Numerical Evaluation of Callable Bonds Using Green's Function

by

HANS-JüRG BüTTLER [*] AND JöRG WALDVOGEL [‡]
Swiss National Bank
and
University of Zurich
Switzerland

Seminar for Applied Mathematics
Swiss Federal Institute of Technology
8092 Zurich
Switzerland

Abstract:

The purpose of this paper is to describe the numerical algorithm to implement the analytical price of the semi-American callable bond which has been derived in a companion paper. As an example, the algorithm is applied to an exchange-traded callable bond with ten call dates and with a maximum life of twenty years.

First version: November, 1993

[*] Address: Swiss National Bank, 8022 Zurich, Switzerland. Phone (direct dialling): country code–1–6313 417, Fax: country code–1–6313 901.

[‡] Phone (direct dialling): country code–1–632 3460, Fax: country code–1–252 3401, E-mail: waldvoge@math.ethz.ch.

1. Introduction

The purpose of this paper is to describe the numerical algorithm to implement the analytical price of the semi-American callable bond which has been derived in a companion paper [1993]. In that paper, the price of the semi-American callable bond is given by a multiple integral of the normal probability density function or the non-central chi-square probability density function, respectively. The dimension of the integral is equal to the number of call dates. Since each of the limits of this multiple integral is the zero of a monotone function which involves the computation of a lower-dimensional integral, the computation of a semi-American callable bond with ten call dates, say, would require to compute a sequence of integrals with a dimension ranging from one to nine, before one could finally compute the ten-dimensional integral. Since the computation of a multiple integral is rather time-consuming for higher dimensions, we choose to split the multiple integral into a sequence of one-dimensional integrals involving GREEN's function. The price we have to pay for this simplification is that, for each time step, the resulting price function can only be evaluated for discrete interest-rate points. Hence, the price function which has been obtained from the preceding time step must be interpolated in order that it can serve as an input in the current integration.

The outline of the paper is as follows. The numerical algorithm is described in the next section. Sections 3 to 6 deal with particular numerical problems, namely, the computation of a probability distribution, the interpolation and extrapolation of the price function, the computation of the "break-even" interest rate, and the numerical quadrature. A numerical example is given in the last section.

2. The Numerical Algorithm

Suppose there are n call dates identical to the coupon dates $[t_r - j]$ for $j = 1, 2, \ldots, n$, with t_r the last possible redemption date if the bond is never called. The notice period is denoted as τ_n, which is, in general, two months. Let the call prices per face value be denoted as $X_j := X(t_r - j)$ for $j = 1, 2, \ldots, n$, the instantaneous interest rates prevailing on the notice dates $[t_r - \tau_n - j]$ as $r_j := r(t_r - \tau_n - j)$ for $j = 1, 2, \ldots, n$, the instantaneous interest rate today as $r := r(t_0)$, the time periods between call dates as τ (one year), the time period between today and the last notice date (when moving backwards in time) as $\hat{\tau}$, the number of whole years of the remaining time until the last possible redemption date as m, and the annual coupon as η. Let $P(\cdot, \cdot)$ denote the price of a discount bond and $K(\cdot, \cdot)$ the price of a callable bond.

The numerical algorithm follows the method of equation (7.1) of the companion paper. We proceed backwards in time. The starting point is the *second* notice day for which we compute the price of the callable bond an instant before this notice day with the help of the formula for the European callable bond (equations (6.7a & b) of the companion paper) for a set of inter-

polating points $\{r_j^{(1)}, r_j^{(2)}, \ldots, r_j^{(\nu)}\}$ on the notional call date j. The computation of the probability distributions for the European callable bond will be described in the next section. The computation of the price of the European callable bond can be accomplished with approximate machine precision. There are three numerical operations to be done on each of the remaining notice days.

First, the set of callable bond prices an instant before each of the remaining notice days $\{K_j(r_j^{(1)}, \cdot), K_j(r_j^{(2)}, \cdot), \ldots, K_j(r_j^{(\nu)}, \cdot)\}$ for $j := 2, 3, \ldots, n$, which is obtained from the preceding time step, is interpolated numerically. The two interpolation methods which will be discussed in the fourth section are the polynomial interpolation and the exponential cubic spline. Since the analytical price of the callable bond has been derived in terms of series in orthogonal polynomials, it seems natural to use these series again for the interpolation. The callable bond price is interpolated on the finite interval $[r_l, r_u]$, but extrapolated for instantaneous interest rates which are greater than the upper interpolation boundary r_u. There is no extrapolation needed if the instantaneous interest is less than the lower interpolation boundary because the integration in the range $(-\infty, r_l]$ will be done analytically. The interpolated price function an instant before each notice day, denoted as $K_j(\cdot, \cdot)^+$ for $j := 2, 3, \ldots, n$, may be written as:

$$K_j(r_j, j\cdot\tau + \tau_n)^+ := \begin{cases} \mathrm{e}^{br_j} \sum_{i=0}^{N_p} d_i \, \mathcal{P}_i(x), \quad r_j := \vartheta_1 + \vartheta_2 x \quad \text{(polynomial interpolation)}, \\ \exp\big(\mathcal{S}_3(r_j)\big), \quad \mathcal{S}_3(r_j) := \sum_{i=0}^{3} a_i \, r_j^i \quad \text{(exponential cubic spline)}, \\ \exp\big(y(r_j)\big), \quad y(r_j) := y(r_u) + y'(r_u)\big[r_j - r_u\big] \quad \text{(extrapolation)}. \end{cases} \tag{2.1}$$

The orthogonal polynomial $\mathcal{P}_i$ of ith degree is the HERMITE polynomial in the case of VASICEK's model (first case) or the generalized LAGUERRE polynomial in the case of the CIR model (second case). The constant b has been defined in equations (2.5a & b) of the companion paper, whereas $\{d_i, \vartheta_1, \vartheta_2\}$ will be determined in the fourth section. The exponential spline method with a cubic polynomial $\mathcal{S}_3$ computes the first derivatives $y'(\cdot)$ of the logarithm of the callable bond price function, whereas the coefficients a_i are not needed. The extrapolation assumes that the logarithm of the callable bond price function grows linearly. An inspection of the graph of this function reveals that $\ln(K_j(\cdot, \cdot)^+)$ grows indeed approximately linearly for large interest rates. The first derivative of the callable bond price function evaluated at the upper interpolation boundary, $y'(r_u)$, can be taken from the exponential cubic spline.

Secondly, the "break-even" interest-rate on each notice day, r_j^*, equates the time value of the sum of the call price and the coupon payment, $[X_j + \eta]\cdot P(r_j, \tau_n)$, with the callable bond price an instant before the notice day, $K_j(\cdot, \cdot)^+$:

$$\mathcal{F}_j(r_j) := \big[X_j + \eta\big] P\big(r_j, \tau_n\big) - K_j\big(r_j, j\cdot\tau + \tau_n\big)^+, \quad j = 2, 3, \ldots, n. \tag{2.2}$$

$\mathcal{F}_j(r_j)$ is a monotone increasing function of the instantaneous interest rate, that is, r_j^* is the unique solution of $\mathcal{F}_j(r_j) = 0$. The root can be bracketed with the help of the same set of callable

bond prices $\{K_j(r_j^{(1)}, \cdot), K_j(r_j^{(2)}, \cdot), \ldots, K_j(r_j^{(\nu)}, \cdot)\}$ which is used for the interpolation mentioned above. The combined NEWTON-bisection method to find the "break-even" interest rate will be described in the fifth section.

Thirdly, the integration involving GREEN's function over the next time period yields the price of the callable bond an instant before the next notice day, using the price of the callable bond an instant after the current notice day as initial data:

$$\begin{aligned} K_{j+1}(r_{j+1}, [j+1]\tau + \tau_n)^+ &= \int_{-\infty}^{+\infty} G(r_{j+1}, \tau, r_j)\, K_j(r_j, j\cdot\tau + \tau_n)^- \, \mathrm{d}r_j + \eta\, P(r_{j+1}, \tau_n) \\ &:= \Im_1 + \Im_2 + \eta\, P(r_{j+1}, \tau_n), \quad (j = 2, 3, \ldots, n-1). \end{aligned} \tag{2.3}$$

GREEN's function, $G(\cdot, \cdot, \cdot)$, is given in equations (5.6a & b) of the companion paper and the callable bond price function an instant after the current notice day, $K_j(\cdot, \cdot)^-$, in equation (7.1) of the companion paper. The integral of GREEN's function can be split into two parts. The first integral can be performed analytically (see the lemmas 1a & b of the companion paper):

$$\begin{aligned} \Im_1 &:= [X_j + \eta] \int_{-\infty}^{r_j^*} G(r_{j+1}, \tau, r_j)\, P(r_j, \tau_n)\, \mathrm{d}r_j \\ &= [X_j + \eta]\, P(r_{j+1}, \tau + \tau_n) \cdot \begin{cases} \mathcal{N}(d_j), & d_j := \dfrac{r_j^* - \mu - s(\tau)^2 [t(r_{j+1}, \tau) + g(\tau_n)]}{s(\tau)}, \\ \mathcal{H}(d_j \mid 2\alpha + 2, \Lambda(r_{j+1}, \tau, \tau_n)), & d_j := 2\, r_j^* [p(\tau) - g(\tau_n)]. \end{cases} \end{aligned} \tag{2.4}$$

All the symbols and functions have been defined in equations (6.7a & b) of the companion paper. The computation of the GAUSSian probability integral $\mathcal{N}(\cdot)$ in the first case and of the non-central chi-square probability integral $\mathcal{H}(\cdot \mid \nu, \lambda)$ in the second case will be shown in the next section. In the sixth section, we will show how the second integral

$$\Im_2 := \int_{r_j^*}^{+\infty} G(r_{j+1}, \tau, r_j)\, K_j(r_j, j\cdot\tau + \tau_n)^+ \, \mathrm{d}r_j\,, \tag{2.5}$$

will be integrated numerically. The last step of the algorithm is equal to those described so far except for the time period: replace τ by $\hat{\tau}$. While all the call dates are relevant in the case of VASICEK's model, the algorithm stops in the case of the CIR model as soon as the "break-even" interest rate becomes zero or negative. In that event, the final time step $\hat{\tau}$ has to be redefined.

The computer program has been written in PASCAL (JENSEN AND WIRTH [1978]) and runs on the APPLE® MACINTOSH™ family, the machine precision of which is 19 – 20 decimal digits (the range of real numbers is from $1.9\cdot10^{-4951}$ to $1.1\cdot10^{4932}$).

3. Probability Distributions and Related Functions

The algorithm starts on the second notice day for which we compute the price of the European callable bond with $[2\,\tau + \tau_n]$ years to expiration. This requires the numerical evaluation of the GAUSSian or non-central chi-square probability integral, respectively, which we write as an infinite power series for small arguments or as a continued fraction for large arguments (HENRICI [1974, 1982]). In the case of the normal probability integral, all the terms are positive, that is, no deletion of digits occurs. In the case of the non-central chi-square probability integral, the terms may, unfortunately, alternate in sign. In order to obtain a result with machine precision without using any machine constant, the summation is stopped as soon as the current partial sum becomes equal to the previous partial sum.

The GAUSSian probability integral may be written in terms of the KUMMER function $M(\cdot, \cdot, \cdot)$ for small arguments ($x < 3$) (ABRAMOWITZ AND STEGUN [1965, equ. (26.2.37)]):

$$\begin{aligned} \mathcal{N}(x) - \frac{1}{2} &= x\,\mathcal{N}'(x)\,M\left(1, \frac{3}{2}, \frac{x^2}{2}\right), \quad (x > 0) \\ &:= x\,\mathcal{N}'(x) \sum_{n=0}^{\infty} a_n\,, \quad \text{with } a_0 = 1, \quad a_n = \frac{x^2}{1+2n}\,a_{n-1} > 0, \quad (n = 1, 2, \ldots). \end{aligned} \tag{3.1}$$

Here, $\mathcal{N}'(\cdot)$ denotes the normal probability density function. The terms a_n may be obtained recursively as shown in the second line of the above equation. For large arguments ($x \geqq 3$), a continued fraction of the complementary normal probability integral is used (ABRAMOWITZ AND STEGUN [1965, equ. (26.2.14)]):

$$\mathcal{N}_c(x) := 1 - \mathcal{N}(x) = \mathcal{N}'(x)\left[\frac{1}{x+}\,\frac{1}{x+}\,\frac{2}{x+}\,\frac{3}{x+}\,\frac{4}{x+}\,\ldots\right], \quad (x > 0). \tag{3.2}$$

This continued fraction is evaluated forwards by means of the well-known recursive algorithm (see e. g. PRESS ET AL. [1989]). Evaluating the continued fraction from the tail is, of course, much faster but has the disadvantage that the number of terms, which is necessary to achieve machine precision, is a function of the argument x.

The non-central chi-square probability integral is written as an infinite series for all arguments because no useful continued fraction representation is known.

$$\begin{aligned} \mathcal{H}(2x \mid 2\nu, 2\lambda) &= \int_0^{2x} H(t \mid 2\nu, 2\lambda)\,\mathrm{d}t = \mathrm{e}^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\,\frac{1}{2^{\nu+n}\,\Gamma(\nu+n)} \int_0^{2x} t^{\nu+n-1}\,\mathrm{e}^{-t/2}\,\mathrm{d}t \\ &= \mathrm{e}^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\,\mathcal{H}(2x \mid 2\nu+2n) = \mathrm{e}^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\,\frac{\gamma(\nu+n,\, x)}{\Gamma(\nu+n)} := \mathrm{e}^{-\lambda} \sum_{n=0}^{\infty} a_n\,. \end{aligned} \tag{3.3}$$

Here, the non-central chi-square distribution $\mathcal{H}(\cdot \mid \nu, \lambda)$ has been written in terms of the χ^2-distribution, denoted as $\mathcal{H}(\cdot \mid \nu)$, and in terms of the incomplete gamma function, denoted as $\gamma(\cdot, \cdot)$ (ABRAMOWITZ AND STEGUN [1965, equ. (26.4.1) and (26.4.19)]). The terms a_n may be obtained recursively as follows:

$$\begin{aligned} a_0 &= \frac{\gamma(\nu,\, x)}{\Gamma(\nu)} = \mathcal{H}(2x \mid 2\nu) > 0, \quad b_1 = \frac{\lambda\, x^{\nu}\, \mathrm{e}^{-x}}{\nu\, \Gamma(\nu)} > 0, \\ b_n &= \frac{\lambda x}{n\,[\nu + n - 1]}\, b_{n-1}, \quad (n = 2, 3, \ldots), \quad a_n = \frac{\lambda}{n}\, a_{n-1} - b_n, \quad (n = 1, 2, \ldots). \end{aligned} \tag{3.4}$$

The above expansion relies on the recurrence relation for the incomplete gamma function (ABRAMOWITZ AND STEGUN [1965, equ. (6.5.22)]). The term b_n is merely an auxiliary variable to compute the sequence a_n. We find the algorithm (3.4) simpler than the one given by TIKU [1981]. The chi-square distribution $\mathcal{H}(\cdot \mid \nu)$ or the incomplete gamma function, respectively, has been computed by means of the KUMMER function for small arguments ($x < \nu + 1$) as follows (ABRAMOWITZ AND STEGUN [1965, equ. (6.5.12)]):

$$\gamma(\nu,\, x) = \frac{x^{\nu}\, \mathrm{e}^{-x}}{\nu}\, M(1,\, 1+\nu,\, x) := \frac{x^{\nu}\, \mathrm{e}^{-x}}{\nu} \sum_{n=0}^{\infty} a_n\,, \quad a_0 = 1, \quad a_n = \frac{x}{n+\nu}\, a_{n-1}\,. \tag{3.5}$$

All the terms a_n are positive. The complementary chi-square probability distribution is computed for large arguments ($x \geqq \nu + 1$) by means of a continued fraction (ABRAMOWITZ AND STEGUN [1965, equ. (26.4.10)]):

$$\mathcal{H}_c(2x \mid 2\nu) := 1 - \mathcal{H}(2x \mid 2\nu) = \frac{x^{\nu}\, \mathrm{e}^{-x}}{\Gamma(\nu)} \left[\frac{1}{x+} \frac{1-\nu}{1+} \frac{1}{x+} \frac{2-\nu}{1+} \frac{2}{x+} \ldots \right], \quad (x > 0). \tag{3.6}$$

Again, the continued fraction is evaluated forwards. Both the power series and the continued fraction are mentioned in PRESS ET AL. [1989].

The non-central chi-square density function, which is needed for GREEN's function in the case of the CIR model, may also be computed recursively from an infinite power series representation:

$$\begin{aligned} H(2x \mid 2\nu,\, 2\lambda) &= \mathrm{e}^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\, \frac{[2x]^{\nu+n-1}\, \mathrm{e}^{-x}}{2^{\nu+n}\, \Gamma(\nu+n)} \\ &:= \frac{\mathrm{e}^{-[\lambda+x]}}{2} \sum_{n=0}^{\infty} a_n\,, \quad a_0 = \frac{x^{\nu-1}}{\Gamma(\nu)}\,, \quad a_n = \frac{\lambda x}{n\,[\nu+n-1]}\, a_{n-1} > 0. \end{aligned} \tag{3.7}$$

All the terms are positive, that is, no deletion of digits occurs. It is obvious from the first term that the non-central chi-square density function, $H(\cdot \mid \nu, \lambda)$, has a singularity at the origin if $\nu < 2$.

The EULER gamma function, $\Gamma(x)$, is computed from an asymptotic formula as follows (ABRAMOWITZ AND STEGUN [1965, equ. (6.1.40)]):

$$\Gamma(x) = \frac{\Gamma(z)}{(x)_n} \sim \frac{\sqrt{2\pi}}{(x)_n} \exp\left(\left[z - \frac{1}{2} \right] \ln(z) - z + \sum_{m=1}^{M} \frac{B_{2m}}{2m\,[2m-1]\, z^{2m-1}} \right), \quad z := x + n. \tag{3.8}$$

The expression on the right-hand side approximates the value of the EULER gamma function for sufficiently large arguments z. The expression $(x)_n := x\,[x+1]\,\ldots\,[x+n-1]$ denotes POCH-

HAMMER's symbol (ABRAMOWITZ AND STEGUN [1965, equ. (6.1.22)]) and the symbol B_m the BERNOULLI number (ABRAMOWITZ AND STEGUN [1965, Table 23.2]). In order to get a value of the gamma function with 19-digit precision, the integer n is chosen such that the argument z becomes greater than fifteen and the series involving BERNOULLI numbers, the BINET function, is truncated after $M = 14$ terms.

4. The Interpolation and Extrapolation of the Price Function

One of the two interpolation methods considered in this paper is the *polynomial interpolation*. The constants d_m of equation (2.1) are obtained from the orthogonality conditions (4.1a & b) of the companion paper:

$$d_m = a_m \int_{-\infty}^{+\infty} \omega(x)\, F(x)\, \mathcal{P}_m(x)\, \mathrm{d}x, \quad F(x) := K_j\left(r_j,\, j\cdot\tau+\tau_n\right)^+ \mathrm{e}^{-b\, r_j}, \quad m = 0,\, 1,\, \ldots,\, N_p,$$
$$a_m := \begin{cases} \left[2^m\, m!\, \sqrt{\pi}\right]^{-1} & \text{if } \mathcal{P}_m(x) = H_m(x), \\ \dfrac{m!}{\Gamma(\alpha+m+1)} & \text{if } \mathcal{P}_m(x) = L_m^{(\alpha)}(x), \end{cases} \qquad \omega(x) := \begin{cases} \mathrm{e}^{-x^2} & \text{if } \mathcal{P}_m(x) = H_m(x), \\ x^\alpha\, \mathrm{e}^{-x} & \text{if } \mathcal{P}_m(x) = L_m^{(\alpha)}(x). \end{cases} \tag{4.1}$$

The transformation of variable $r_j := r(x)$, which was defined in equation (2.1) and which is used in the above equation, will be determined later. The orthogonal polynomials are computed by means of their recurrence relations (ABRAMOWITZ AND STEGUN [1965, Table (22.7)]). Again, the terms a_m may be computed recursively:

$$a_m = \begin{cases} \dfrac{a_{m-1}}{2m}, & a_0 = \left[\sqrt{\pi}\right]^{-1}, & \text{if } \mathcal{P}_m(x) = H_m(x), \\ \dfrac{m\, a_{m-1}}{\alpha+m}, & a_0 = \left[\Gamma(\alpha+1)\right]^{-1}, & \text{if } \mathcal{P}_m(x) = L_m^{(\alpha)}(x), \end{cases} \qquad m = 1,\, \ldots,\, N_p. \tag{4.2}$$

The equation (4.1) is integrated with the help of the GAUSSian quadrature rule:

$$\int_{-\infty}^{+\infty} \omega(x)\, F(x)\, \mathcal{P}_m(x)\, \mathrm{d}x \approx \sum_{n=1}^{\nu} w_n\, \omega\left(x_n^{(\nu)}\right) F\left(x_n^{(\nu)}\right) \mathcal{P}_m\left(x_n^{(\nu)}\right), \quad m = 0,\, 1,\, \ldots,\, N_p\,. \tag{4.3}$$

Observe that $x_n^{(\nu)}$ denotes the nth root of the νth-degree orthogonal polynomial under consideration, where the roots (which are all real and single, SZEGö [1939]) are in increasing order $x_1^{(\nu)} < x_2^{(\nu)} < \ldots < x_\nu^{(\nu)}$, $\omega(\cdot)$ denotes the weight function associated with a particular orthogonal polynomial, and w_n denote the GAUSSian weights which correspond to the roots of the orthogonal polynomial in question. The GAUSSian quadrature algorithm of PRESS ET AL. [1989, section 4.5] to compute the GAUSSian weights is *not* reliable because the accuracy of their algorithm deteriorates very rapidly with higher polynomial degrees. To bear out our assertion, here is an example for $\nu = 20$, 30 and 40: first, the GAUSSian weight w_{20} for the HERMITE polynomial of degree $\nu = 20$ is accurate up to only 5 decimal digits. Secondly, the weight w_{29} for the HER-

MITE polynomial of degree $\nu = 30$ turns out to be negative (!) and the weight w_{30} "explodes". Thirdly, with degree $\nu = 40$ the situation becomes even worse, because the weights w_{35} to w_{40} behave like an undampened oscillation with alternating sign. Also, the combined NEWTON-RAPHSON-bisection method for real roots or the LAGUERRE method for complex roots, which are the proposed algorithms of PRESS ET AL. [1989, sections 9.4 and 9.5] to compute the roots of the orthogonal polynomials, are *not* reliable for large polynomial degrees as well. In fact, the roots of the HERMITE polynomial have been computed for the above example with these two algorithms of PRESS ET AL. [1989]. The bracketing values of the roots, which are required for the former method, are provided by SZEGö [1939].

A computationally efficient and numerically reliable method to compute both the roots and the GAUSSian weights has been proposed by GOLUB AND WELSH [1969]. In fact, we do not know a better algorithm. The basic idea is to write the three-term reccurrence relationship of an orthogonal polynomial of νth degree as an equation system which consists of ν equations. The unknowns are the values of the orthogonal polynomials of degree 0 to $[\nu - 1]$, evaluated at arguments x. If the polynomials of lower degrees are evaluated at the roots of the orthogonal polynomial of degree ν, then this equation system transforms into an eigenvalue problem for which the roots of the orthogonal polynomial of degree ν are the eigenvalues of a real, symmetric, and tridiagonal matrix **J**:

$$\mathbf{J} = \begin{bmatrix} \delta_1 & \beta_1 & & & \\ \beta_1 & \delta_2 & \beta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \beta_{\nu-2} & \delta_{\nu-1} & \beta_{\nu-1} \\ & & & \beta_{\nu-1} & \delta_\nu \end{bmatrix}, \quad \text{with} \begin{cases} \left.\begin{aligned} \delta_j &= 0 \\ \beta_j &= \sqrt{j/2} \end{aligned}\right\} & \text{if } \mathcal{P}_m(x) = H_m(x), \\ \left.\begin{aligned} \delta_j &= 2j + \alpha - 1 \\ \beta_j &= \sqrt{j\,[j + \alpha]} \end{aligned}\right\} & \text{if } \mathcal{P}_m(x) = L_m^{(\alpha)}(x). \end{cases} \tag{4.4}$$

GOLUB AND WELSH have shown that the GAUSSian weights are equal to the square of the first element of the normalized eigenvectors, multiplied by the integral of the polynomial weight function, and divided by the polynomial weight function (because we use the transformed GAUSSian weights):

$$w_j = \frac{q_{1,j}^2\,\mu_0}{\omega\left(x_j^{(\nu)}\right)}, \quad \text{with} \quad \mu_0 := \int_{-\infty}^{+\infty} \omega(x)\,\mathrm{d}x = \begin{cases} \sqrt{\pi} & \text{if } \mathcal{P}_m(x) = H_m(x), \\ \Gamma(\alpha+1) & \text{if } \mathcal{P}_m(x) = L_m^{(\alpha)}(x), \end{cases} \tag{4.5}$$

where the weight function $\omega(\cdot)$ has been introduced in equation (4.1) and $q_{1,j}$ denotes the first element of the eigenvector which is associated with the jth eigenvalue. The eigenvectors have unit length, that is, $\mathbf{q}_j' \mathbf{q}_j = 1$ for $j = 1, 2, \ldots, \nu$. Both the eigenvalues and eigenvectors of the real, symmetric and tridiagonal matrix **J** have been computed with the QL algorithm with implicit shifts of PRESS ET AL. [1989, section 11.3]. As a control for accuracy, we reproduced exactly those results for the GAUSS-LAGUERRE quadrature rule which are given in CONCUS ET AL. [1963] and in ABRAMOWITZ AND STEGUN [1965, Table (25.9)], as well as those results for the GAUSS-HERMITE quadrature rule which are given in ABRAMOWITZ AND STEGUN

[1965, Table (25.10)]. GOLUB AND WELSH themselves found a minor difference for the last significant digits which is due to the fact that they applied the QR rather than the QL algorithm with implicit shifts to the tridiagonal matrix **J**.

How many polynomial terms should be considered? Although N_p could be twice the polynomial degree ν in order to yield an exact GAUSSian quadrature, our own computations indicate that the accuracy of the interpolated price function, $K_j(\cdot, \cdot)^+$, turns out to be best for the choice $N_p = \nu - 1$.

Since we need the callable bond prices at the transformed roots $\{r_j^{(k)} := r(x_k^{(\nu)}),$ for $k = 1, \ldots, \nu\}$ in order to perform the GAUSSian quadrature of equation (4.3), it seems natural to use the same roots for the numerical quadrature of GREEN's function of equation (2.5) on each of the call dates as well, that is, $\{r_j^{(k)} := r^{(k)},$ for $j = 2, \ldots, n,$ and $k = 1, \ldots, \nu\}$. Hence, this defines the transformation of equation (2.1): given the choice of the polynomial degree ν, we map linearly all the polynomial roots onto the region of interpolation such that the first root corresponds to the lower interpolation boundary and the last root corresponds to the upper interpolation boundary, that is,

$$\vartheta_1 = \frac{r_l\, x_\nu^{(\nu)} - r_u\, x_1^{(\nu)}}{x_\nu^{(\nu)} - x_1^{(\nu)}}, \quad \vartheta_2 = \frac{r_u - r_l}{x_\nu^{(\nu)} - x_1^{(\nu)}}. \tag{4.6}$$

Observe that the analytical solution (3.18) of the companion paper has also a linear transformation of variable. It may be perceivable to apply the same transformation to the cubic spline too. Our own computations, however, indicate that such a transformation does not improve considerably the interpolation accuracy of the callable bond price function. Hence, we do *not* use this transformation of variable for the exponential cubic spline.

The second interpolation method, which is considered in this paper, is the *cubic spline* for the logarithm of the callable bond prices $y(\cdot) := \ln(K_j(\cdot, \cdot)^+)$. Again, we did *not* use the cubic spline algorithm of PRESS ET AL. [1989, section 3.3] because it leaves open the question of how to find the two missing equations of the cubic spline approach in terms of first derivatives, $y'(\cdot)$. Our choice to complete the equation system is to follow DE BOOR [1978] who equates the third derivatives of two neighbouring third-degree polynomials $\mathfrak{P}_3$ both at the second interpolating point and at the last but one interpolating point. The equation system of the cubic spline may then be written as follows:

$$\begin{bmatrix} b_1 & a_2/2 & & & \\ b_1 & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & b_{\nu-2} & a_{\nu-1} & b_{\nu-1} \\ & & & a_{\nu-1}/2 & b_{\nu-1} \end{bmatrix} \begin{bmatrix} y'_1 \\ y'_2 \\ \vdots \\ y'_{\nu-1} \\ y'_\nu \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{\nu-1} \\ d_\nu \end{bmatrix}, \tag{4.7a}$$

where $y'_j := y'(r^{(j)})$. Here, $\{r_k^{(j)} := r^{(j)},$ for $k = 2, \ldots, n,$ and $j = 1, \ldots, \nu\}$ denote the ν interpolating points or *knots*, respectively. Again, the knots are the same for all the notional call dates $k =$

2, …, n. The different matrix and vector elements are defined as follows, given the abbreviation $y_j := y(r^{(j)})$:

$$\begin{aligned} a_j &= \frac{2}{h_{j-1}} + \frac{2}{h_j}, \quad b_j = \frac{1}{h_j}, \quad d_j = 3\left[\frac{y_{j+1}-y_j}{h_j^2} + \frac{y_j - y_{j-1}}{h_{j-1}^2}\right], \quad h_j = r_k^{(j+1)} - r_k^{(j)}, \\ d_1 &= 2\frac{y_2 - y_1}{h_1^2} + \frac{h_1}{h_1 + h_2}\frac{d_2}{3}, \quad d_v = 2\frac{y_v - y_{v-1}}{h_{v-1}^2} + \frac{h_{v-1}}{h_{v-1} + h_{v-2}}\frac{d_{v-1}}{3}. \end{aligned} \tag{4.7b}$$

The tridiagonal matrix of equation (4.7) is solved by means of the tridiagonal matrix algorithm of PRESS ET AL. [1989, section 2.6]. Given the partial derivatives from equation (4.7), the interpolated callable bond price can be written as:

$$y\big(t(r_j)\big) = a_0 + \big[b_0 + \big[c_0 + d_0 t\big]\big[t - 1\big]\big]\, t, \quad r^{(i)} \leqq r_j < r^{(i+1)}, \quad t := \frac{r_j - r^{(i)}}{r^{(i+1)} - r^{(i)}}, \tag{4.8a}$$

with the meaning that t is the local variable, that $\{r^{(i)}$ for $i = 1, \ldots, v\}$ are the given knots, and that the constants are determined from the following difference scheme:

$$\begin{array}{c} \vdots \\ a_0 = y_i \\ a_1 = y_{i+1} \\ \vdots \end{array} \Big\rangle \begin{array}{c} b_{-1} = h_i\, y'_i \\ b_0 = a_1 - a_0 \\ b_1 = h_i\, y'_{i+1} \end{array} \Big\rangle \begin{array}{c} \\ c_0 = b_0 - b_{-1} \\ c_1 = b_1 - b_0 \end{array} \Big\rangle d_0 = c_1 - c_0. \tag{4.8b}$$

How should the knots $\{r^{(i)}$ for $i = 1, \ldots, v\}$ be chosen? Following DE BOOR [1978], a good choice for the knots are the roots of the CHEBYSHEV polynomial. However, if the number of knots, v, is large, also equidistant knots are feasible. We have tried both types of knots and could not find a significant difference in accuracy for a large number of knots. It should be noted that, in view of the extrapolation approach considered in this paper, it may be conceivable to set the second derivative evaluated at the last knot, y''_v, equal to zero.

Some numerical experiments show that the interpolation by means of series in orthogonal polynomials is extremely accurate, that is, accurate up to 16 – 19 decimal digits, for a moderate number of polynomials, for example $N_p = v - 1 \approx 20$, if the interpolation interval is small, say, [– 0.5, 0.5] in the first case or [0.0, 1.0] in the second case. For comparison, the exponential cubic spline with the same number of knots would yield a much lower accuracy of 5 – 6 decimal digits only. However, if the interval of interpolation grows, the accuracy of the polynomial interpolation deteriorates very rapidly for large interest rates because the orthogonal polynomials under consideration grow very large too. Our own computations indicate that the polynomial interpolation fails if the interpolation interval has a length of more than two, as in the example to follow. Considering the fact that most of the variation of the callable bond price function occurs in the two intervals just mentioned, it seems reasonable to combine the polynomial interpolation with the cubic spline interpolation in real applications. This hybrid interpolation method should reduce the number of interpolating points considerably. In a first subinterval where most of the price variation occurs, the interpolation could be done with relatively few orthogonal polyno-

mials, whereas in the remaining subinterval, the exponential cubic spline could also be applied with relatively few knots.

The *extrapolation* of the callable bond price function assumes that the logarithm of this function is approximately linear for large interest rates. This defines the upper interpolation boundary, r_u. In the example to follow, an inspection of the graph of this function reveals that $r_u = 6$ might be a reasonable choice for both interest-rate models. The first derivative of the logarithm of the callable bond price at the last knot is obtained from the cubic spline of equation (4.7). If the polynomial interpolation is used exclusively, an extra cubic spline may be applied to the interpolating callable bond prices with the polynomial roots as spline knots.

5. The Determination of the "Break-even" Interest Rate

The "break-even" interest rate on each notional call date is the root of equation (2.2). A modified algorithm of PRESS ET AL. [1989, section 9.4] is used, which is a combination of bisection and NEWTON-RAPHSON. The hybrid algorithm takes a bisection step whenever NEWTON-RAPHSON would take the solution out of bounds, or whenever NEWTON-RAPHSON is not reducing the size of the brackets rapidly enough.

The NEWTON-bisection algorithm requires as input both brackets for the real root and the first derivative of the function under consideration. The bracketing of the root is easily done because the interpolating values of both the callable bond price function and the time value of the call price, including the current coupon payment, is altogether the necessary information.

If the polynomial interpolation applies, the partial derivative of the monotone increasing function $\mathcal{F}_j(r_j)$ is given by

$$\frac{\partial \mathcal{F}_j(r_j)}{\partial r_j} = \left[X_j + \eta\right] \frac{\partial P(r_j,\, \tau_n)}{\partial r_j} - \frac{\partial K_j(r_j,\, j\cdot\tau + \tau_n)^+}{\partial r_j}, \quad \frac{\partial P(r_j,\, \tau_n)}{\partial r_j} = g(\tau_n)\, P(r_j,\, \tau_n),$$

$$\frac{\partial K_j(r_j,\, j\cdot\tau + \tau_n)^+}{\partial r_j} = b\, K_j(r_j,\, j\cdot\tau + \tau_n)^+ + \frac{\mathrm{e}^{br_j}}{\vartheta_2} \cdot \begin{cases} 2 \sum_{i=2}^{N_p} d_i \left[i-1\right] H_{i-2}(x), \\ \frac{1}{x} \sum_{i=1}^{N_p} d_i \left[i\, L_i^{(\alpha)}(x) - \left[\alpha + i\right] L_{i-1}^{(\alpha)}(x)\right]. \end{cases} \tag{5.1}$$

The function $g(\cdot)$ is given in equations (1.3a & b) of the companion paper, b is given in equations (2.5a & b) of the companion paper, and x is defined in equation (2.1). The first and second half-lines correspond to the first and second case, respectively. If the exponential cubic spline applies, then

$$\frac{\partial K_j(r_j,\, j\cdot\tau + \tau_n)^+}{\partial r_j} = \frac{\mathrm{d}y(t(r_j))}{\mathrm{d}r_j} \exp\left(y(t(r_j))\right), \quad \frac{\mathrm{d}y(t(r_j))}{\mathrm{d}r_j} = \frac{1}{h_i} \left\{b_0 + c_0\left[2t-1\right] + d_0\left[3t^2 - 2t\right]\right\}, \tag{5.2}$$

should be substituted into equation (5.1). All the symbols have been introduced in equations (4.7) and (4.8) above.

6. The Numerical Quadrature Involving Green's Function

The key operation is the computation of the integral in equation (2.5) by means of numerical quadrature. We suggest to apply the trapezoidal rule as described by WALDVOGEL [1988]. In the case of an analytic integrand, this operation is computationally much more efficient than the ROMBERG integration algorithm which is highly recommended by PRESS ET AL. [1989, section 4.3]. The use of the trapezoidal rule for analytic integrands has been suggested earlier by several authors, see e. g. SCHWARTZ [1969] or IRI ET AL. [1970], but the method entered the textbooks only recently, see e. g. SCHWARZ [1986]. For instance, WALDVOGEL's algorithm achieves machine precision for the probability integrals over the whole range of definition for the normal density function and the non-central chi-square density function, whereas the ROMBERG algorithm fails. However, one cannot expect to accomplish machine precision for equation (2.5) due to the interpolation errors. Similarly, the loss of accuracy might be considerable for, say, a ten-dimensional integral of an analytic function.

The numerical quadrature algorithm considers three standard types of ranges of integration, namely $\mathcal{I}_1 = (-\infty, +\infty)$, $\mathcal{I}_2 = (0, +\infty)$ and $\mathcal{I}_3 = (-1, +1)$. Of course, any integral can be transformed into an integral with one of these three ranges by means of a linear transformation. Suppose that the integral $\mathcal{J} := \int_{t \in \mathcal{I}} f(t)\,\mathrm{d}t$ exists and that $f(t) \to 0$ as $t \to \pm\infty$ if the limits of integration are infinite. Integrable singularities can be treated at one or both limits of integration in the case of the single-infinite range $\mathcal{I} = \mathcal{I}_2$ or the finite range $\mathcal{I} = \mathcal{I}_3$. Each of the three types of integrals is transformed into an improper integral with both limits unbounded, that is

$$\mathcal{J} := \int_{t_{n+1} \in \mathcal{I}} f(t_{n+1})\,\mathrm{d}t_{n+1} = \int_{t_0=-\infty}^{t_0=+\infty} g(t_0)\,\mathrm{d}t_0, \quad (\mathcal{I} = \mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3). \tag{6.1}$$

This is exactly the reverse strategy of what is usually done, see e. g. PRESS ET AL. [1989]. The transformation of variables in the above equation consists of n optional transformations with the sinh-function and one compulsory transformation with a particular function which is suited for each of the three ranges of integration:

$$t_j = \sinh(t_{j-1}), \quad (j = 1, \ldots, n), \quad t_{n+1} = \begin{cases} t_n & \text{if } \mathcal{I} = \mathcal{I}_1, \\ \exp(t_n) & \text{if } \mathcal{I} = \mathcal{I}_2, \\ \tanh(t_n) & \text{if } \mathcal{I} = \mathcal{I}_3. \end{cases} \tag{6.2}$$

In the case of the first type of integral with range $\mathcal{I} = \mathcal{I}_1$, there are, of course, just the n optional trasformations with the sinh-function because the original integral is already improper.

Given the transformations of equation (6.2), the function $g(\cdot)$ of equation (6.1) can be written as:

$$g(t_0) = \begin{cases} f(\sinh(\ldots\sinh(t_0)))\prod_{j=0}^{n-1}\cosh(t_j) & \text{if } \mathcal{I} = \mathcal{I}_1, \\ f(\exp(\sinh(\ldots\sinh(t_0))))\,e^{t_n}\prod_{j=0}^{n-1}\cosh(t_j) & \text{if } \mathcal{I} = \mathcal{I}_2, \\ f(\tanh(\sinh(\ldots\sinh(t_0))))\dfrac{1}{\cosh^2(t_n)}\prod_{j=0}^{n-1}\cosh(t_j) & \text{if } \mathcal{I} = \mathcal{I}_3. \end{cases} \tag{6.3}$$

The integral (6.1) is now evaluated by means of the simplest quadrature scheme, that is, the trapezoidal rule. Given a choice of the initial step size h, the function values, which are evaluated at equally spaced abscissas, are summed in both the positive and negative direction of integration. Given a center of integration c as the starting point, the summation is stopped as soon as the sum does not change any more:

$$T(h) = h\,S(h), \quad \text{with } S(h) := \sum_{\substack{t_0 = c,\\ t_i = t_{i-1} + h}}^{|g(\cdot)| < \varepsilon} g(t_i) + \sum_{\substack{t_0 = c - h,\\ t_i = t_{i-1} - h}}^{|g(\cdot)| < \varepsilon} g(t_i)\,, \tag{6.4}$$

where $\varepsilon = 1.0\cdot 10^{-19}$ is the machine tolerance. Next, the step size is cut in half and the new center of integration is equal to the old center shifted by the new step size. Then the summation is repeated in both directions as in equation (6.4). Next, the step size is cut in half again, the summation is continued, and so forth. The repeated reduction of the step size is stopped when $|\,T(h) - T(h/2)\,| < \sqrt{\varepsilon}$. Then, $T(h/2)$ has accuracy ε. The convergence of the trapezoidal value to the integral is given by $T(h) - \mathfrak{I} = \mathbb{O}(e^{-\gamma/h})$ with γ a positive constant, given an analytic function $g(\cdot)$.

How many transformations with the sinh-function should be used? The number of sinh-transformations, n, which depends on the type of both the integral and the integrand, may be chosen such that the numerical quadrature uses as few function calls as possible. As an example, the number of function evaluations is smallest for the normal probability integral if there is no sinh-transformation used in the case of interval $\mathcal{I}_1$, or if there are two sinh-transformations used in the case of interval $\mathcal{I}_2$, or if there is one sinh-transformation used in the case of interval $\mathcal{I}_3$. In the case of the non-central chi-square probability integral, one or two sinh-transformations are generally efficient. Numerical experiments show that the trapezoidal rule is generally quite efficient if the decay of the integrand $g(t_0)$ as $t_0 \to \pm\infty$ is doubly exponential (see TAKAHASHI AND MORI [1974]).

The second integral (2.5) is transformed into an integral with range $\mathcal{I}_2$,

$$\begin{aligned}\mathfrak{I}_2 &:= \int_{r_j = r_j^*}^{r_j = +\infty} G(r_{j+1},\, \tau,\, r_j)\, K_j\left(r_j,\, j\cdot\tau + \tau_n\right)^+ \mathrm{d}r_j \\ &= \int_{t_{n+1}=0}^{t_{n+1}=+\infty} G(r_{j+1},\, \tau,\, t_{n+1} + r_j^*)\, K_j\left(t_{n+1} + r_j^*,\, j\cdot\tau + \tau_n\right)^+ \mathrm{d}t_{n+1}\,,\end{aligned} \tag{6.5}$$

by means of the shift $r_j = t_{n+1} + r_j^*$.

7. An Example

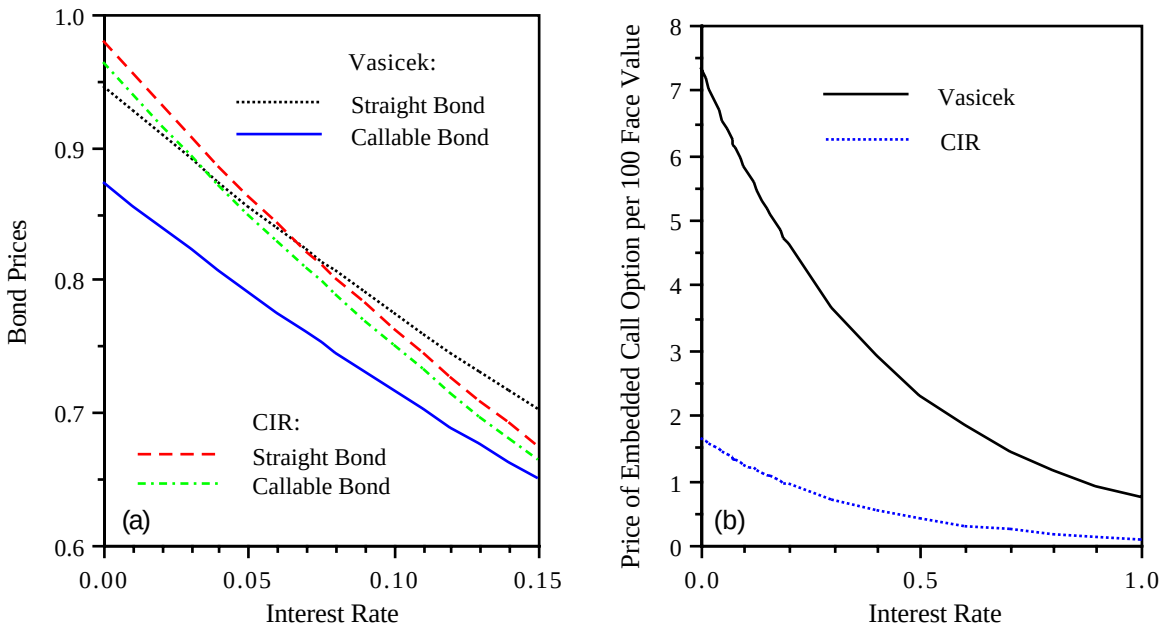
In order to obtain comparable results, the bond valuation models of both VASICEK [1977] and COX, INGERSOLL AND ROSS [1985a & b] have been calibrated to the *same* term structure of interest rates which was *observed* on a particular trading day. In VASICEK's model, the partial derivatives of the theoretical yield curve with respect to two pairs of parameters are linearly dependent. These pairs are either the long-run "equilibrium" value of the instantaneous interest rate, θ, and the market price of interest-rate risk, q, or the the long-run "equilibrium" value of the instantaneous interest rate, θ, and the instantaneous interest rate, r. Since we want to compute the price of the callable bond primarly for the *current* value of the instantaneous interest rate, r_0, this value is chosen as a data of the observed yield curve. Hence, the long-run "equilibrium" value of the instantaneous interest rate, θ, is estimated as the mean value of the instantaneous interest rate over the last twenty years. Although none of the partial derivatives of the theoretical yield curve with respect to the model parameters are linearly dependent in the CIR model, an inspection of these derivatives reveals that the partial derivatives with respect to the speed of adjustment, κ, the risk premium, λ, and the long-run "equilibrium" value of the instantaneous interest rate, θ, are almost linearly dependent. Moreover, these derivatives are linearly dependent when $\kappa = \theta$. Therefore, we estimate again the long-run "equilibrium" value of the instantaneous interest rate, θ, as the mean value of the instantaneous interest rate over the last twenty years. There remain three parameters to be estimated in both bond models. Given three *observed* discount bond yields and the current value of the instantaneous interest rate, these three parameters have been computed from the theoretical yield curves of each of the two models by means of a modified NEWTON-RAPHSON method. The estimated model parameters are reported in Table A.1. Both the market price of interest-rate risk, q, and the risk premium, λ, have the expected sign.

We report the numerical results for an exchange-traded callable bond with ten call dates and with a maximum life of more than twenty years. The characteristics of this callable bond are given in Table A.2. An inspection of the graph of the callable bond price function reveals that $r_u = 6$ might be a reasonable choice for both interest-rate models. The lower interpolation boundary has been chosen to be $r_l = -0.5$ in VASICEK's model in order to accommodate the negative "break-even" interest rates. In the CIR model, a natural choice is zero, that is, $r_l = 0$.

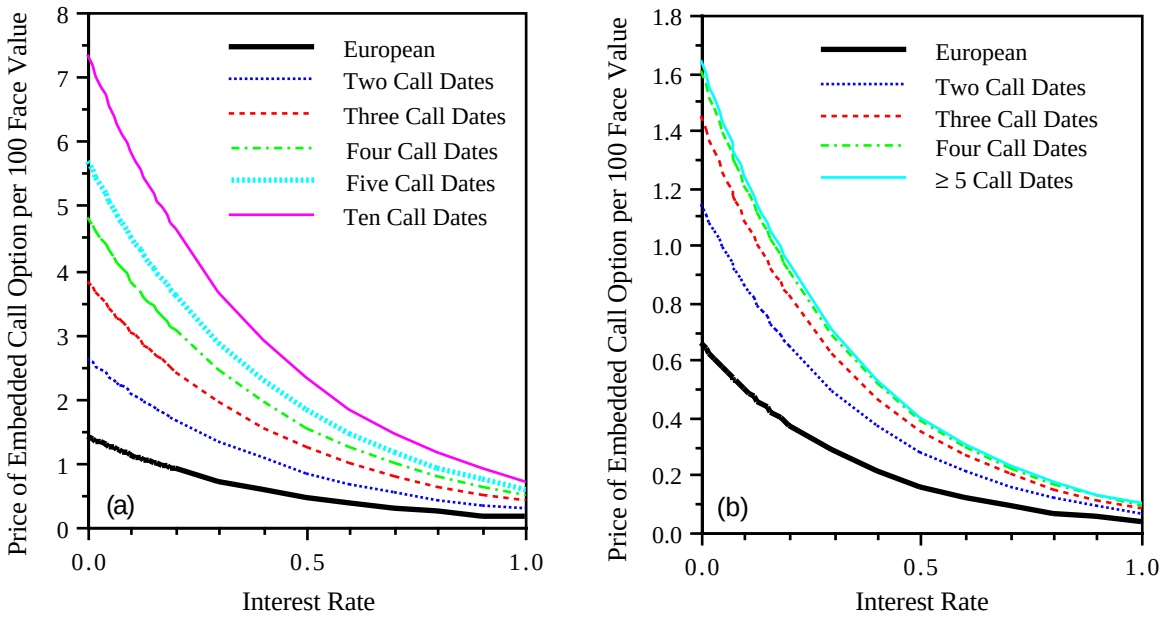
We report the numerical results for the exponential cubic spline interpolation with equidistant knots.

The price of the callable bond given the current value of the instantaneous interest rate, $K(r_0, \hat{\tau} + n\,\tau)$ where $r_0 = 0.07522$ and $\hat{\tau} + n\,\tau = 20.172$ years, is shown in Table A.3 for various numbers of interpolating knots. In VASICEK's model, the price is accurate up to two decimal digits only when 325 interpolating knots are used. A computation with a larger interpolation interval shows that the price of the callable bond which is shown in Table A.3 for 650 knots is accurate up to three significant digits. In the model of CIR, the price is accurate up to six significant digits when 325 knots are used. The difference in accuracy of these two models is due to the fact that there are twice as many integrals to be solved in VASICEK's model because there are only five positive "break-even" interest rates in the model of CIR. The "break-even" interest rates are reported in Table A.4 for the case of 650 interpolating knots. In VASICEK's model, the smallest "break-even" interest rate is – 13.91% and, surprisingly, there is only one positive "break-even" interest rate. As mentioned before, the numerical algorithm leaves the loop to compute the integral involving GREEN's function whenever the "break-even" interest becomes zero or negative in the model of CIR.

The price of the callable bond, the price of the underlying straight bond, and the price of the embedded call option are reported in Table A.5 for both models as a function of the instantaneous interest rate. These results are depicted in Figures 1a & b. More information on the price of various embedded call options is given in Figures 2a & b. These call options refer to callable bonds the characteristics of which are identical to the ones given in Table A.2 except for the number of call dates.



Figures 1a & b: The Price of the Callable Bond, the Underlying Straight Bond (a), and the Embedded Call Option (b). The price of the callable bond has been obtained for 650 knots. The characteristics of the callable bond are given in Table A.2.



Figures 2a & b: The Price of Embedded Call Options for VASICEK's model (a) and for the CIR model (b). The call options refer to callable bonds the characteristics of which are identical to the ones given in Table A.2 except for the number of call dates. The call dates are counted backwards in time. The price of the callable bonds have been obtained for 650 knots in panel (a) and for 325 knots in panel (b).

References

ABRAMOWITZ, MILTON AND IRENE A. STEGUN (eds.) [1965]: *Handbook of Mathematical Functions*, New York: Dover Publications, Inc., ninth edition, 1972.

BÜTTLER, HANS-JÜRG AND JÖRG WALDVOGEL [1993]: "Pricing the European and Semi-American Callable Bond by means of Series Solutions of Parabolic Differential Equations", *Working Paper*, Swiss National Bank, and Applied Mathematics, Swiss Federal Institute of Technology.

CONCUS, P., D. CASSATT, G. JAEHNIG, AND E. MELBY [1963]: "Tables for the Evaluation of $\int_0^{+\infty} x^{\beta} e^{-x} f(x) dx$ by Gauss-Laguerre quadrature", *Mathematics of Computation*, 17: 245 – 265.

COX, JOHN C., JONATHAN E. INGERSOLL, JR., AND STEPHEN A. ROSS [1985a]: "An Inter-temporal General Equilibrium Model of Asset Prices", *Econometrica*, 53: 363 – 384.

COX, JOHN C., JONATHAN E. INGERSOLL, JR., AND STEPHEN A. ROSS [1985b]: "A Theory of the Term Structure of Interest Rates", *Econometrica*, 53: 385 – 407.

DE BOOR, CARL [1978]: *A Practical Guide to Splines*, New York: Springer-Verlag.

GOLUB, GENE H. AND JOHN H. WELSH [1969]: "Calculation of Gauss Quadrature Rules", *Mathematics of Computation*, 23: 221 – 230.

IRI, M., S. MORIGUTI, AND Y. TAKASAWA [1970]: "On a certain Quadrature Formula", *RIMS Kokyuroku Kyoto University*, 91: 82 – 118.

JENSEN, KATHLEEN AND NIKLAUS WIRTH [1982]: *Pascal: User Manual and Report*, New York: Springer-Verlag, second edition.

HENRICI, PETER [1974]: *Applied and Computational Complex Analysis*, New York: John Wiley and Sons, volume 1.

HENRICI, PETER [1977]: *Applied and Computational Complex Analysis*, New York: John Wiley and Sons, volume 2.

HENRICI, PETER [1982]: *Essentials of Numerical Analysis with Pocket Calculator Demonstrations*, New York: John Wiley and Sons.

PRESS, WILLIAM H., BRIAN P. FLANNERY, SAUL A. TEUKOLSKY, AND WILLIAM T. VETTERLING [1989]: *Numerical Recipes in Pascal: The Art of Scientific Computing*, Cambridge (Mass.): Cambridge University Press.

SCHWARTZ, C. [1969]: "Numerical Integration of Analytic Functions", *Journal of Computational Physics*, 4: 19 – 29.

SCHWARZ, HANS RUDOLF [1986]: *Numerische Mathematik*, Stuttgart: B. G. Teubner, 2nd edition.

SZEGÖ, GABOR [1939]: *Orthogonal Polynomials*, Providence (R. I.): American Mathematical Society, Colloquium Publications, volume XXIII.

TAKAHASHI, H. AND M. MORI [1974]: "Double Exponential Formulas for Numerical Integration", *Publications RIMS Kokyuroku Kyoto University*, 9: 721 – 741.

TIKU, M. [1981]: "Noncentral Chi-Square Distribution", in SAMUEL KOTZ AND NORMAN L. JOHNSON (eds.): *Encyclopedia of Statistical Sciences*, New York: John Wiley & Sons.

VASICEK, OLDRICH [1977]: "An Equilibrium Characterization of the Term Structure", *Journal of Financial Economics*, 5: 177 – 188.

WALDVOGEL, JÖRG [1988]: "Numerical Quadrature in Several Dimensions", *International Series of Numerical Mathematics*, 85: 295 – 309, Basel: Birkhäuser Verlag.

Appendix

Table A.1: The Estimated Model Parameters. [1]

Parameter [2]	VASICEK	CIR
κ	0.44178462	0.54958046
σ	0.13264223	0.38757496
q or λ	0.21166329	– 0.40663675
θ	0.0348468515	
r_0	0.0752280589	

[1] Trading day December 23, 1991. The theoretical term structure which is implied by the two models under consideration was perfectly fitted to the four observed yields $\hat{R}_i = \{r_0,\ 0.775395775,\ 0.0664769812,\ 0.0629805885\}$ with $\tau_i = \{0,\ 1,\ 7.175,\ 10.2555555\}$ years to expiration by means of a modified NEWTON-RAPHSON method.

[2] The underlying interest-rate processes are the ORNSTEIN-UHLENBECK process in the case of VASICEK's model, $dr = \kappa(\theta - r)\,dt + \sigma\,dz$, and the square-root process in the case of the CIR model, $dr = \kappa(\theta - r)\,dt + \sigma\sqrt{(r)}\,dz$, where r denotes the instantaneous interest rate, t a point in time, κ the speed of adjustment, θ the long-run "equilibrium" value of the instantaneous interest rate, σ the instantaneous volatility, and dz the GAUSS-WIENER process. The long-run "equilibrium" value of the instantaneous interest rate was estimated as the mean value over the last twenty years; its discrete-time equivalent is 3.546% p. a. The parameters q and λ denote the market price of interest-rate risk in VASICEK's model and the risk premium of the single factor in the CIR model, respectively. Finally, r_0 denotes the current value of the instantaneous interest rate, approximated by the tomorrow-next rate, which was observed on the trading day in question; its discrete-time equivalent is 7.813% p. a.

Table A.2: The Characteristics of the Numerical Example.

Name of security [1]	4 1/4% Swiss Confederation 1987–2012	Year [2]	Call Price
Maximum life of callable bond	20.172 years	1. – 5.	1.000
Notice period	0.1666 years (2 months)	6.	1.005
Interpolation method	Exponential spline with equidistant knots	7.	1.010
Center of integration	0.0	8.	1.015
Initial step size of integration	1.0	9.	1.020
Number of sinh-transformations	1	10.	1.025

[1] Security number 15 718. The bond bears an annual coupon of 4 1/4%. Trading day December 23, 1991.

[2] Ordinal number of the call date when counted backwards in time. The ten call dates are equal to the ten coupon dates prior to the last possible redemption day. The computations are based on the assumption that the debtor makes his call decision on the notice days (notional call dates) which lie two months ahead of the effective call dates.

Table A.3: The Price of the Callable Bond for Various Numbers of Knots. [1]

Number of Knots	VASICEK	CIR
20	7.69009349390688 5004e-1	7.97791748204131 1071e-1
41	7.44975087722577 6772e-1	7.98072875297476 8909e-1
81	7.44965790483799 0726e-1	7.98146760278869 3622e-1
162	7.50069425664897 8132e-1	7.98155146022675 2178e-1
325	7.51150354812794 0533e-1	7.98155680567579 2171e-1
650	7.52082229856677 1664e-1	7.98155703081802 7044e-1
r_u	6.0	6.5
r_l	– 0.5	0.0

[1] Price of the callable bond for the current value of the instantaneous interest rate, r_0. r_u denotes the upper interpolation boundary and r_l the lower interpolation boundary.

Table A.4: The Break-even Interest Rates. [1]

Year [2]	VASICEK	CIR
1.	0.0270644976	0.0338871564
2.	- 0.0101266321	0.0179273733
3.	- 0.0333187039	0.0097892562
4.	- 0.0557203925	0.0048817260
5.	- 0.0743127490	0.0015784739
6.	- 0.0935658091	0
7.	- 0.1088988626	0
8.	- 0.1214749074	0
9.	- 0.1314698228	0
10.	- 0.1391392481	0

[1] Break-even interest rates obtained for 650 knots.

[2] Ordinal number of notional call date (i. e., notice date) counted backwards in time.

Table A.5: The Price of the Callable Bond, the Underlying Straight Bond and the Embedded Call Option. [1]

	VASICEK's model			CIR model		
Interest [2]	Straight Bond	Callable Bond	Call [4]	Straight Bond	Callable Bond	Call [4]
0.000	9.462e-1	8.728e-1	7.346	9.796e-1	9.631e-1	1.644
0.010	9.274e-1	8.556e-1	7.178	9.552e-1	9.392e-1	1.599
0.020	9.089e-1	8.388e-1	7.014	9.315e-1	9.159e-1	1.554
0.030	8.908e-1	8.223e-1	6.853	9.084e-1	8.933e-1	1.511
0.040	8.731e-1	8.062e-1	6.696	8.859e-1	8.712e-1	1.469
0.050	8.558e-1	7.904e-1	6.543	8.641e-1	8.498e-1	1.428
0.060	8.389e-1	7.749e-1	6.393	8.428e-1	8.289e-1	1.388
0.070	8.223e-1	7.598e-1	6.246	8.220e-1	8.085e-1	1.350
0.075 [3]	8.137e-1	7.520e-1	6.171	8.114e-1	7.981e-1	1.330
0.080	8.060e-1	7.450e-1	6.103	8.018e-1	7.887e-1	1.312
0.090	7.901e-1	7.305e-1	5.963	7.822e-1	7.694e-1	1.276
0.100	7.746e-1	7.163e-1	5.827	7.631e-1	7.507e-1	1.240
0.110	7.594e-1	7.024e-1	5.693	7.444e-1	7.324e-1	1.206
0.120	7.445e-1	6.888e-1	5.563	7.263e-1	7.146e-1	1.172
0.130	7.299e-1	6.755e-1	5.436	7.086e-1	6.973e-1	1.140
0.140	7.156e-1	6.625e-1	5.311	6.915e-1	6.804e-1	1.108
0.150	7.016e-1	6.497e-1	5.189	6.747e-1	6.640e-1	1.077
0.160	6.879e-1	6.372e-1	5.070	6.584e-1	6.480e-1	1.047
0.170	6.745e-1	6.250e-1	4.954	6.426e-1	6.324e-1	1.018
0.180	6.614e-1	6.130e-1	4.841	6.271e-1	6.172e-1	0.990
0.190	6.486e-1	6.013e-1	4.730	6.121e-1	6.025e-1	0.962
0.200	6.360e-1	5.898e-1	4.622	5.974e-1	5.881e-1	0.936
0.300	5.240e-1	4.874e-1	3.666	4.705e-1	4.635e-1	0.706
0.400	4.334e-1	4.044e-1	2.907	3.731e-1	3.678e-1	0.532
0.500	3.601e-1	3.370e-1	2.306	2.982e-1	2.942e-1	0.402
0.600	3.006e-1	2.823e-1	1.829	2.405e-1	2.375e-1	0.303
0.700	2.523e-1	2.378e-1	1.450	1.959e-1	1.937e-1	0.228
0.800	2.130e-1	2.015e-1	1.150	1.614e-1	1.597e-1	0.172
0.900	1.810e-1	1.719e-1	0.912	1.345e-1	1.332e-1	0.130
1.000	1.549e-1	1.476e-1	0.724	1.136e-1	1.126e-1	0.098
1.500	7.986e-2	7.759e-2	0.227	5.923e-2	5.899e-2	0.024
2.000	5.028e-2	4.957e-2	0.071	4.056e-2	4.050e-2	0.006
2.500	3.718e-2	3.696e-2	0.022	3.244e-2	3.243e-2	0.001
3.000	3.039e-2	3.032e-2	0.007	2.787e-2	2.786e-2	0.000
3.500	2.623e-2	2.620e-2	0.002	2.471e-2	2.471e-2	0.000
4.000	2.327e-2	2.327e-2	0.001	2.225e-2	2.224e-2	0.000
4.500	2.096e-2	2.096e-2	0.000	2.019e-2	2.019e-2	0.000
5.000	1.904e-2	1.904e-2	0.000	1.841e-2	1.841e-2	0.000
5.500	1.737e-2	1.737e-2	0.000	1.683e-2	1.683e-2	0.000
6.000	1.590e-2	1.590e-2	0.000	1.542e-2	1.542e-2	0.000
6.500	1.458e-2	1.458e-2	0.000	1.414e-2	1.414e-2	0.000
7.000	1.339e-2	1.339e-2	0.000	1.297e-2	1.297e-2	0.000
7.500	1.230e-2	1.230e-2	0.000	1.190e-2	1.190e-2	0.000
8.000	1.131e-2	1.131e-2	0.000	1.093e-2	1.093e-2	0.000

[1] The price of the callable bond is obtained for 650 knots. The figures given below are *not* rounded to four significant digits but merely cut off from the 19-digit results.

[2] Instantaneous interest rate.

[3] Current value of the instantaneous interest rate on the trading day in question, denoted as r_0.

[4] Price of the embedded call option per 100 face value.