The Term Structure of Interest Rates
and
the Classification of Corporate Bonds

by

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1 Purpose

- Calculate the **zero-coupon rates** (zero rates for short) or **spot interest rates**, respectively, from a set of quoted prices of coupon-bearing bonds.
- Draw the zero rates against the term to maturity in a graph → **Term structure of interest rates** or **yield curve**, respectively, see Fig. 1-1.

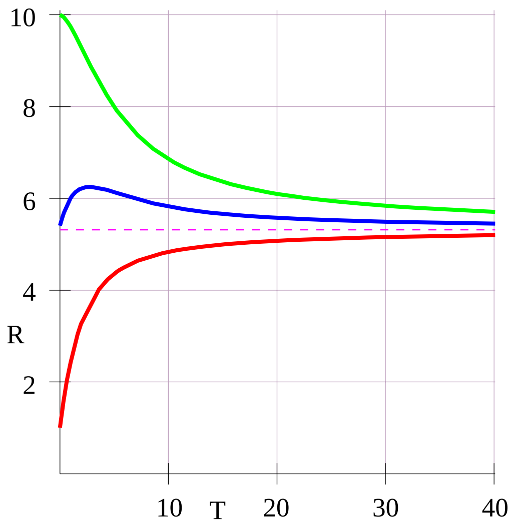

Figure 1-1: Term structure of interest rates

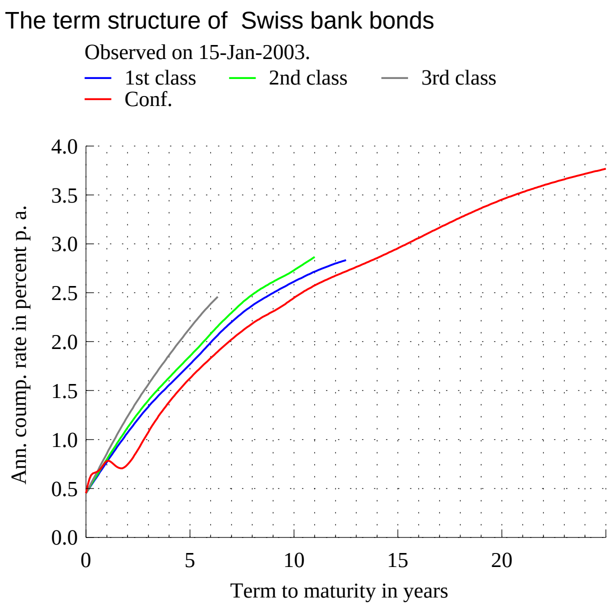

Figure 1-2

- The yield curve is relevant for macro-econometric models, discounting of future cash flows (capital investments, par yield) as well as the pricing of futures, swaps and options written on securities (either variable-income securities or fixed-income securities).
- We use the **"forward-rate" method** to estimate the yield curve. Its focus lies on the instantaneous forward interest rate rather than the spot interest rate.
- The "forward-rate" method uses tolerances given by the bid prices and ask prices of the coupon-bearing bonds considered.
- The bid-ask spread determines ranges of possible yield curves → **Classification of corporate bonds**, see Fig. 1-2.

2 Basic Relationships

- All the interest rates are meant to be continuously compounded ("c. c." for short).

Spot interest rates

- τ Term to maturity of a bond measured in years.
- $P(\tau)$ $\equiv P(t, t+\tau)$. Spot price of a zero-coupon bond or a discount bond, respectively, with a face value of 1 monetary unit ("MU"). The price of the bond is fixed at date t. The bond is delivered immediatly and matures at date $(t+\tau)$.
- $R(\tau)$ $\equiv R(t, t+\tau)$. Zero rate or spot interest rate, respectively, per annum (p. a.) corresponding with the zero-coupon bond $P(\tau)$. Physical dimension: [1/year].

$$P(\tau) \equiv \exp\left(-R(\tau)\,\tau\right) \cdot 1\ \text{MU} \qquad (2\text{-}1)$$

- Example 2-1. Suppose that the c. c. spot interest rate is 10% p. a. and the term to maturity of the corresponding discount bond is 7.5 years, then the price of the disount bond denominated in EUR becomes

$$P(\tau) \equiv \exp\left(-0.1 \cdot 7.5\right) \cdot 1\ \text{EUR} = 0.4724\ \text{EUR}$$

- Solve eq. (2-1) for the zero rate to get the term structure of interest rates:

$$R(\tau) = -\frac{\ln\left(\frac{P(\tau)}{1\ \text{MU}}\right)}{\tau} \qquad (2\text{-}2)$$

- Example 2-2. Suppose that the prices of three discount bonds denominated in EUR are 0.89, 0.7 and 0.5 with corresponding terms to maturity of 3.2, 6.5 and 10 years, then the corresponding c. c. spot interest rates become (see Fig. 2-1):

$$R(3.2) = -\frac{\ln\left(\frac{0.89}{1\ \text{EUR}}\right)}{3.2} = 0.0364 \quad (3.64\%)$$

$$R(6.5) = -\frac{\ln\left(\frac{0.7}{1\ \text{EUR}}\right)}{6.5} = 0.0549 \quad (5.49\%)$$

$$R(10) = -\frac{\ln\left(\frac{0.5}{1\ \text{EUR}}\right)}{10} = 0.0693 \quad (6.93\%)$$

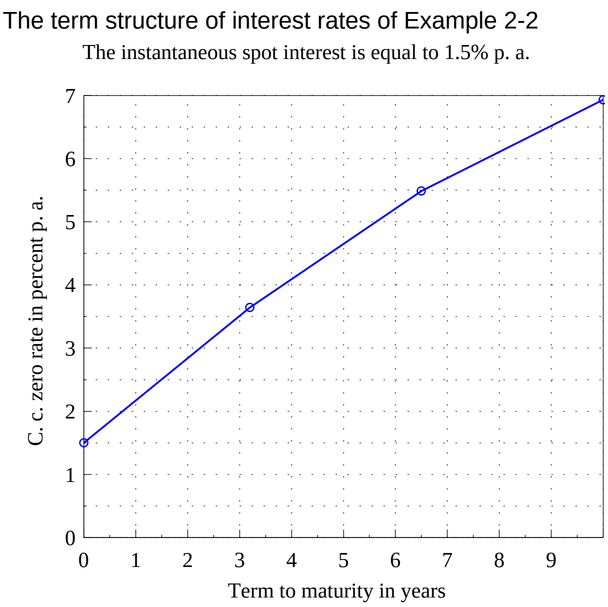

Figure 2-1

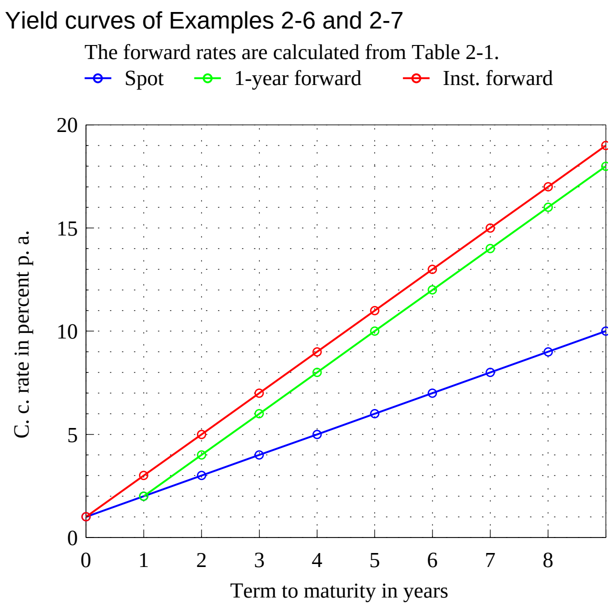

Figure 2-2

- In the sequel, we omit the monetary unit.
- r $\equiv R(0)$. Instantaneous spot interest rate ($\approx$ overnight interest rate).

$$r \equiv R(0) = \lim_{\tau \to 0} \left\{ -\frac{\ln\left(P(\tau)\right)}{\tau} \right\} \tag{2-3}$$

Forward interest rates

- $\mathcal{P}(\tau, \omega)$ $\equiv \mathcal{P}(t, t+\tau, t+\tau+\omega)$. The τ-period forward price of a ω-period zero-coupon bond, fixed at date t and paid upon delivery date $(t+\tau)$. The discount bond delivered matures at the later date $(t+\tau+\omega)$.
- $F(\tau, \omega)$ $\equiv F(t, t+\tau, t+\tau+\omega)$. The ω-period forward interest rate applicable in τ periods from now corresponding with the forward contract as given by $\mathcal{P}(t, t+\tau, t+\tau+\omega)$.
- Relationship between forward price and forward rate:

$$\mathcal{P}(\tau, \omega) \equiv \exp\left(-F(\tau, \omega)\,\omega\right) \tag{2-4}$$

- Example 2-3. Suppose that the c. c. 7-year forward interest rate applicable in 3 years from now is 10% p. a., then the 3-year forward price of a 7-year zero-coupon bond denominated in EUR becomes

$$\mathcal{P}(3, 7) \equiv \exp\left(-0.1 \cdot 7\right) = 0.4966$$

- Solve eq. (2-4) for the forward rate to get the term structure of forward rates:

$$F(\tau,\,\omega) = -\frac{\ln\left(\mathcal{P}(\tau,\,\omega)\right)}{\omega} \tag{2-5}$$

- Example 2-4. Suppose that the 4-year forward price of a 5-year discount bond denominated in EUR is 0.6, then the c. c. 5-year forward interest rate applicable in 4 years becomes

$$F(4,\,5) = -\frac{\ln(0.6)}{5} = 0.1022 \quad (10.22\%)$$

- No arbitrage: The price of a discount bond fixed at date t with maturity date $(t + \tau + \omega)$ should be equal to the price of a portfolio at date t, which consists of a discount bond maturing at date $(t + \tau)$ plus a ω-period discount bond with τ periods forward.

$$P(\tau + \omega) = P(\tau)\,\mathcal{P}(\tau,\,\omega) \quad \text{or} \quad \mathcal{P}(\tau,\,\omega) = \frac{P(\tau + \omega)}{P(\tau)} \tag{2-6}$$

- Example 2-5. Suppose that the spot price a 5-year discount bond denominated in EUR is 0.7 and that the spot price of a 4-year discount bond denominated in EUR is 0.8, then the 4-year forward price of a one-year discount bond becomes

$$\mathcal{P}(4,\,1) = \frac{0.7}{0.8} = 0.8750$$

- Insert eqs. (2-1) and (2-4) into the above equation to get the ω-period forward interest rate with τ years forward.

$$\begin{aligned} F(\tau,\,\omega) &= \frac{R(\tau + \omega)\,[\tau + \omega] - R(\tau)\,\tau}{\omega} \\ &= R(\tau + \omega) + \frac{\tau}{\omega}\left[R(\tau + \omega) - R(\tau)\right] \end{aligned} \tag{2-7}$$

- Example 2-6. Suppose that we observe the following spot interest rates as given in the second column of Table 2-1 and wish to calculate the one-year forward interest rate for various forward periods.

Table 2-1

τ	$R(\tau)$	$F(\cdot,\,1)$	$f(\tau)$
0.00	0.01	–	0.01
1.00	0.02	$F(0,\,1) = 0.02$	0.03
2.00	0.03	$F(1,\,1) = 0.04$	0.05
3.00	0.04	$F(2,\,1) = 0.06$	0.07
4.00	0.05	$F(3,\,1) = 0.08$	0.09
5.00	0.06	$F(4,\,1) = 0.10$	0.11
6.00	0.07	$F(5,\,1) = 0.12$	0.13
7.00	0.08	$F(6,\,1) = 0.14$	0.15
8.00	0.09	$F(7,\,1) = 0.16$	0.17
9.00	0.10	$F(8,\,1) = 0.18$	0.19

Example 2-6 continued. Consider the one-year forward interest which applies to the period between year 8 and year 9, that is, $F(8, 1)$:

$$F(8, 1) = \frac{0.1 \cdot 9 - 0.09 \cdot 8}{1} = 0.18 \quad (18\%)$$

Example 2-6 continued. Repeat for the other forward periods to obtain the third column in Table 2-1 and Fig. 2-2 above.

- $f(\tau) \equiv F(\tau, 0)$. Instantaneous forward interest rate.
- From eq. (2-7) as $\omega \to 0$, we get

$$f(\tau) = \lim_{\omega \to 0} F(\tau, \omega) = R(\tau) + \frac{\partial R(\tau)}{\partial \tau}\, \tau \qquad (2\text{-}8)$$

- As the forward period vanishes, the instantaneous forward interest rate becomes a spot interest rate from eq. (2-8).

$$f(0) = r \; as \; \tau \to 0, \quad \text{for } \frac{\partial R(\tau)}{\partial \tau} \text{ finite.} \qquad (2\text{-}9)$$

- Example 2-7. Consider the spot interest rates as given in the second column of Table 2-1 and approximate the derivative in eq. (2-8) by a first difference to obtain the fourth column in Table 2-1. For instance, for $\tau = 1$, we calculate

$$f(1) = R(1) + \frac{R(2) - R(0)}{2}\, 1 = 0.02 + \frac{0.03 - 0.01}{2} = 0.03$$

Example 2-7 continued. Repeat for the other forward periods to obtain the fourth column in Table 2-1 and Fig. 2-2 above.

Relationship between spot interest rate and instantaneous forward interest rate

- Integrate by parts eq. (2.8) to get eq. (2-10).

$$R(\tau) = \frac{1}{\tau} \int_{s=0}^{s=\tau} f(s)\, \mathrm{d}s \qquad (2\text{-}10)$$

- Exercise. Recover the spot interest rates in Table 2-1 from numerical integration of the instantaneous forward interest rate as given in Table 2-1, e. g., by Simpson's rule.
- The "forward-rate" method: estimation errors in the instantaneous forward interest rate are averaged for the spot interest rate.

Relationship between ω-period forward rate and instantaneous forward rate

- Insert eq. (2.10) into eq. (2-7) to get eq. (2-11).

$$F(\tau,\,\omega) = \frac{1}{\omega}\int_{s=\tau}^{s=\tau+\omega} f(s)\,\mathrm{d}s \qquad (2\text{-}11)$$

- Exercise. Recover the one-year forward interest rates in Table 2-1 from numerical integration of the instantaneous forward interest rate as given in Table 2-1, e. g., by Simpson's rule.

Consistency of τ-period forward price of a ω-period zero-coupon bond

- Insert eq. (2-11) into eq. (2-4) to get eq. (2-12) which is the no-arbitrage equation (2-6).

$$\begin{aligned}\mathcal{P}(\tau,\,\omega) &\stackrel{\text{def}}{=} \exp\left(-F(\tau,\,\omega)\,\omega\right)\\ &= \exp\left(-\int_{s=\tau}^{s=\tau+\omega} f(s)\,\mathrm{d}s\right)\\ &= \exp\left(-\int_{s=0}^{s=\tau+\omega} f(s)\,\mathrm{d}s + \int_{s=0}^{s=\tau} f(s)\,\mathrm{d}s\right)\\ &= \frac{P(\tau+\omega)}{P(\tau)}\end{aligned} \qquad (2\text{-}12)$$

Relationship between c. c. and discretly compounded interest rates

- m Compounding frequency per year. For c. c., $m \to \infty$.
- $R_m(\tau)$ $\equiv R_m(t,\,t+\tau)$. Spot interest rate with compounding frequency per year, m.

$$R_m(\tau) = m\left[\exp\left(\frac{R(\tau)}{m}\right) - 1\right], \quad R(\tau) = m\,\ln\left(1 + \frac{R_m(\tau)}{m}\right) \qquad (2\text{-}13)$$

- Example 2-8. Suppose that you wish to transform a quarterly compounded spot interest rate which is 10% p. a. into a c. c. interest rate p. a. Then with $m = 4$, we get

$$R(\tau) = 4\,\ln\left(1 + \frac{0.1}{4}\right) = 0.0988 \quad (9.88\%)$$

3 The "Forward-rate" Method of the Term Structure of Interest Rates

- The estimated term structure of interest rates should have two properties at least.
  1. It should be model-independent.
  2. The estimated prices should lie within the observed corresponding bis-ask spreads.
- Nelson-Siegel [1987] and Svensson [1995] violate both conditions.
- The "forward-rate" method fulfils both conditions.
- All the prices of coupon bearing bonds are meant to be *cash* prices. The cash price is equal to the quoted price plus the interest which has accrued since the last coupon payment date.
- $B(\tau)$ $\equiv B(t, t+\tau)$. The theoretical price of a coupon-bearing bond.
- c_k The kth coupon, $k = 1, \ldots, K$. The last coupon, c_K, includes the redemption value of the coupon bearing bond.
- τ_k The kth cash flow period of a particular coupon-bearing bond, $k = 1, \ldots, K$.

$$B(\tau) = \sum_{k=1}^{K} c_k\, P(\tau_k) = \sum_{k=1}^{K} c_k \exp\left(-\int_0^{\tau_k} f(t)\, \mathrm{d}t\right) \tag{3-1}$$

- $B^{\text{obs}}(\tau_m)$ The observed price of the mth coupon-bearing bond with its corresponding term to maturity τ_m, $m = 1, \ldots, M$.
- $B^{\text{bid}}(\tau_m)$ The observed bid price of the mth coupon-bearing bond with its corresponding term to maturity τ_m, $m = 1, \ldots, M$.
- $B^{\text{ask}}(\tau_m)$ The observed ask price of the mth coupon-bearing bond with its corresponding term to maturity τ_m, $m = 1, \ldots, M$.
- ϵ_m $\equiv [B^{\text{ask}}(\tau_m) - B^{\text{bid}}(\tau_m)]\,/\,[B^{\text{ask}}(\tau_m) + B^{\text{bid}}(\tau_m)]$. The tolerance of the mth bond.
- T $\equiv \max(\tau_m)$, $m = 1, \ldots, M$. The maximum term to maturity of all bonds in the sample considered.
- Constrained optimization of the "forward-rate" method.

$$\min_{\{f(t)\}} G(T) = \int_0^T \left(\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right)^2 \mathrm{d}t\,, \quad T = \max(\tau_1, \ldots, \tau_M)\,, \quad \text{subject to}$$
$$\epsilon_m \geqq \left|\frac{B(\tau_m)}{B^{\text{obs}}(\tau_m)} - 1\right|\,, \quad \text{for } m = 1, 2, \ldots, M, \tag{3-2}$$
$$f(t) \geqq 0, \quad \text{for every } t.$$

- Macrostructure data: $B^{\text{bid}} < B^{\text{obs}} < B^{\text{ask}}$.
- Microstructure data: $B^{\text{bid}} \leqq B^{\text{obs}} \leqq B^{\text{ask}}$.

Discrete-time version

- Büttler [1999, 2000]: Constrained optimization in eq. (3-2). Delbaen and Lorimier [1992]: Multi-objective goal attainment problem.
- N — The number of time increments for the maximum term to maturity, T.
- Δt — $\equiv T / N$. The time increment for the maximum term to maturity, T.
- f_i — $\equiv f(i \cdot \Delta t)$, $i = 0, 1, 2, \ldots, N$. The discrete values of the instananeous forward interest rate. Recall $f_0 = r$ which is observed.
- The discrete objective function:

$$\min_{\{f_1, \ldots, f_N\}} G(N) = \sum_{i=1}^{N} [f_i - f_{i-1}]^2 \qquad (3\text{-}3)$$

- In principal, the integral of the instantaneous forward interest rate is approximated by a simple sum:

$$\int_0^{\tau_k} f(t)\, dt \approx \sum_{i=0}^{N_k} f_i \,\Delta t \qquad (3\text{-}4)$$

- Example 3-1: Fig. 3-1 - 3-4. Büttler [1999]: Variety of yield curves (normal, inverse, humped).

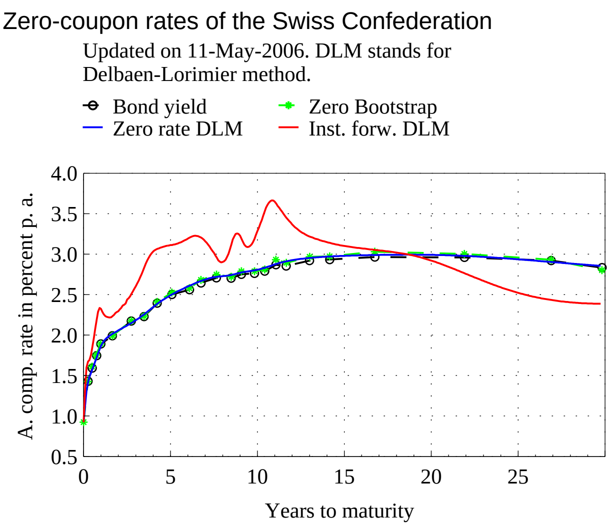

Figure 3-1

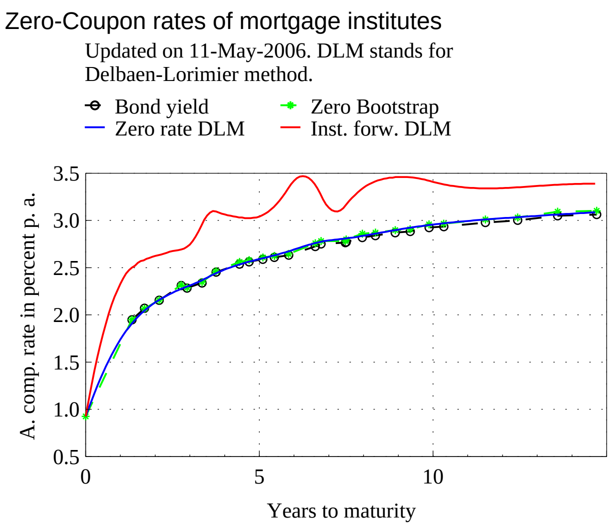

Figure 3-2

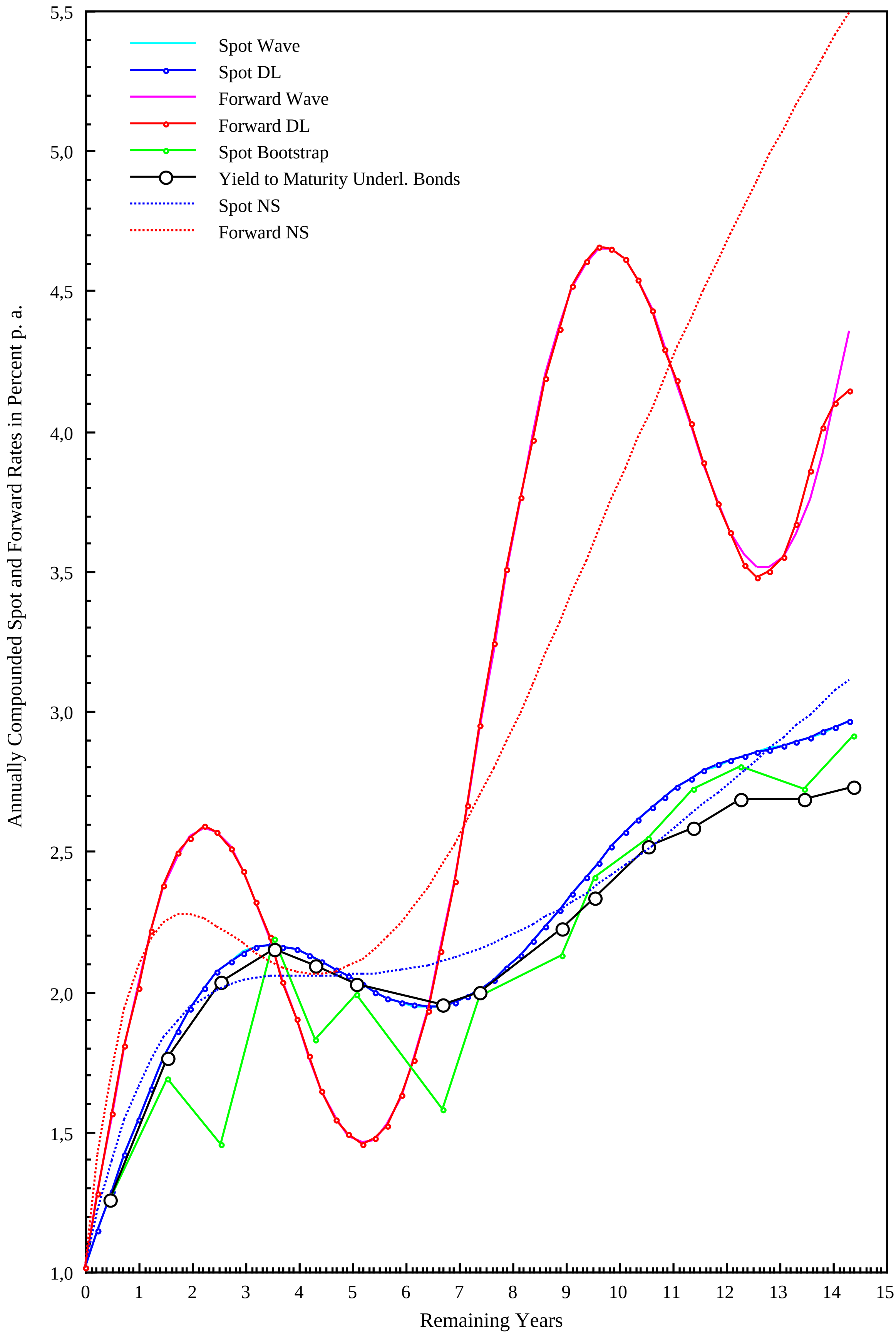
5,5
5,0
4,5
4,0
3,5
3,0
2,5
2,0
1,5
1,0
Spot Wave
Spot DL
Forward Wave
Forward DL
Spot Bootstrap
Yield to Maturity Underl. Bonds
Spot NS
Forward NS
Annually Compounded Spot and Forward Rates in Percent p. a.
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
Remaining Years

Figure 3-3

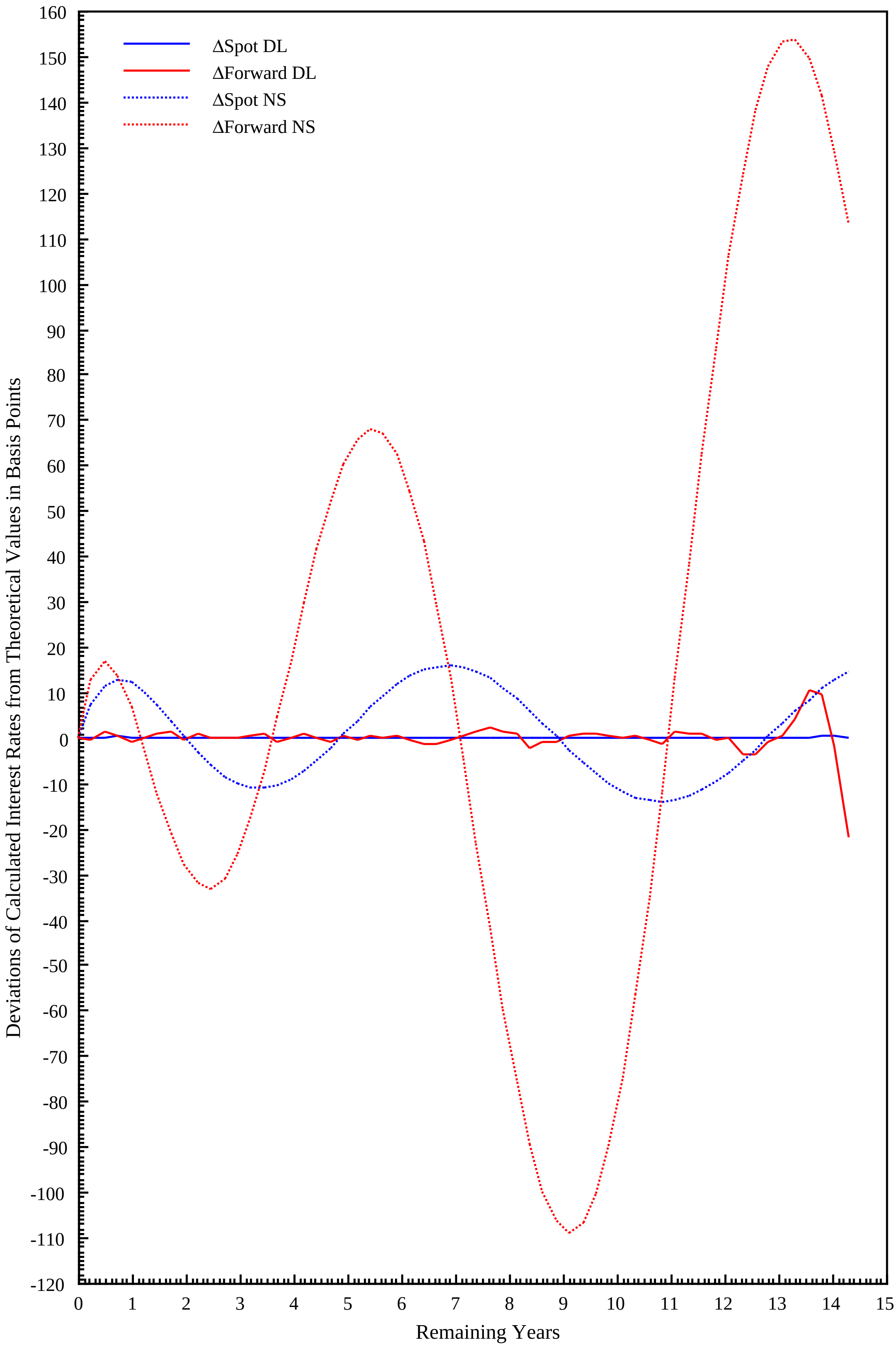
ΔSpot DL
ΔForward DL
ΔSpot NS
ΔForward NS
Deviations of Calculated Interest Rates from Theoretical Values in Basis Points
Remaining Years
160
150
140
130
120
110
100
90
80
70
60
50
40
30
20
10
0
-10
-20
-30
-40
-50
-60
-70
-80
-90
-100
-110
-120
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15

Figure 3-4

Continuous-time version

- Büttler, Hans-Jürg [2004]: Orthogonal polynomial.
- N The number of orthogonal polynomials considered.
- $L_n(\tau)$ The ordinary Laguerre polynomial of degree n and argument τ, $n = 0, 1, \ldots, N$.
- c_n The constants (*Laguerre constants* for short) to be estimated, $n = 0, 1, \ldots, N$.
- $f(\infty)$ The (unknown) instantaneous forward interest rate with an infinite term to maturity.
- Approximate the unknown function of the instantaneous forward interest rate by a finite series of Laguerre polynomials.

$$f(\tau) = f(\infty) + \mathrm{e}^{-\tau/2} \sum_{n=0}^{N} c_n L_n(\tau) \tag{3-5}$$

- Recall $f(0) = r$, the instantaneous spot interest rate, we get for $f(\infty)$:

$$f(\infty) = r - \sum_{n=0}^{N} c_n \geqq 0 \tag{3-6}$$

- The continuous objective function becomes a quadratic form, where **E** and **Q** are known matrices, and **c** is the vector of Laguerre constants.

$$\min_{\langle c_0, \ldots, c_N \rangle} G(T) = \int_0^T \left(\frac{\mathrm{d}f(t)}{\mathrm{d}t} \right)^2 \mathrm{d}t = = \frac{1}{4}\, \mathbf{c}'\, \mathbf{E}'\, \mathbf{Q}(T)\, \mathbf{E}\, \mathbf{c} \tag{3-7}$$

- The spot interest rate becomes:

$$\begin{aligned} R(\tau) &= \frac{1}{\tau} \int_0^\tau f(t)\, \mathrm{d}t = f(\infty) + \frac{1}{\tau} \sum_{n=0}^{N} c_n J_n(\tau)\,, \quad \text{where} \\ J_n(\tau) &\overset{\text{def}}{=} \int_0^\tau \mathrm{e}^{-t/2} L_n(t)\, \mathrm{d}t = (-1)^n\, 2\left[1 - S_n(\tau)\right], \quad \text{where} \\ S_n(\tau) &\overset{\text{def}}{=} \sum_{m=0}^{n} (-1)^{n-m}\, 2^m\, \mathrm{e}^{-\tau/2} L_{n-m}^{(m)}(\tau)\,, \quad \text{with } S_n(0) = 1, \quad S_n(\infty) = 0 \end{aligned} \tag{3-8}$$

- $L_n^{(\alpha)}(\tau)$ The generalized Laguerre polynomial of degree n, parameter α and argument τ.
- Since we need to calculate the integral $J_n(\tau)$ many times during the optimization, we calculate the sum in the equation above recursively by the following relationships.

$$\begin{aligned} &S_n(\tau) = S_{n-1}(\tau) + (-1)^n\, \mathrm{e}^{-\tau/2} L_n(\tau) + (-1)^{n-1}\, \mathrm{e}^{-\tau/2} L_{n-1}(\tau)\,, \quad \text{for } n = 1, 2, \ldots, \quad \text{where} \\ &S_0(\tau) = \mathrm{e}^{-\tau/2}, \quad \text{and} \\ &L_n(\tau) = \frac{1}{n} \left\{ \left[2\,n - 1 - \tau\right] L_{n-1}(\tau) - \left[n - 1\right] L_{n-2}(\tau) \right\}, \quad \text{for } n = 2, 3, \ldots, \quad \text{where} \\ &L_0(\tau) = 1, \quad L_1(\tau) = 1 - \tau. \end{aligned} \tag{3-9}$$

- Note that as $\tau \to \infty$, $J_n(\infty) = (-1)^n\, 2$.

4 The Classification of Corporate Bonds

- Rating agencies (S & P, Moody's) evaluate the status of a particular bond rather infrequently due to the high cost.
- The bond market reviews all the relevant information immediately. Hence, the bond prices should reflect the *actual* credit rating.
- Example: the bond market downgraded General Motors bonds long before the rating agencies changed their "official" rating from "AAA" to "BBB".
- We propose a classification of bonds based on quoted prices, quoted bid prices and quoted ask prices.
- In general, the yield curve and the classes of corporate bonds should be calculated simultaneously.
- Büttler [2005]: We propose to separate one task from the other.
  1. Divide the sample of corporate bonds into classes using bond yields.
  2. Given the different classes, calculate the yield curve for each class by means of the "forward-rate" method.

The Algorithm to Classify Corporate Bonds

- Again, all the bond prices are meant to be *cash* prices.
- $B^{\text{obs}}(\tau)$ The observed price of a coupon-bearing bond with its corresponding term to maturity τ.
- c_k The kth coupon, $k = 1, \ldots, K$. The last coupon, c_K, includes the redemption value of the coupon bearing bond.
- $y(\tau)$ The bond yield or the yield to maturity, respectively, of a coupon-bearing bond with a term to maturity of τ years. The bond yield is defined as that *constant* discount rate which equates the present value of the cash flows with the observed bond price.

$$B^{\text{obs}}(\tau) = \sum_{k=1}^{K} c_k \exp\left(-y(\tau)\,\tau_k\right), \quad (0 \leqq \tau_k \leqq \tau) \tag{4-1}$$

- Require the yield curve of each class to lie within the corresponding bid-ask prices of corporate bonds.
- y^{bid} The bid-price based bond yield of a corporate bond.
- y^{ask} The ask-price based bond yield of a corporate bond.

$$\begin{aligned} B^{\text{bid}}(\tau) &= \sum_{k=1}^{K} c_k \exp\left(-y^{\text{bid}}(\tau)\,\tau_k\right) \\ B^{\text{ask}}(\tau) &= \sum_{k=1}^{K} c_k \exp\left(-y^{\text{ask}}(\tau)\,\tau_k\right) \end{aligned} \tag{4-2}$$

- $y^*(\tau_m)$ The smoothed bond yield of the government bonds which corresponds with the term to maturity of the mth corporate bond. The government bonds are considered to be risk-free.
- π The bond-yield based credit risk premium or the spread of corporate bond yields over government bond yields, respectively.
- π^{bid} The bid-price based credit risk premium.
- π^{ask} The ask-price based credit risk premium.

$$\begin{aligned} \pi^{\text{bid}}(\tau) &= y^{\text{bid}}(\tau) - y^*(\tau) \\ \pi(\tau) &= y(\tau) - y^*(\tau) \\ \pi^{\text{ask}}(\tau) &= y^{\text{ask}}(\tau) - y^*(\tau), \quad (\pi^{\text{ask}} \leqq \pi \leqq \pi^{\text{bid}}) \end{aligned} \tag{4-3}$$

- Exclude corporate bonds from the sample whose credit risk premium, π, is negative.
- Exclude outliers from the sample:
- $\pi^{\text{bid-ask}}$ The spread of the bid-price based credit risk premium over the ask-price based credit risk premium, or the bid-ask premium spread for short.

$$\pi^{\text{bid-ask}}(\tau) = \pi^{\text{bid}}(\tau) - \pi^{\text{ask}}(\tau) \tag{4-4}$$

- The bid-ask premium spread is approximately lognormally distributed (see Fig. 4-1a & 4-1b).

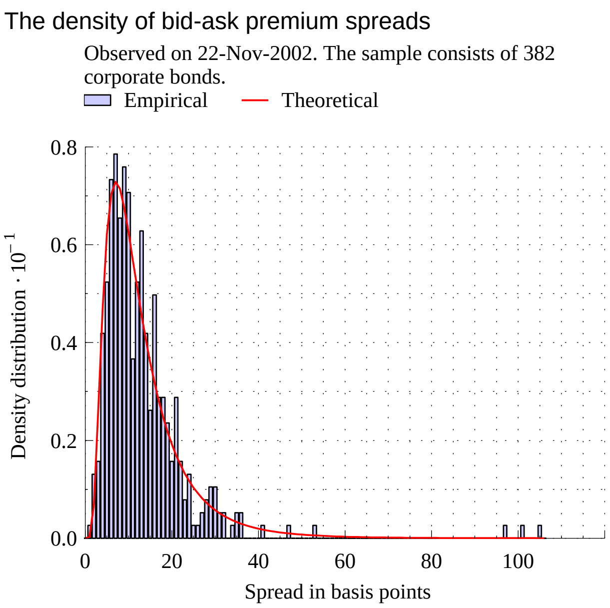

Figure 4-1a

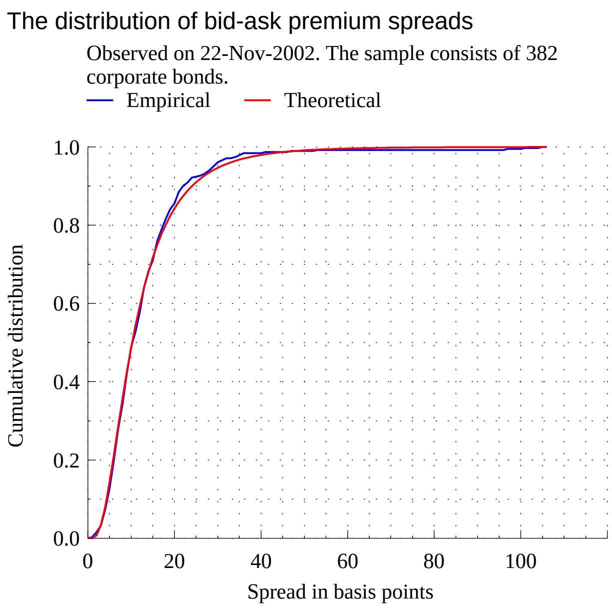

Figure 4-1b

- $\Phi(\cdot)$ The lognormal cumulative distribution function.
- μ The mean of the lognormal density function.
- σ The standard deviation of the lognormal density function.
- p The probability of an outlier.

- π^{c} The critical value of an outlier.

$$\pi^{c}(\tau) = \Phi^{-1}(1 - p;\, \mu,\, \sigma) \tag{4-5}$$

- Exclude corporate bonds whose bid-ask premium spread is greater than the critical value of an outlier.
- Example 4-1. Suppose the mean of the logarithm of the bid-ask premium spread of the sample of corporate bonds considered is equal to 0.5 bps, the corresponding standard deviation is equal to 2.5 bps and the probability of outliers is 5%, then critical value of an outlier is equal to 100.7 bps.
- M^{*} The number of remaining corporate bonds in the sample considered.
- λ The "premium factor". For a liquid bond market, set $\lambda = 0$.
- η The "premium tolerance". For a liquid bond market, we get $\eta = 0$.
- The "premium tolerance" is related to the standard deviation of the bid-ask premium spread.

$$\eta = \sigma\, \lambda, \quad (\lambda \geqq 0) \tag{4-6}$$

- θ^{low} The lower threshold value of the corporate bonds remaining in the sample considered.
- The lower threshold value is equal to the minimum of the ask-price based credit risk premia:

$$\theta^{\text{low}} = \min\left(\pi^{\text{ask}}(\tau_1),\, \ldots,\, \pi^{\text{ask}}(\tau_{M^*})\right) \tag{4-7}$$

- $\mathcal{I}(\cdot)$ The index function which returns the ordinal number of the corporate bonds which fulfil the inequality within the round brackets.
- $\mathcal{I}^{\text{low}}$ The lower index set found.
- The lower index set consist of those bonds whose ask-price based credit risk premia are less than or equal to the sum of the lower threshold value and the "premium tolerance".

$$\mathcal{I}^{\text{low}} = \mathcal{I}\left(\pi^{\text{ask}}(\tau_j) \leqq \theta^{\text{low}} + \eta\right), \quad (j = 1, \ldots, M^{*}) \tag{4-8}$$

- θ^{up} The upper threshold value of the corporate bonds remaining in the sample considered.
- The upper threshold value is the maximum of the bid-price based credit risk premia which belong to the lower index set:

$$\theta^{\text{up}} = \max\left(\pi^{\text{bid}}(\mathcal{I}^{\text{low}})\right) \tag{4-9}$$

- $\mathcal{I}_1$ The index set of the bonds belonging to the *first* class.

- In order to keep the yield curve of a particular class within the bid-ask spread, we select those bonds for the first class, whose ask-price based credit risk premia lie within the range given by the lower and upper threshold values:

$$\mathcal{J}_1 = \mathcal{J}\left(\theta^{\mathrm{low}} \leqq \pi^{\mathrm{ask}}(\tau_j) \leqq \theta^{\mathrm{up}}\right), \quad (j = 1, \ldots, M^*) \qquad (4\text{-}10)$$

- If it turns out that there are several classes, remove the first-class corporate bonds from the sample and repeat the procedure given by equ. (4-7) - (4-10) until there are no bonds left.
- Example 4-2. Sample of 92 coupon-bearing bonds of the commercial banks in Switzerland outstanding on 15-Jan-2003. See Figures 4-2a & 4.2b.

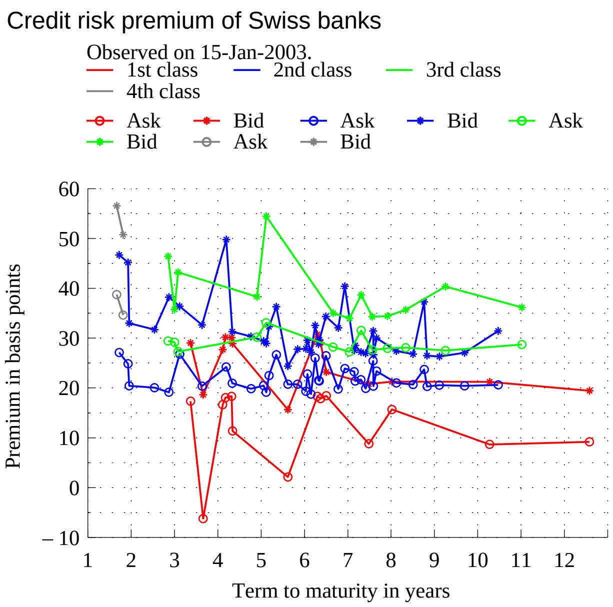

Figure 4-2a

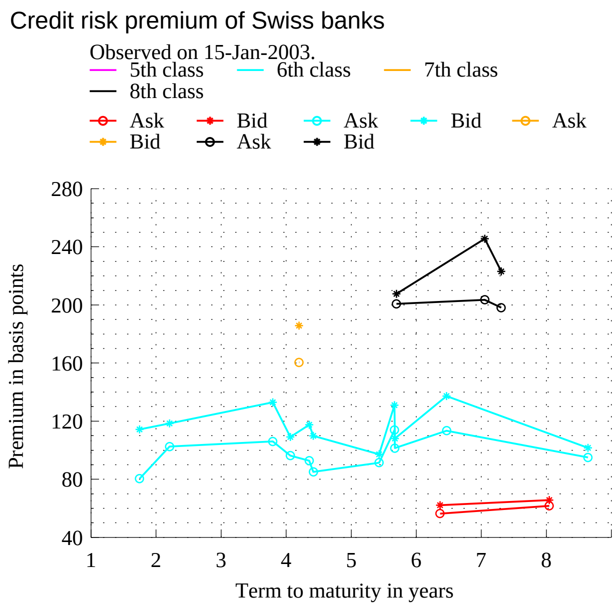

Figure 4-2b

- The estimated term structure of the first three classes using the "forward-rate" method. See Figures 4-3a - 4-3d.

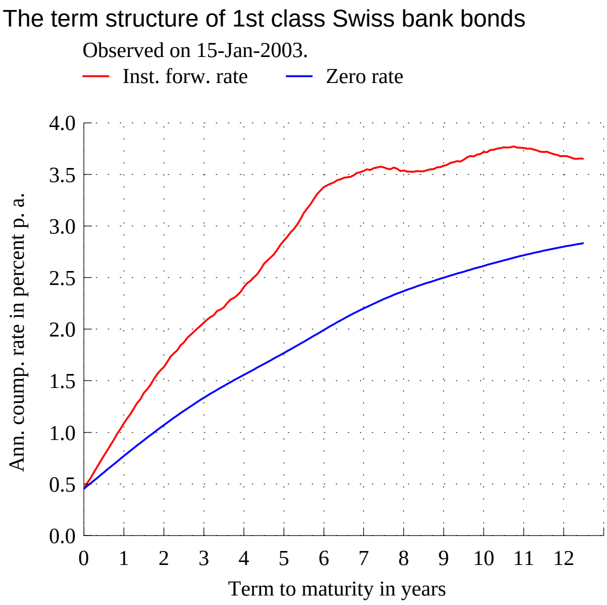

Figure 4-3a

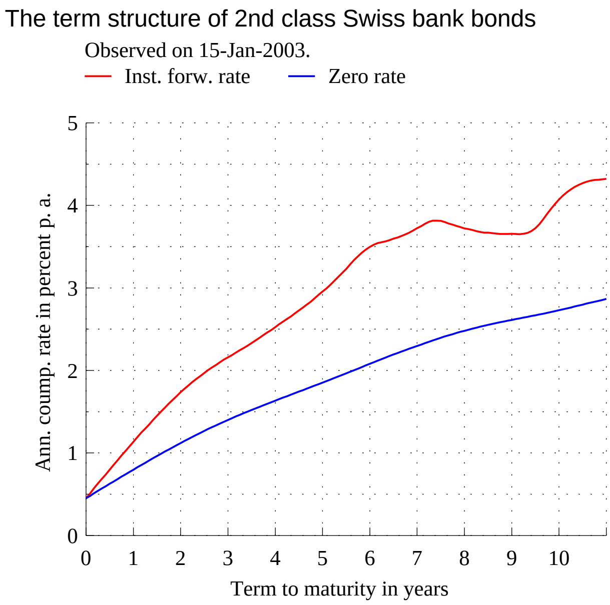

Figure 4-3b

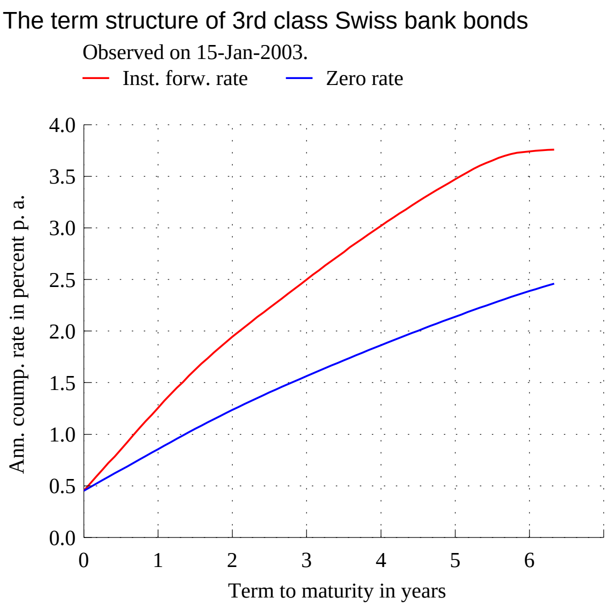

Figure 4-3c

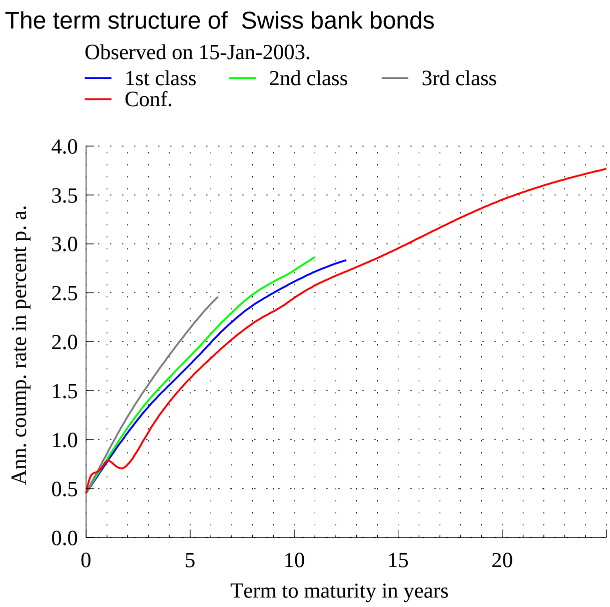

Figure 4-3d

- The evolution of the credit risk premium over time for three classes of corporate bonds selected for some economic sectors in Switzerland. See Figures 4-4 to 4-6.

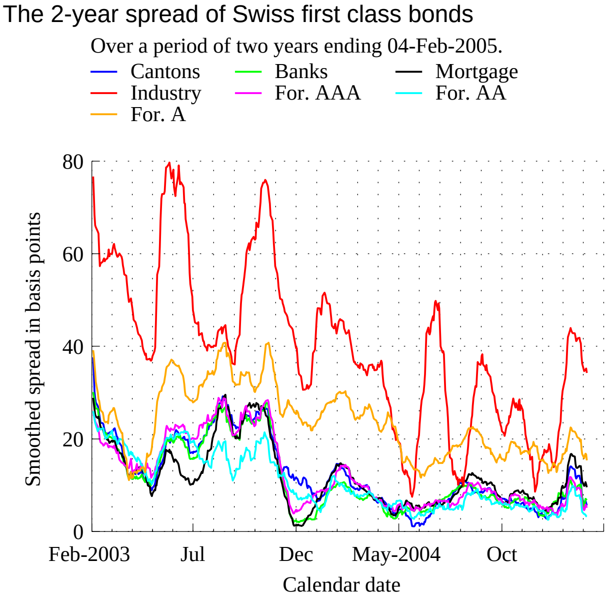

Figure 4-4

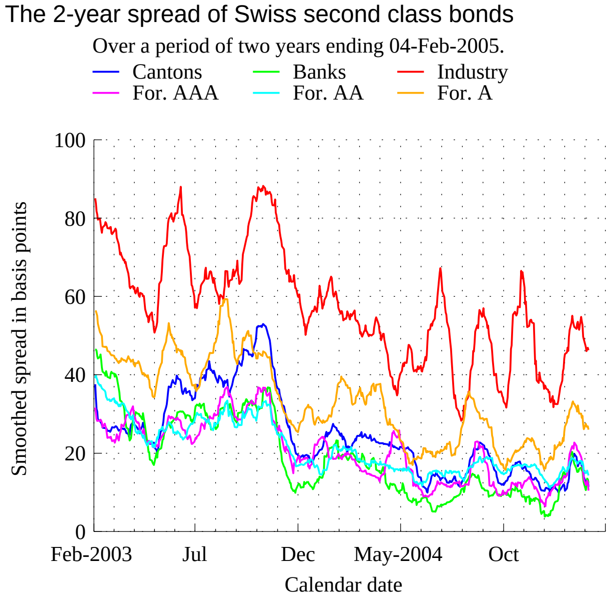

Figure 4-5

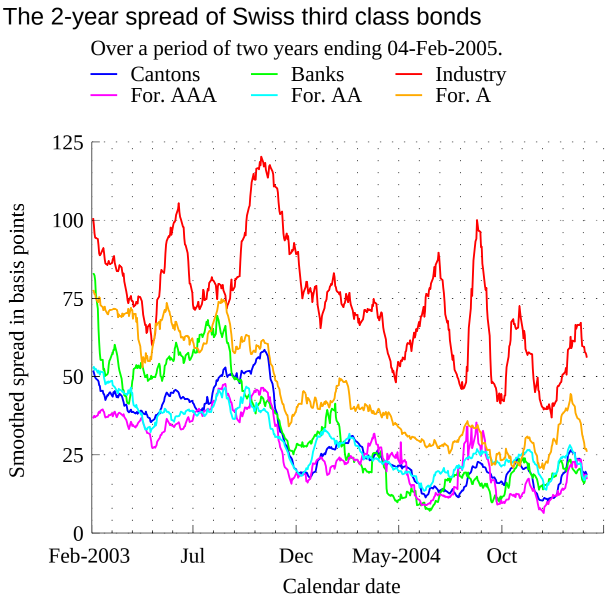

Figure 4-6

5 Demonstration

- The computer package is written in MATLAB code and calculates
  - The yield curves for the bonds of the Swiss confederation, the Swiss Cantons, the industry, the mortgage institutes, the Swiss banks as well as foreign debtors rated “AAA”, “AA” and “A”.
  - The ex ante term structure of real interest rates and expected inflation rates.
  - The future path of the expected three-month spot interest rates.
  - The various credit classes of corporate bonds and their yield curves as well as their credit risk premia.
  - Various forward interest rates and futures interest rates.
- The demonstration includes the yield curves and credit classes for a recent day.

6 Concluding Remarks

- The “forward-rate” method is a powerful tool to estimate the yield curves.
- It allows for a straightforward extention to classify corporate bonds.

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