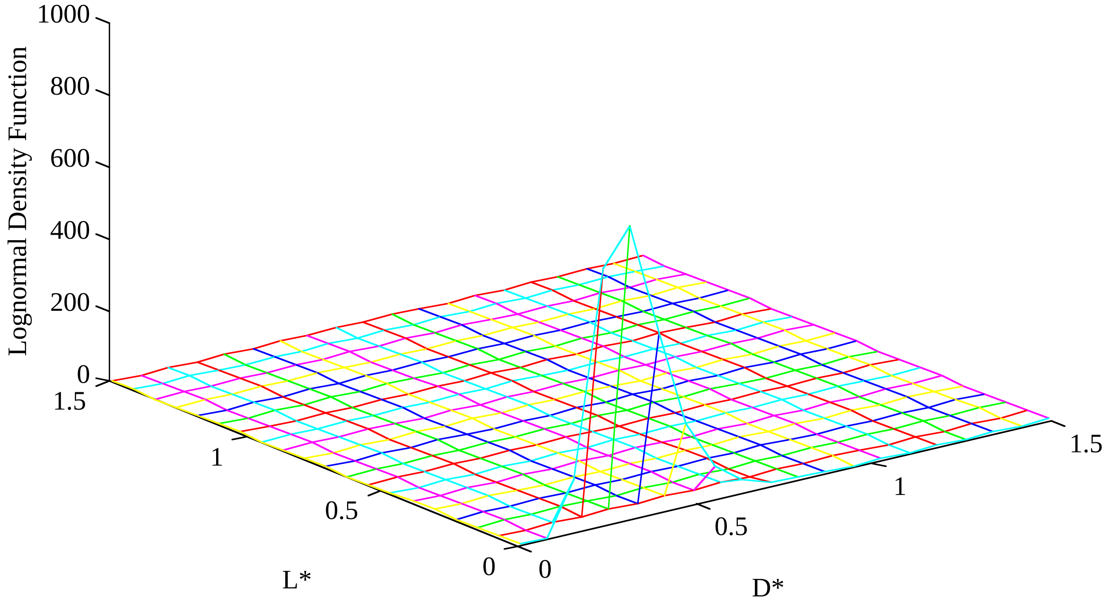
… D = 0.01 and L = 0.33.



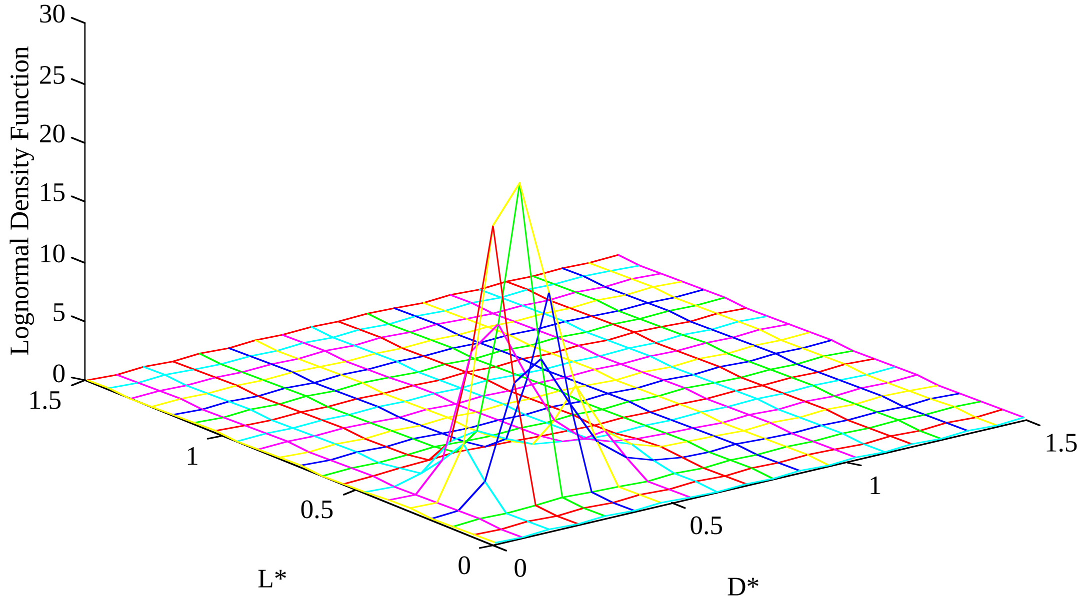
… *D* = 0.01 and *L* = 0.66.



… *D* = 0.01 and *L* = 0.99.



… *D* = 0.33 and *L* = 0.01.



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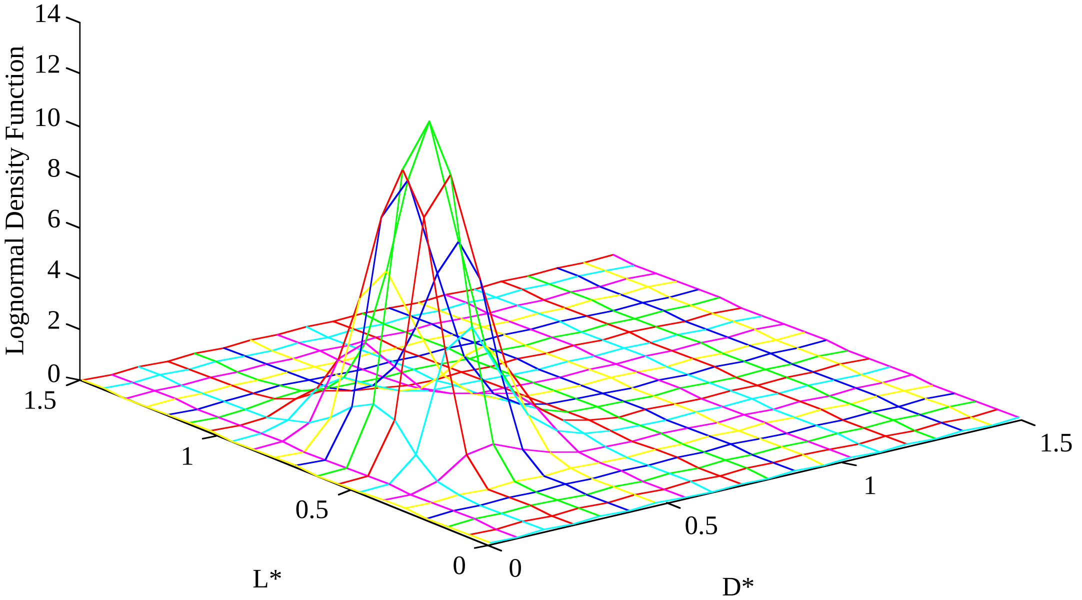

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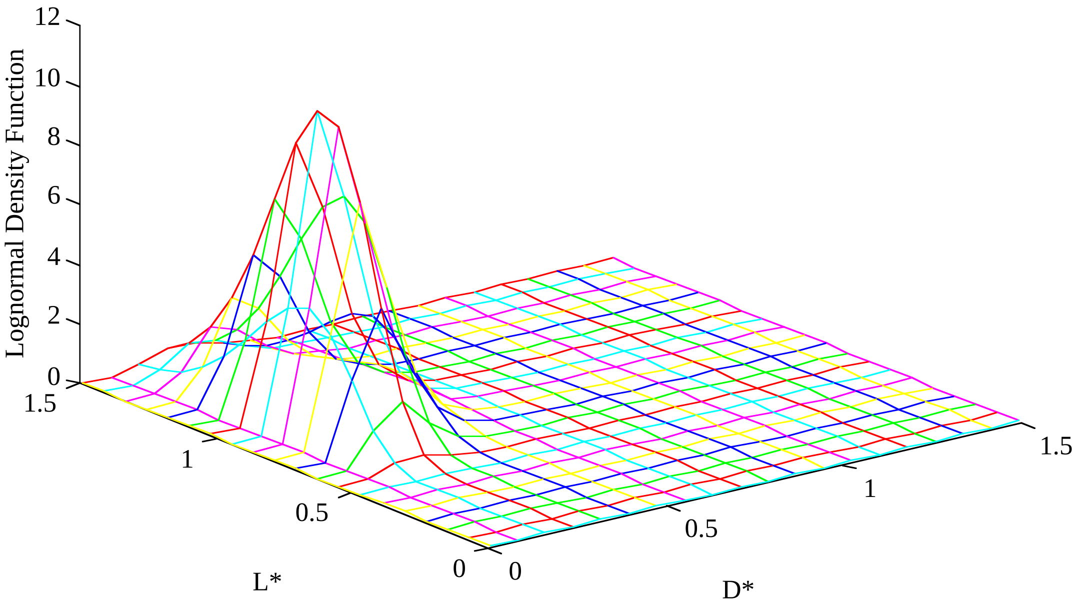

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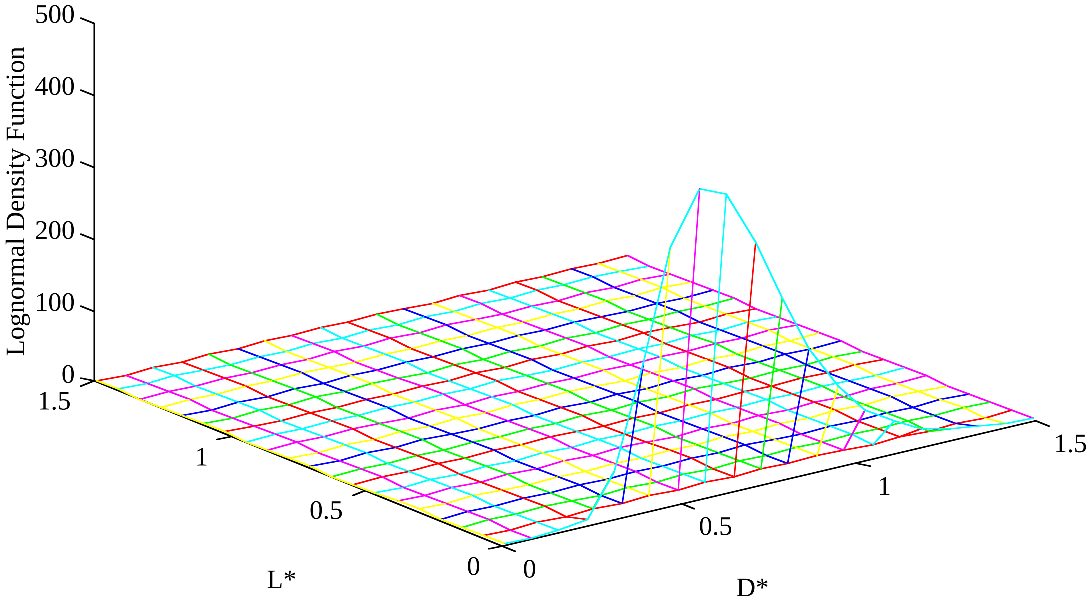
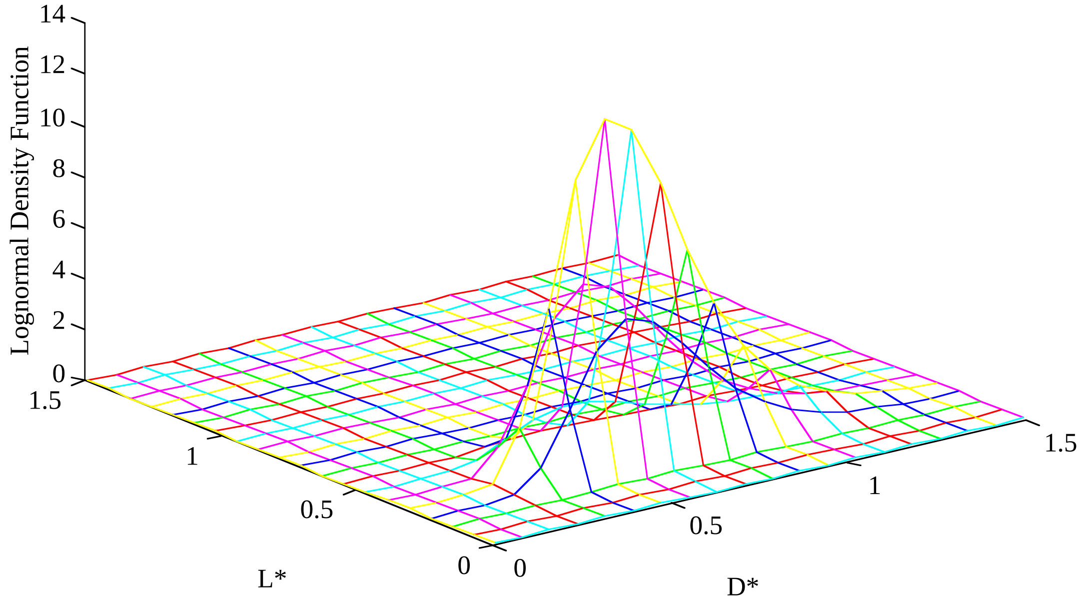


… *D* = 0.66 and *L* = 0.01.



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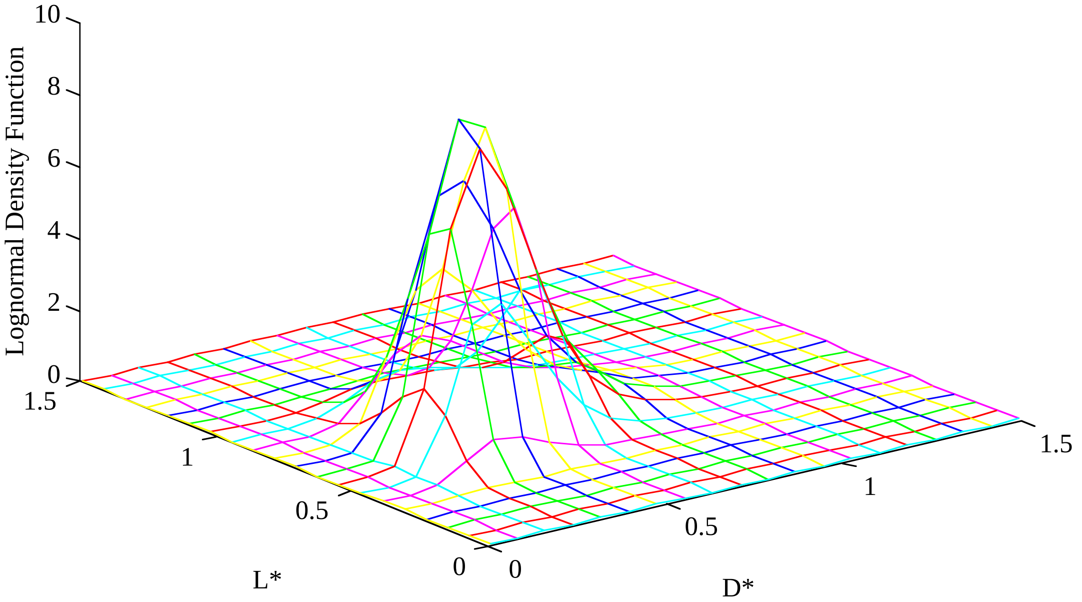

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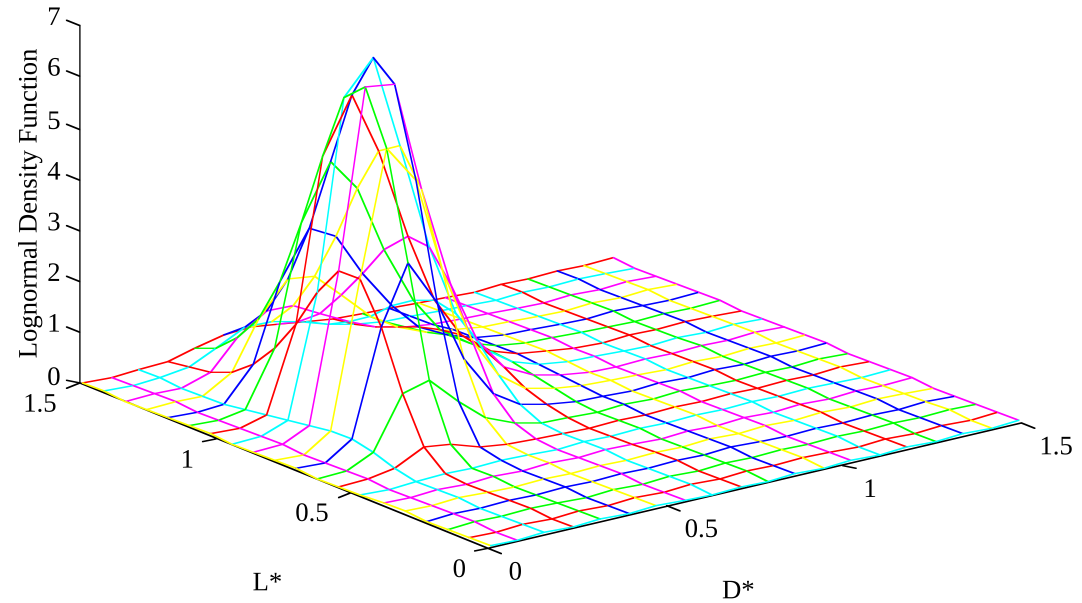

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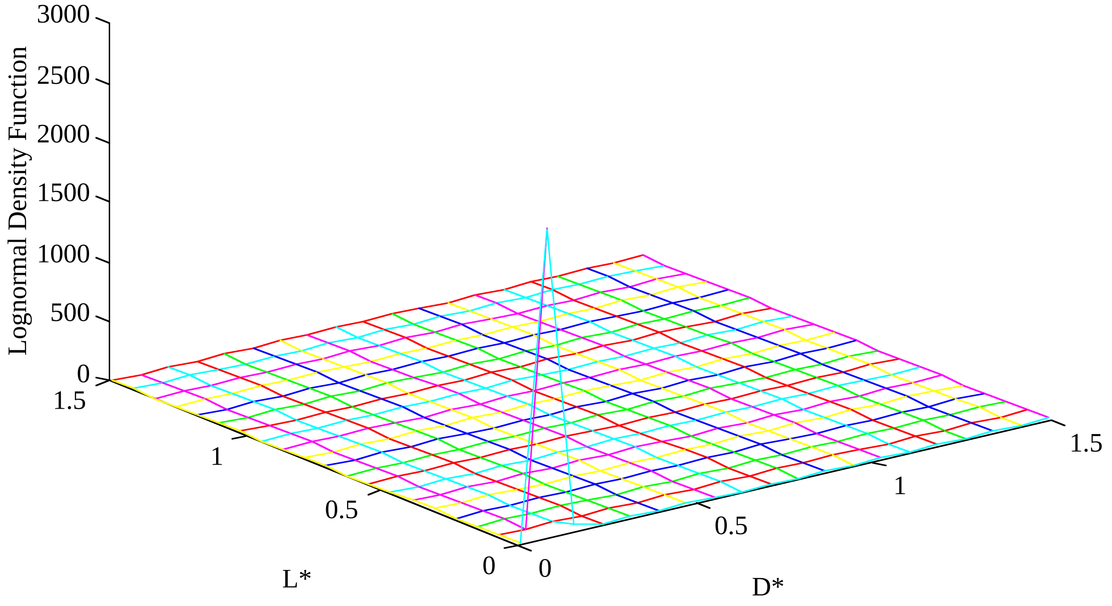
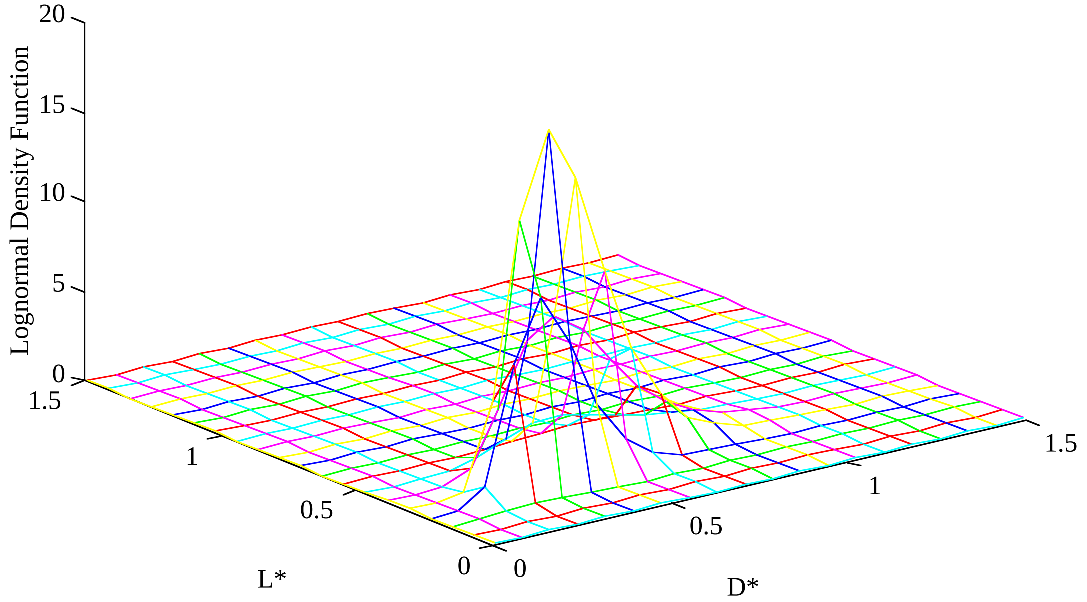


… D = 0.99 and L = 0.01.



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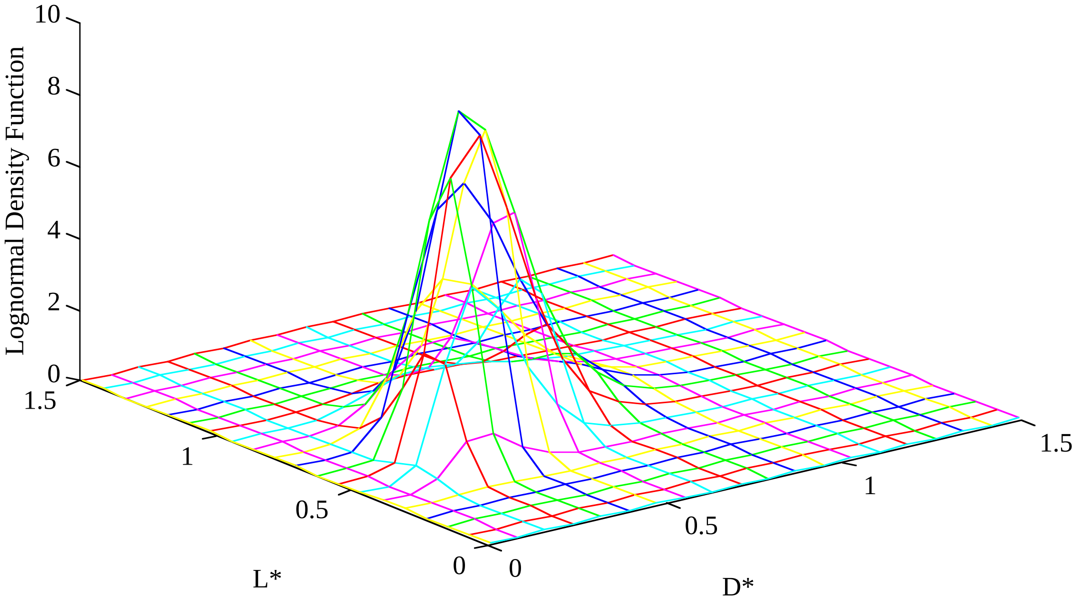

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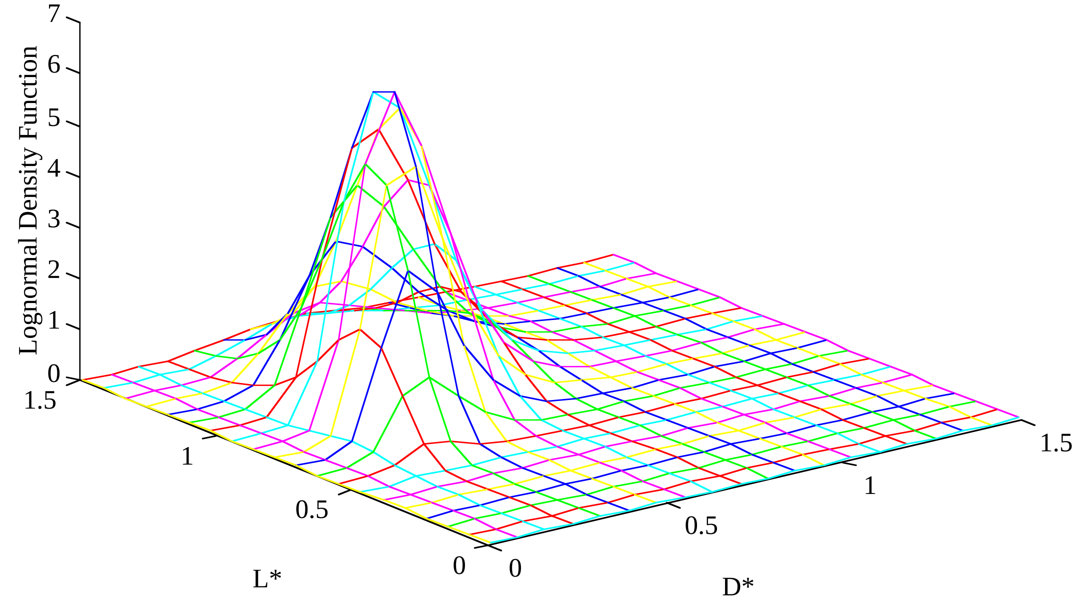

… *D* = 0.99 and *L* = 0.99.