A New Algorithm for the Kummer Function

by

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ABSTRACT

In this paper, we propose a new algorithm to compute the Kummer function that takes care of the possible cancellation of digits. The algorithm relies on three new recurrence relationships which shift the parameters of the Kummer function into the positive region.

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0 Introduction

The Kummer function or the confluent hypergeometric function, respectively, is widely used in statistics and financial economics (see, e. g., Abadir, 1996, or Cox, Ingersoll and Ross, 1985). The computation of the Kummer function may suffer from a considerable loss of significant digits if the two parameters are of opposite sign. An example is given below. Therefore, we propose a new algorithm that takes care of the possible cancellation of digits. The algorithm relies on three new recurrence relationships which shift the parameters of the Kummer function into the positive region.

1 The Definition of the Kummer Function

The Kummer function, $M(a, b, z)$, with the two parameters a and b as well as the argument z is defined by the following infinite series

$$M(a, b, z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!}, \quad (b \neq 0, -1, -2, -3, \ldots, \text{ for } a \neq b)$$
$$= 1 + \frac{a z}{b} + \frac{(a)_2 z^2}{(b)_2 2!} + \ldots + \frac{(a)_n z^n}{(b)_n n!} + \ldots \tag{1a}$$

where $(\cdot)_n$ denotes Pochhammer's symbol, which is defined as

$$(a)_n = a (a+1)(a+2) \ldots (a+n-1), \quad (a)_0 \equiv 1. \tag{1b}$$

If $a = b$, the Kummer function degenerates to the exponential function $\exp(z)$. In the following, we restrict ourselves to *real* values of a, b, and z. If $b = 0, -1, -2, \ldots$ for $a \neq b$, the Kummer function has poles or is not defined. However, the transformed Kummer function $\mathcal{M}(a, b, z) \equiv M(a, b, z) / \Gamma(b)$, where $\Gamma(\cdot)$ denotes Euler's gamma function, is an entire function of both parameters as well as of the variable (Erdélyi et al., 1953). The limit is shown to be (Erdélyi et al., 1953, vol. 1, eq. 6-7-12)

$$\lim_{b \to 1-n} \mathcal{M}(a, b, z) = \frac{(a)_n}{n!} z^n M(a+n, 1+n, z), \quad n = 1, 2, 3, \ldots, \quad \mathcal{M}(a, b, z) \equiv \frac{M(a, b, z)}{\Gamma(b)}$$

For negative arguments z, the following transformation is useful (Abramowitz and Stegun, 1972, eq. 13.1.27).

$$M(a, b, -z) = e^{-z} M(b-a, b, z) \tag{2}$$

For large and positive arguments given fixed parameter values, the following relationship is in general be used (Abramowitz and Stegun, 1972, eq. 13.5.1)

$$M(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)} \, e^z z^{a-b} \left\{ \sum_{n=0}^{S} \frac{(b-a)_n (1-a)_n z^{-n}}{n!} + \mathbb{O}\left(z^{-(S+1)}\right) \right\}, \quad z > 0 \tag{3}$$

where again $\Gamma(\cdot)$ denotes Euler's gamma function.

For large and negative arguments given fixed parameter values, the following relationship is in general be used (Abramowitz and Stegun, 1972, eq. 13.5.1).

$$M(a, b, z) = \frac{\Gamma(b)}{\Gamma(b-a)} (-z)^{-a} \left\{ \sum_{n=0}^{R} \frac{(a)_n (1+a-b)_n (-z)^{-n}}{n!} + \mathbb{O}\left((-z)^{-(R+1)}\right) \right\}, \quad z < 0 \tag{4}$$

Using the series expansion of equation (1) to compute $M(-6.9, 0.1, 26)$ we obtain $-1.0431 \cdot 10^7$, and using the asymptotic expansion of equation (3) to compute $M(-6.9, 0.1, 26)$ we obtain $-4.4128 \cdot 10^6$: a considerable difference. With a positive $a = 6.9$, we obtain $M(6.9, 0.1, 26) = 4.7112 \cdot 10^{19}$ for both the equations (1) and (3). We attribute this fact to the existence of cancellation of digits.

2 Three Recurrence Relationships

In this section, we develop three new recurrence relationships which shift the parameters of the Kummer functions into the positive region. The first recurrence relationship applies to the infinite series as given in equation (1), the other two apply to the asymptotic expansions (3) and (4).

2-1 The Infinite Series

If both parameters a and b as well as the argument z are positive, the evaluation of the infinite series as given by the equation (1) should not suffer from a loss of significant digits. The series can be evaluated until the partial sum does not change any more. If both parameters a and b are positive but the argument z is negative, then apply the transformation (2).

However, if one of the parameters is negative but the argument is positive, then a cancellation of digits may occur. In this case, we propose to compute the Kummer function according to the first recurrence relationship below. This recurrence establishes a relationship between the Kummer function with negative parameters and a series of Kummer functions with positive parameters only. In this case, all the terms in the series of equation (1) are positive. Hence, no cancellation of digits occurs in the computation of the various Kummer functions.

First recurrence relationship: *Suppose that $a, b < 0$ and $z > 0$. Choose the smallest positive integer n greater than or equal to the maximum of either a or b in absolute value, i. e., $n = \text{ceiling}(\max(|a|, |b|))$, then the Kummer function $M(a, b, z)$ can be represented by the following series of Kummer functions with positive parameters and argument, where the various Kummer functions are evaluated according to equation* (1).

$$\begin{aligned} M(a, b, z) &= \sum_{j=0}^{n} (-1)^j A(a+n, b+n+j, z)\, M(a+n, b+n+j, z), \quad n \geqq 1, \\ A(a+n, b+n+j, z) &= A(a+n-1, b+n+j-1, z) \\ &\quad + A(a+n-1, b+n+j-2, z)\, B(a+n-1, b+n+j-2, z), \\ B(a, b, z) &\equiv \frac{z\,(b-a)}{b\,(b+1)}, \\ A(a+1, b+2, z) &\equiv B(a, b, z), \\ A(a+n, b+n, z) &\equiv 1, \\ A(a+n, b+n+j, z) &\equiv 0 \quad \text{for } j > n \text{ or } j < 0. \end{aligned} \tag{5a}$$

The above relationship may be written explicitly as follows.

$$\begin{aligned} M(a, b, z) &= M(a+n, b+n, z) - A(a+n, b+n+1, z)\, M(a+n, b+n+1, z) \\ &\quad + A(a+n, b+n+2, z)\, M(a+n, b+n+2, z) - \ldots \\ &\quad + (-1)^n A(a+n, b+2n, z)\, M(a+n, b+2n, z)\,, \end{aligned} \tag{5b}$$

where the coefficients $A(\cdot)$ *may be written as*

$$\begin{aligned} &A(a+n, b+n+1, z) = A(a+n-1, b+n, z) + B(a+n-1, b+n-1, z), \quad j = 1, \\ &1 < j < n-1: \\ &A(a+n, b+n+j, z) = A(a+n-1, b+n+j-1, z) \\ &\qquad + A(a+n-1, b+n+j-2, z)\, B(a+n-1, b+n+j-2, z), \\ &A(a+n, b+2n, z) = A(a+n-1, b+2n-2, z)\, B(a+n-1, b+2n-2, z), \quad j = n. \end{aligned} \tag{5c}$$

If $a \geqslant 0$ *and* $b \leqslant 0$, *then n is the smallest positive integer greater than or equal to* $|b|$ *or* $|a|$, *respectively*. ◪

Proof: by induction. Using equations (13.4.8), (13.4.12), (13.4.5) and (13.4.2) in Abramowitz and Stegun [1972], we obtain

$$M(a, b, z) = M(a+1, b+1, z) - B(a, b, z)\, M(a+1, b+2, z), \quad B(a, b, z) \equiv \frac{z\,(b-a)}{b\,(b+1)}\,. \tag{6}$$

This equation is the building block for the induction. First, we set $n = 1$ in the equation (5a) to get:

$$M(a, b, z) = A(a+1, b+1, z)\, M(a+1, b+1, z) - A(a+1, b+2, z)\, M(a+1, b+2, z) \tag{7}$$

Since $A(a+1, b+1, z) \equiv 1$ and $A(a+1, b+2, z) \equiv B(a, b, z)$, the equation (7) reduces to the equation (6). Next, we assume that the equation (5a) is correct for any n and derive a formula for $n+1$. By equation (6) we can write

$$M(a+n, b+n+j, z) = M(a+n+1, b+n+j+1, z) - B(a+n, b+n+j, z)\, M(a+n+1, b+n+j+2, z) \tag{8}$$

Inserting the above relationship into the equation (5a), we obtain the following expression.

$$\begin{aligned} M(a, b, z) = {} & A(a+n, b+n, z)\, M(a+n+1, b+n+1, z) \\ & + \sum_{j=1}^{n} \Big\{ \big[(-1)^j A(a+n, b+n+j, z) \\ & \quad - (-1)^{j-1} A(a+n, b+n+j-1, z)\, B(a+n, b+n+j-1, z) \big] \\ & \quad M(a+n, b+n+j, z) \Big\} \\ & - (-1)^n A(a+n, b+2n, z)\, B(a+n, b+2n, z)\, M(a+n+1, b+2n+2, z) \end{aligned} \tag{9}$$

Since $A(a + n, b + n, z) \equiv 1$, since the expression in square brackets is equal to $(-1)^j A(a + n + 1, b + n + 1 - j, z)$ by the equation (5a), and since the expression in last row of the above equation is equal to $+(-1)^{n+1} A(a + n + 1, b + 2n + 2, z)$ after adding $A(a + n, b + 2n + 1, z) \equiv 0$, we arrive at the following expression.

$$M(a, b, z) = \sum_{j=0}^{n+1} (-1)^j A(a+n+1, b+n+1+j, z)\, M(a+n+1, b+n+1+j, z) \tag{10}$$

which is the equation (5a) for $n + 1$. This completes the proof. ◑

It is clear that the above recurrence relationship does not avoid any cancellation of digits because it involves sign changes. However, the various Kummer functions $M(a + n, b + n + j, z)$ decline with increasing $j = 1, 2, 3, \ldots, n$. Since the cancellation of digits is smaller, if the various terms are not of the same order of magnitude, the recurrence relationship proposed above avoids some of the cancellation faced with the direct computation according to equation (1).

Note that the recurrence relationship (5) holds true for $n = 0$, that is, when $a, b > 0$. In this case, no shift of the parameters is necessary.

If both parameters and the argument are negative, then use the transformation (2) and apply the first recurrence relationship to the transformed Kummer function.

2-2 Asymptotic Expansions

For large arguments, we use the asymptotic expansions (3) and (4). Consider the case of a positive argument first.

Second recurrence relationship: *Suppose that* $a > b$, $a > 1$ *and* $z > 0$. *Choose the smallest positive integer n greater than or equal to the maximum of either* $(a - 1)$ *or* $(a - b)$, *i. e.,* $n = \text{ceiling}(\max(a - 1, a - b))$, *then the Kummer function* $M(a, b, z)$ *can be represented by the following series of Kummer functions with positive terms, where the various Kummer functions are evaluated according to equation* (3).

$$
\begin{aligned}
&M(a, b, z) = \sum_{j=0}^{n} (-1)^j\, C(a-n, b+j, z)\, M(a-n, b+j, z), \quad n \geqq 1, \\
&C(a-n, b+j, z) = C(a-n+1, b+j, z)\, D(a-n, z) + C(a-n+1, b+j-1, z)\, E(a-n, b+j, z)\,, \\
&D(a, z) := \frac{a+z}{a}\,, \quad E(a, b, z) := \frac{(b-a-1)\,z}{(b-1)\,a}\,, \\
&C(a-1, b, z) := D(a-1, z) = \frac{a-1+z}{a-1}\,, \\
&C(a-1, b+1, z) := E(a-1, b+1, z) = \frac{(b+1-a)\,z}{b\,(a-1)}\,, \\
&C(a-n, b+j, z) := 0 \quad \text{for } j > n \text{ or } j < 0.
\end{aligned}
\tag{11a}
$$

The above relationship may be written explicitly as follows.

$$
\begin{aligned}
M(a, b, z) = {} & C(a-n, b, z)\, M(a-n, b, z) - C(a-n, b+1, z)\, M(a-n, b+1, z) \\
& + C(a-n, b+2, z)\, M(a-n, b+2, z) - \ldots + (-1)^n\, C(a-n, b+n, z)\, M(a-n, b+n, z)\,,
\end{aligned}
\tag{11b}
$$

where the coefficients $C(\cdot)$ *may be written as*

$$
\begin{aligned}
&C(a-n, b, z) = C(a-n+1, b, z)\, D(a-n, z), \quad j = 0, \\
&\quad 0 < j < n-1: \\
&C(a-n, b+j, z) = C(a-n+1, b+j, z)\, D(a-n, z) + C(a-n+1, b+j-1, z)\, E(a-n, b+j, z)\,, \\
&C(a-n, b+n, z) = C(a-n+1, b+n-1, z)\, E(a-n, b+n, z)\,,\ j = n.
\end{aligned}
\tag{11c}
$$

If $a \lesseqgtr 1$ *and* $b \lesseqgtr a$, *then n is the smallest positive integer greater than or equal to* $(a - b)$ *or* $(a - 1)$, *respectively.* ◪

Proof: by induction. Using equations (13.4.4) and (13.4.6) in Abramowitz and Stegun [1972], we obtain

$$
\begin{aligned}
&M(a, b, z) = D(a-1, z)\, M(a-1, b, z) - E(a-1, b+1, z)\, M(a-1, b+1, z), \\
&D(a, z) := \frac{a+z}{a}\,, \quad E(a, b, z) := \frac{(b-a-1)\,z}{(b-1)\,a}\,.
\end{aligned}
\tag{12}
$$

This equation is the building block for the induction. First, we set $n = 1$ in the equation (11a) to get:

$$
M(a, b, z) = C(a-1, b, z)\, M(a-1, b, z) - C(a-1, b+1, z)\, M(a-1, b+1, z) \tag{13}
$$

Since $C(a-1, b, z) \equiv D(a-1, z)$ and $C(a-1, b+1, z) \equiv E(a-1, b+1, z)$, the equation (13) reduces to the equation (12). Next, we assume that the equation (11a) is correct for any n and derive a formula for $n + 1$. By equation (12) we can write

$$
\begin{aligned}
M(a-n, b+j, z) = {} & D(a-n-1, z)\, M(a-n-1, b+j, z) \\
& - E(a-n-1, b+1+j, z)\, M(a-n-1, b+1+j, z)
\end{aligned}
\tag{14}
$$

Inserting the above relationship into the equation (11a), we obtain the following expression.

$$\begin{aligned} M(a, b, z) &= C(a-n, b, z)\, D(a-n-1, z)\, M(a-n-1, b, z) \\ &+ \sum_{j=1}^{n} \Big\{ \big[(-1)^j C(a-n, b+j, z)\, D(a-n-1, z) \\ &\quad - (-1)^{j-1} C(a-n, b+j-1, z)\, E(a-n-1, b+j, z) \big] M(a-n-1, b+j, z) \Big\} \\ &- (-1)^n C(a-n, b+n, z)\, E(a-n-1, b+1+n, z)\, M(a-n-1, b+1+n, z) \end{aligned} \tag{15}$$

Using $C(a-n, b-1, z) \equiv 0$ for $j = 0$ and $C(a-n, b+n+1, z) \equiv 0$ for $j = n+1$ to complete the recurrence relationships for $C(\cdot)$, we arrive at the following expression.

$$M(a, b, z) = \sum_{j=0}^{n+1} (-1)^j C(a-n-1, b+j, z)\, M(a-n-1, b+j, z) \tag{16}$$

which is the equation (11a) for $n + 1$. This completes the proof. ◑

Note that the recurrence relationship (11) holds true for $n = 0$ if we define $C(a, b, z) \equiv 1$, that is, when $a < b$, $a < 1$. In this case, no shift of the parameters is necessary.

Next, consider the case of a large negative argument.

Third recurrence relationship: *Suppose that* a, $(1 + a - b)$, $z < 0$. *Choose the smallest positive integer n greater than or equal to the maximum of either* $(-a)$ *or* $(b - a - 1)$, *i. e.*, $n = \text{ceiling}(\max(|a|, b - a - 1))$, *then the Kummer function* $M(a, b, z)$ *can be represented by the following series of Kummer functions with positive terms, where the various Kummer functions are evaluated according to equation* (4).

$$\begin{aligned} M(a, b, z) &= \sum_{j=0}^{n} (-1)^j F(a+n, b-j, z)\, M(a+n, b-j, z), \quad n \geqq 1, \\ F(a+n, b-j, z) &= F(a+n-1, b-j, z)\, G(a+n, b-j, z) + F(a+n-1, b-j+1, z)\, H(a+n, b-j), \\ G(a, b, z) &:= \frac{a-1+z}{a-b}, \quad H(a, b) := \frac{b}{a-b-1}, \\ F(a+1, b, z) &:= G(a+1, b, z) = \frac{a+z}{1+a-b}, \\ F(a+1, b-1, z) &:= H(a+1, b-1) = \frac{b-1}{1+a-b}, \\ F(a+n, b-j, z) &:= 0 \quad \text{for } j > n \text{ or } j < 0. \end{aligned} \tag{17a}$$

The above relationship may be written explicitly as follows.

$$\begin{aligned} M(a, b, z) &= F(a+n, b, z)\, M(a+n, b, z) - F(a+n, b-1, z)\, M(a+n, b-1, z) \\ &+ F(a+n, b-2, z)\, M(a+n, b-2, z) - \ldots + (-1)^n F(a+n, b-n, z)\, M(a+n, b-n, z), \end{aligned} \tag{17b}$$

where the coefficients $F(\cdot)$ *may be written as*

$$\begin{aligned} F(a+n, b, z) &= F(a+n-1, b, z)\, G(a+n, b, z), \quad j = 0, \\ &0 < j < n-1: \\ F(a+n, b-j, z) &= F(a+n-1, b-j, z)\, G(a+n, b-j, z) + F(a+n-1, b-j+1, z)\, H(a+n, b-j), \\ F(a+n, b-n, z) &= F(a+n-1, b-n+1, z)\, H(a+n, b-n), \; j = n. \end{aligned} \tag{17c}$$

If $a \lesseqgtr 0$ *and* $(1 + a - b) \gtreqless 0$*, then n is the smallest positive integer greater than or equal to* $|a|$ *or* $(b - a - 1)$*, respectively.* ◪

Proof: by induction. Using equations (13.4.3) and (13.4.5) in Abramowitz and Stegun [1972], we obtain

$$\begin{aligned} &M(a, b, z) = G(a+1, b, z)\, M(a+1, b, z) - H(a+1, b-1)\, M(a+1, b-1, z), \\ &G(a, b, z) := \frac{a - 1 + z}{a - b}, \quad H(a, b) := \frac{b}{a - b - 1}\,. \end{aligned} \tag{18}$$

This equation is the building block for the induction. First, we set $n = 1$ in the equation (17a) to get:

$$M(a, b, z) = F(a+1, b, z)\, M(a+1, b, z) - F(a+1, b-1, z)\, M(a+1, b-1, z) \tag{19}$$

Since $F(a + 1, b, z) \equiv G(a + 1, b, z)$ and $F(a + 1, b - 1, z) \equiv H(a + 1, b - 1)$, the equation (19) reduces to the equation (18). Next, we assume that the equation (17a) is correct for any n and derive a formula for $n + 1$. By equation (18) we can write

$$\begin{aligned} M(a+n, b-j, z) = {} & G(a+n+1, b-j, z)\, M(a+n+1, b-j, z) \\ & - H(a+n+1, b-j-1, z)\, M(a+n+1, b-j-1, z) \end{aligned} \tag{20}$$

Inserting the above relationship into the equation (17a), we obtain the following expression.

$$\begin{aligned} M(a, b, z) = {} & F(a+n, b, z)\, G(a+n+1, b, z)\, M(a+n+1, b, z) \\ & + \sum_{j=1}^{n} \Big\{ \big[(-1)^j F(a+n, b-j, z)\, G(a+n+1, b-j, z) \\ & \qquad - (-1)^{j-1} F(a+n, b-j+1, z)\, H(a+n+1, b-j) \big]\, M(a+n+1, b-j, z) \Big\} \\ & - (-1)^n F(a+n, b-n, z)\, H(a+n+1, b-n-1, z)\, M(a+n+1, b-n-1, z) \end{aligned} \tag{21}$$

Using $F(a + n, b + 1, z) \equiv 0$ for $j = 0$ and $F(a + n, b - n - 1, z) \equiv 0$ for $j = n + 1$ to complete the recurrence relationships for $F(\cdot)$, we arrive at the following expression.

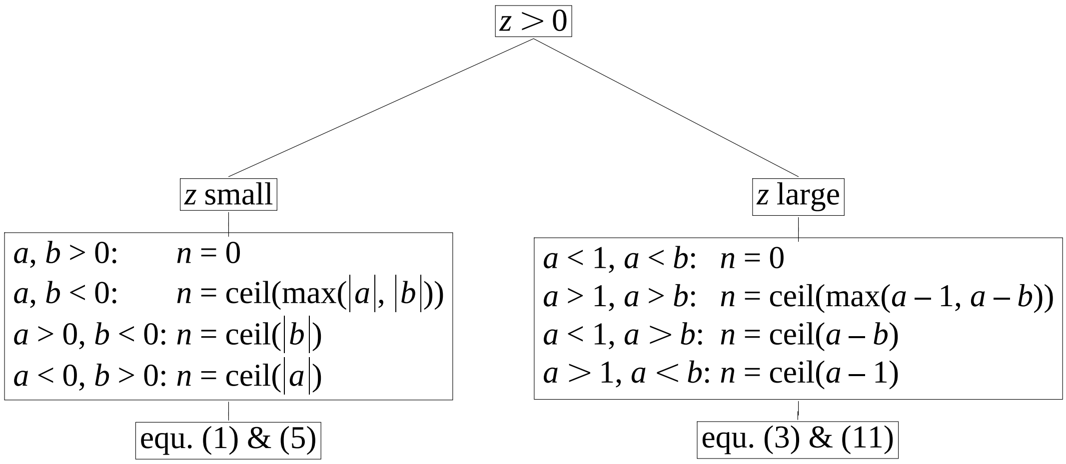
$$M(a, b, z) = \sum_{j=0}^{n+1} (-1)^j F(a+n+1, b-j, z)\, M(a+n+1, b-j, z) \tag{22}$$

which is the equation (17a) for $n + 1$. This completes the proof. ◑

Note that the recurrence relationship (17) holds true for $n = 0$ if we define $F(a, b, z) \equiv 1$, that is, when $a > 0$ and $(1 + a - b) > 1$. In this case, no shift of the parameters is necessary.

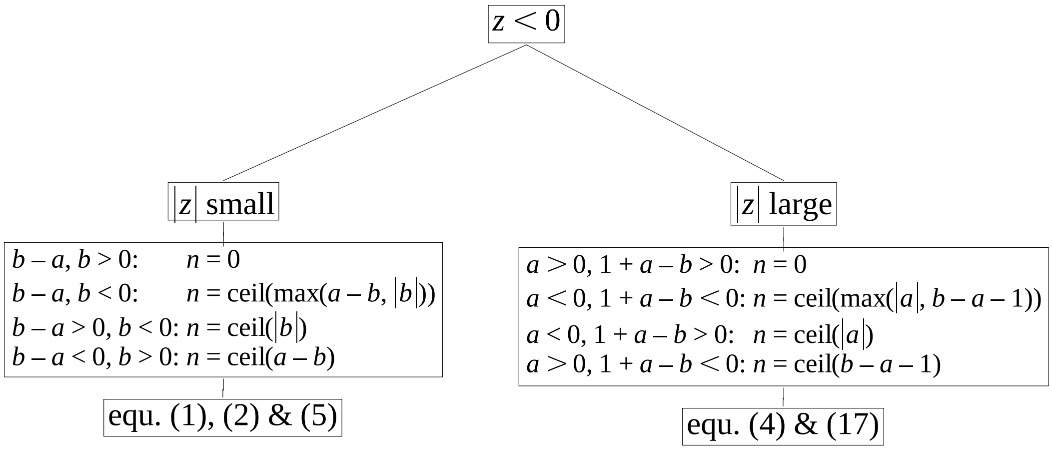
3 The Algorithm

Text


z > 0
z small
z large
a, b > 0: n = 0
a, b < 0: n = ceil(max(|a|, |b|))
a > 0, b < 0: n = ceil(|b|)
a < 0, b > 0: n = ceil(|a|)
a < 1, a < b: n = 0
a > 1, a > b: n = ceil(max(a – 1, a – b))
a < 1, a > b: n = ceil(a – b)
a > 1, a < b: n = ceil(a – 1)
equ. (1) & (5)
equ. (3) & (11)

(23a)

Text


z < 0
|z| small
|z| large
b – a, b > 0: n = 0
b – a, b < 0: n = ceil(max(a – b, |b|))
b – a > 0, b < 0: n = ceil(|b|)
b – a < 0, b > 0: n = ceil(a – b)
a > 0, 1 + a – b > 0: n = 0
a < 0, 1 + a – b < 0: n = ceil(max(|a|, b – a – 1))
a < 0, 1 + a – b > 0: n = ceil(|a|)
a > 0, 1 + a – b < 0: n = ceil(b – a – 1)
equ. (1), (2) & (5)
equ. (4) & (17)

(23b)

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