Pricing the European and Semi-American Callable Bond by means of Series Solutions of Parabolic Differential Equations

by

HANS-JÜRG BÜTTLER *
Swiss National Bank
and
University of Zurich
Switzerland

AND

JÖRG WALDVOGEL ‡
Seminar for Applied Mathematics
Swiss Federal Institute of Technology
8092 Zurich
Switzerland

Abstract:
This paper derives a closed-form solution of the price of the European and Semi-American callable bond. It uses two popular one-factor models of the term structure of interest rates which have been proposed by VASICEK and COX, INGERSOLL AND ROSS. The series approach to the parabolic partial differential equations allows for both the analytical treatment of the call policy condition and the identification of the boundary conditions which lead to the well-known formulas for the price of the discount bond. The paper is motivated by the fact that the numerical solution of the partial differential equation, which describes the evolution of the price of the callable bond, may cause slowly decaying finite oscillations after each call date, when the price of the callable bond as a function of the prevailing interest rate is a kinked curve due to the early redemption condition.

First version: January, 1993

* Address: Swiss National Bank, 8022 Zurich, Switzerland. Phone (direct dialling): country code–1–6313 417, Fax: country code–1–6313 901.

‡ Phone (direct dialling): country code–1–632 3460, Fax: country code–1–252 3401, E-mail: waldvoge@math.ethz.ch.

1. Introduction

The callable bond is a straight bond with the provision that allows the debtor to buy back or to "call" the bond for a specified amount, the call price, plus the accrued interest since the last coupon date at some time, the call date(s), during the life of the bond (see, e. g., BRENNAN AND SCHWARTZ [1977]). As regards the call dates, three types of callable bonds can be observed in financial markets. The American callable bond may be repurchased at any time on or before the final redemption date, in contrast to the European or Semi-American counterparts, which may only be called at one or several specific dates, respectively. For instance, American callable bonds have predominantly been issued by corporations in the United States, whereas Semi-American callable bonds have solely been issued by both public and private institutions in Switzerland. In the case of Swiss financial markets, the single call date of the European callable bond is the last but one coupon date, whereas the additional call dates of the Semi-American callable bond are the preceding coupon dates. The number of call dates of outstanding Semi-American callable bonds varies between two and ten in Switzerland. Moreover, the Swiss debtor has to consider a notice period of two months ahead of the call date. For any type of the callable bond, the initial call price may be somewhat above the face value of the bond and may decline gradually over time. In the case of the American or Semi-American callable bond, the call provision will be deferred sometimes until the bond has been outstanding for some length of time.

The callable bond can be viewed as a compound security, which consists of an otherwise identical straight bond and of an embedded call option, which is not traded and whose price is, therefore, not observable. The embedded call option, which is written on the underlying straight bond, can be viewed as being "sold" by the initial bondholder to the issuer of the callable bond, the debtor. Hence, the price of the callable bond must be equal to the price of the underlying straight bond less the price of the embedded call option at any time. Or, the callable bond is worth less than the underlying straight bond.

The purpose of this paper is to derive the price of the European and Semi-American callable bond in an analytical form for two different one-factor models of the term structure of interest rates. The two models used in this paper are the one proposed by VASICEK [1977] and the one by COX, INGERSOLL AND ROSS [1985a & b]. Our paper is motivated by the fact that the numerical solution of the partial differential equation, which describes the evolution of the price of the callable bond, may cause slowly decaying finite oscillations after each call date, when the price of the callable bond as a function of the prevailing interest rate is a kinked curve due to the early redemption condition.[1] For instance, GIBSON-ASNER [1990] reports some *negative* computed prices of the embedded call option which must be due to numerical errors,

[1] A description of the partial differential equation of the callable bond's price and of the early redemption condition can be found in BRENNAN AND SCHWARTZ [1977].

and our own computations show some negative computed prices, too. Hence, the analytical solution of the price of the callable bond to be derived in this paper may serve as a benchmark for the numerical evaluation of the partial differential equation.

In VASICEK's model, the instantaneous interest rate, r, serves as the only factor which describes the whole term structure of insterest rates. It is assumed that the instantaneous interest rate can be described by the following ORNSTEIN-UHLENBECK process: [2, 3]

$$\mathrm{d}r = \kappa\,(\theta - r)\,\mathrm{d}t + \sigma\,\mathrm{d}z, \tag{1.1a}$$

where $\kappa > 0$ denotes the speed of adjustment, $\theta > 0$ the long-run "equilibrium" value of the instantaneous interest rate, t the calendar time, $\sigma > 0$ the instantaneous standard deviation of the instantaneous interest rate, and dz the GAUSS-WIENER process. (Our notation is slightly different from that in VASICEK [1977].) The spot interest rate may become negative, but "mean-reverting" will eventually pull it back to its long-run equilibrium value. Given the interest-rate process of equation (1.1a), VASICEK derives the following partial differential equation to determine the price of a (default-free) discount bond, $P(r,\,\tau)$, promising to pay one unit of money on the redemption date:

$$P_\tau = \frac{1}{2}\,\sigma^2\,P_{rr} + \left[\kappa\,(\theta - r) + \sigma q\right] P_r - rP, \tag{1.2a}$$

where the subscripts denote partial derivatives, τ the remaining time period until the redemption of the discount bond, and q the market price of interest-rate risk assumed to be constant. If arbitrage opportunities are ruled out, the market price of interest-rate risk must be the same for all discount bonds of different maturities. Empirically, we would expect q to be positive. Equation (1.2a) is nowhere singular. The analytical solution of the partial differential equation (1.2a) is given in VASICEK as:

$$\begin{aligned} P(r,\,\tau) &= \exp\!\left(f(\tau) + g(\tau)\,r\right), \text{ where} \\ f(\tau) &:= \frac{1}{\kappa}\,R_\infty\left[1 - \mathrm{e}^{-\kappa\tau}\right] - R_\infty\,\tau - \frac{\sigma^2}{4\,\kappa^3}\left[1 - \mathrm{e}^{-\kappa\tau}\right]^2, \\ g(\tau) &:= -\frac{1}{\kappa}\left[1 - \mathrm{e}^{-\kappa\tau}\right], \\ R_\infty &:= \theta + \frac{\sigma\,q}{\kappa} - \frac{1}{2}\,\frac{\sigma^2}{\kappa^2}\,. \end{aligned} \tag{1.3a}$$

The abbreviation R_∞ denotes the yield of a discount bond with infinite time to maturity, i. e., a consol discount bond, and both "e" and "exp" denote the exponential function. In a meaningful model, the (nominal) yield of a consol bond should be positive.

2 Vasicek's derivation of the partial differential equation of the price of a discount bond does not rely on a specific stochastic process for the interest rate. The ORNSTEIN-UHLENBECK process was merely used as an illustration in order to derive an analytical solution of the partial differential equation.

3 The probability density function of the Ornstein-Uhlenbeck process can be found, e. g. , in COX AND MILLER [1965].

In the model of COX, INGERSOLL AND ROSS [1985b], in the following abbreviated as CIR, it is also the instantaneous interest rate, r,[4] that serves as the single factor which describes the whole term structure of insterest rates. The process for the instananeous interest rate is the square-root process:[5]

$$\mathrm{d}r = \kappa\,(\theta - r)\,\mathrm{d}t + \sigma \sqrt{r}\,\mathrm{d}z. \tag{1.1b}$$

All symbols have the same meaning as in equation (1.1a). The spot interest rate cannot become negative. Given the interest-rate process of equation (1.1b), CIR derive the following partial differential equation to determine the price of a discount bond, $P(r, \tau)$:

$$P_\tau = \frac{1}{2}\,\sigma^2\, r\, P_{rr} + \left[\kappa\,\theta - (\kappa + \lambda) r\right] P_r - rP, \tag{1.2b}$$

where λ denotes the risk premium of the single factor which drives the economy, and, similarly, the product $(\lambda\, r)$ is the covariance of changes in the interest rate with percentage changes in optimally invested wealth (the “market portfolio”). The relationship between the market price of interest-rate risk in VASICEK’s model and the factor risk premium in the model of CIR is given by $q = -\,\lambda \sqrt{r}\,/\,\sigma$. Empirically, we would expect λ to be negative. Equation (1.2b) is singular both at $r = 0$ (regular singular) and $r = \infty$ (irregular singular). The analytical solution of the partial differential equation (1.2b) is given in CIR as:

$$P(r, \tau) = f(\tau)\,\exp(g(\tau)\, r), \text{ where}$$

$$\begin{aligned} f(\tau) &:= \left[\frac{2\gamma\, \mathrm{e}^{\frac{\kappa+\lambda+\gamma}{2}\tau}}{2\gamma + [\kappa + \lambda + \gamma]\,[\mathrm{e}^{\gamma\tau} - 1]}\right]^{\frac{2\kappa\,\theta}{\sigma^2}}, \\ g(\tau) &:= -\,\frac{2\,[\mathrm{e}^{\gamma\tau} - 1]}{2\gamma + [\kappa + \lambda + \gamma]\,[\mathrm{e}^{\gamma\tau} - 1]}, \\ \gamma &:= +\sqrt{[\kappa + \lambda]^2 + 2\sigma^2}\,. \end{aligned} \tag{1.3b}$$

Both equations (1.3a) and (1.3b) have been derived for the initial condition $P(r, 0) = 1$, but no boundary conditions have been specified.

The outline of the paper is as follows. In the next section, the differential equations (1.2a & b) will be reduced to the same “normal” form and the two necessary boundary conditions identified which lead to the discount bond prices as given in equations (1.3a & b). In the third section, the separation-of-variables technique will result in KUMMER’s differential equation. In the case of VASICEK’s equation, HERMITE polynomials will be obtained, and in the case of the CIR equation, LAGUERRE polynomials. With the help of the initial condition, the price equa-

---

4 While CIR derive the interest-rate process endogeneously from a general equilibrium model, VASICEK assumes such a process. Moreover, CIR derive the partial differential equation to value a discount bond from the same general equilibrium model, in constrast to VASICEK who develops his partial differential equation from the no-arbitrage condition.

5 The probability density function of the square-root process was first derived by FELLER [1951].

tions (1.3a & b) will be derived in the fourth section. In the fifth section, GREEN's function will be developed. In the last two sections, GREEN's function will be applied to derive the price of the European and Semi-American callable bond for both interest-rate processes as given in equations (1.1a & b).

2. Reduction to Normal Form

In the two models under consideration as given in equations (1.2a & b), the price of the discount bond, $P(r, \tau)$, satisfies the linear parabolic partial differential equation

$$P_\tau = \frac{1}{2}\sigma^2 r^\varepsilon P_{rr} + h(r)\,P_r - rP, \quad \varepsilon \in \{0, 1\}, \tag{2.1}$$

where $h(r) = \kappa\,\theta + \sigma\,q - \kappa\,r$ and $\varepsilon = 0$ in the case of VASICEK's model (to be considered as first case) or $h(r) = \kappa\,\theta - [\kappa + \lambda]\,r$ and $\varepsilon = 1$ in the case of the CIR model (to be considered as second case).

The interval of definition for r is $r \in \mathcal{I}_1$, where $\mathcal{I}_1 = \mathbb{R}$ in the case of VASICEK's model or $\mathcal{I}_1 = (0, \infty)$ in the case of the CIR model, whereas we always assume $\tau \in \mathcal{I}_2 = (0, \infty)$. The region of definition of $P(r, \tau)$ is $\Omega = \mathcal{I}_1 \times \mathcal{I}_2$, i. e., the upper half plane or the first quadrant. In order to obtain a unique solution initial values $P(r, 0)$ must be given for $r \in \mathcal{I}_1$. Also, in general two boundary conditions must be given, namely one on the left and one on the right boundary of $\mathcal{I}_1$ for every $\tau \in \mathcal{I}_2$.

In the following, we will transform equation (2.1) according to

$$P(r, \tau) = \mathrm{e}^{br}\,Q(r, \tau), \quad b < 0, \tag{2.2}$$

which reduces equation (2.1) to "normal" form

$$Q_\tau = \frac{1}{2}\sigma^2 r^\varepsilon Q_{rr} + [-Ar + B]\,Q_r + CQ, \quad \varepsilon \in \{0, 1\}, \tag{2.3}$$

where $A > 0$, $B \gtreqless 0$ and $C < 0$ are given constants.

It can be shown that LONGSTAFF's [1989] one-factor model may be reduced to the normal form of VASICEK's model by means of the following two transformations (i) $\rho = 2\sqrt{r}$ and (ii) $P(\rho, \tau) = \exp(a\,\rho^2 / 2 + b\,\rho)\,Q(\rho, \tau)$.

2.1 First Case

For every $\tau \in \mathcal{I}_2$, the boundary conditions which lead to the price formula (1.3a) are given by:

$$\begin{aligned} &P(r, \tau) \to 0 \text{ as } r \to \infty, \text{ (right boundary)}, \\ &P(r, \tau) = \mathbb{O}\left(\mathrm{e}^{-\vartheta r}\right), \ \vartheta > 0, \text{ as } r \to -\infty, \text{ (left boundary)}. \end{aligned} \tag{2.4a}$$

The right boundary condition has been proposed by BRENNAN AND SCHWARTZ [1977, 1979]. The constants A, B, and C of equation (2.3) as well as the parameter b of equation (2.2) become in the case of VASICEK's model:

$$A = \kappa, \quad B = \kappa\left[R_\infty - \frac{1}{2}\frac{\sigma^2}{\kappa^2}\right], \quad C = -R_\infty, \quad b = -\frac{1}{\kappa} < 0, \tag{2.5a}$$

where R_∞ has been defined in equation (1.3a).

2.2 Second Case

The boundary conditions which lead to the price formula (1.3b) are given by:

$$\begin{aligned} &P(r, \tau) \to 0 \text{ as } r \to \infty, \text{ (right boundary)}, \\ &P_r(r, \tau) \text{ finite as } r \to 0, \text{ (left boundary)}, \end{aligned} \tag{2.4b}$$

for every $\tau \in \mathcal{I}_2$, i. e., only regular solutions will be considered. In the CIR model, the constants A, B, and C as well as the parameter b become

$$A = \gamma, \quad B = \kappa\theta, \quad C = \kappa\theta\,\frac{\kappa + \lambda - \gamma}{\sigma^2}, \quad b = \frac{\kappa + \lambda - \gamma}{\sigma^2} < 0, \tag{2.5b}$$

where γ has been defined in equation (1.3b). The case $b = [\kappa + \lambda + \gamma] / \sigma^2 > 0$ can be excluded in view of the discussion in the next section.

3. Separation of Variables

Separating variables, we construct solutions of equation (2.3) having the product form

$$Q(r, \tau) = R(r)\cdot T(\tau), \tag{3.1}$$

where $R(r)$ satisfies the (homogeneous) boundary conditions (2.4a & b) obtained after the transformation (2.2). This leads to the condition

$$\frac{\frac{1}{2}\sigma^2 r^\varepsilon R''(r) + [-Ar + B]\, R'(r) + CR(r)}{R(r)} = \frac{T'(\tau)}{T(\tau)} = -\zeta, \quad \varepsilon \in \{0, 1\}, \tag{3.2}$$

where ζ is the (constant) separation parameter to be determined, and primes denote derivatives with respect to r or τ, respectively. We first discuss the equation for $R(r)$

$$\frac{1}{2}\sigma^2 r^\varepsilon R''(r) + [-Ar + B]\, R'(r) + [C + \zeta]\, R(r) = 0, \quad \varepsilon \in \{0, 1\}. \tag{3.3}$$

The eigenvalue ζ has to be chosen such that the boundary conditions can be satisfied by an eigenfunction $R(r)$ not identically zero. This is a (generalized) STURM-LIOUVILLE problem on an infinite interval. Both cases may be solved by reducing the above equation to KUMMER's differential equation (ABRAMOWITZ AND STEGUN [1965, chapter 13])

$$z\,\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + [b - z]\,\frac{\mathrm{d}w}{\mathrm{d}z} - a w = 0, \quad (a,\ b \text{ constants}), \tag{3.4}$$

where the transformations $r \to z$ and $w(z) := R(r)$ will be determined below. The (generally) independent solutions are given by

$$\begin{aligned} &w_1(z) = M(a,\ b;\ z), \quad w_2(z) = U(a,\ b;\ z), \text{ where} \\ &M(a,\ b;\ z) := \sum_{n=0}^{\infty} \frac{(a)_n\, z^n}{(b)_n\, n!}, \quad (a)_n := a[a+1][a+2]\ldots[a+n-1], \quad (a)_0 := 1, \\ &U(a,\ b;\ z) := \frac{\Gamma(1-b)}{\Gamma(1-b+a)}\, M(a,\ b;\ z) + \frac{\Gamma(b-1)}{\Gamma(a)}\, z^{1-b}\, M(1-b+a,\ 2-b;\ z), \quad (b \neq 1), \end{aligned} \tag{3.5}$$

where $M(\cdot)$ is the KUMMER function and $U(\cdot)$ the TRICOMI function. We exclude the case $b = 1$.

3.1 First Case

Define the normalized variable x and the transformed eigenfunction $w(x)$ by

$$\sigma \sqrt{A}\ x = A r - B, \quad w(x) := R(r). \tag{3.6a}$$

Then equation (3.3) becomes the differential equation of the HERMITE polynomials

$$\frac{1}{2}\,\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} - x\,\frac{\mathrm{d}w}{\mathrm{d}x} + n w = 0, \quad n := \frac{C + \zeta}{A}, \tag{3.7a}$$

with the transformed boundary conditions obtained from equations (2.4a), (2.2) and (3.1)

$$w(x) = \mathbb{O}(x^k),\ k < \infty \text{ as } x \to \pm\infty. \tag{3.8a}$$

It is well known that the HERMITE polynomials $H_n(x)$

$$w(x) = H_n(x) := n! \sum_{k=0}^{[n/2]} \frac{(-1)^k\,(2x)^{n-2k}}{k!\,(n-2k)!}, \quad n \in \mathbb{N}_0, \tag{3.9a}$$

satisfy the differential equation (3.7a) for every $n \in \mathbb{N}_0$. Here we will consider the eigenvalue problem of equation (3.7a) on the real axis, i. e., we determine all the eigenvalues $n \in \mathbb{R}$ such that the solution of (3.7a) satisfies the boundary conditions (3.8a). It follows that the non-negative integers $n = 0, 1, 2, \ldots$ are eigenvalues, and $w(x) = H_n(x)$ are the corresponding eigenfunctions. We will show that this is the complete spectrum.

First, the general solution of equation (3.7a) is determined by means of the transformation $z = x^2$, thus reducing equation (3.7a) to KUMMER's differential equation (3.4) with $a = -n/2$

and $b = 1/2$. Therefore, the general solution of equation (3.7a) is a linear combination of the (generally) independent solutions

$$w_1(x) = M\left(-\frac{n}{2}, \frac{1}{2}; x^2\right), \quad w_2(x) = U\left(-\frac{n}{2}, \frac{1}{2}; x^2\right), \quad n \in \mathbb{R}. \tag{3.10a}$$

From the asymptotic theory of confluent hypergeometric functions we obtain

$$w_1(x) = \frac{\Gamma\left(\frac{1}{2}\right) e^{x^2}}{\Gamma\left(-\frac{n}{2}\right) x^{n+1}} \left[1 + \mathbb{O}\left(x^{-2}\right)\right] + \mathbb{O}\left(x^n\right) \text{ as } x \to \infty \text{ in } |\arg x| < \frac{3\pi}{4}. \tag{3.11a}$$

Therefore, $w_1(x)$ grows exponentially as $x \to \infty$ except in the cases $n = 0, 2, 4, \ldots$, where $w_1(x)$ is a polynomial and thus satisfies both boundary conditions. However, more eigenfunctions may be produced by means of the solution $w_2(x)$. Using the last line of equation (3.5) with $a = -n/2$, $b = 1/2$ and $z = x^2$ yields

$$w_2(x) = \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1-n}{2}\right)} M\left(-\frac{n}{2}, \frac{1}{2}; x^2\right) + \frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(-\frac{n}{2}\right)} x\, M\left(\frac{1-n}{2}, \frac{3}{2}; x^2\right). \tag{3.12a}$$

From asymptotic theory we have

$$w_2(x) = x^n + \mathbb{O}\left(x^{n-2}\right) \text{ as } x \to \infty \text{ in } |\arg x| < \frac{3\pi}{4}, \tag{3.13a}$$

whereas the asymptotic behaviour of the terms in (3.12a) is given by

$$\begin{aligned} &\frac{\sqrt{\pi}}{2^{n+1}\,\Gamma(-n)} \frac{e^{x^2}}{x^{n+1}} \left[1 + \mathbb{O}\left(x^{-2}\right)\right] + \mathbb{O}\left(x^n\right) \text{ or} \\ &\frac{-\sqrt{\pi}}{2^{n+1}\,\Gamma(-n)} \frac{e^{x^2}}{x^{n+1}} \left[1 + \mathbb{O}\left(x^{-2}\right)\right] + \mathbb{O}\left(x^n\right), \end{aligned} \tag{3.14a}$$

respectively. Since these terms are even or odd functions in x, respectively, their asymptotic behaviour may now be extended to the negative real axis $x < 0$. There follows that the non-negative integers $n = 0, 1, 2, \ldots$ are the only eigenvalues, and that the corresponding eigenfunction $w_2(x)$ is a polynomial. In fact, it may be shown that (ERDéLYI ET AL. [1953, vol. 2, equ. (16), p. 194])

$$w_2(x) = 2^{-n} H_n(x), \quad n \in \mathbb{N}_0. \tag{3.15a}$$

The even eigenfunctions (but not the odd ones) may also be obtained from the solution $w_1(x)$; thus for $n = 0, 2, 4, \ldots$, $w_1(x)$ and $w_2(x)$ are linearly dependent. Other linear combinations of $w_1(x)$ and $w_2(x)$ do not produce new eigenfunctions.

3.2 Second Case

Define the normalized variable x and the transformed eigenfunction $w(x)$ by

$$x = \frac{2Ar}{\sigma^2}, \quad w(x) := R(r). \tag{3.6b}$$

Then equation (3.3) becomes KUMMER's differential equation for $n \in \mathbb{R}$ or the differential equation of the generalized LAGUERRE polynomials for $n \in \mathbb{N}_0$

$$x\frac{d^2w}{dx^2} + [\alpha + 1 - x]\frac{dw}{dx} + nw = 0, \quad n := \frac{C+\zeta}{A}, \quad \alpha + 1 := \frac{2B}{\sigma^2} = \frac{2\kappa\theta}{\sigma^2} > 0, \tag{3.7b}$$

with the transformed boundary conditions obtained from equations (2.4b), (2.2) and (3.1)

$$\begin{aligned} &w(x) = \mathbb{O}(x^k),\ k < \infty \text{ as } x \to +\infty \text{ (right boundary)},\\ &w'(x) \text{ finite as } x \to 0 \text{ (left boundary)}. \end{aligned} \tag{3.8b}$$

It is well known that the generalized LAGUERRE polynomials $L_n^{(\alpha)}(x)$

$$w(x) = L_n^{(\alpha)}(x) := \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!}, \quad n \in \mathbb{N}_0, \tag{3.9b}$$

satisfy the differential equation (3.7b) for every $n \in \mathbb{N}_0$. Here we will consider the eigenvalue problem of equation (3.7b) on the real axis, i. e., we determine all the eigenvalues $n \in \mathbb{R}$ such that the solution of (3.7b) satisfies the boundary conditions (3.8b). It follows that the non-negative integers $n = 0, 1, 2, \ldots$ are eigenvalues, and $w(x) = L_n^{(\alpha)}(x)$ are the corresponding eigenfunctions. We will show that this is the complete spectrum.

The general solution of KUMMER's equation (3.7b) with $a = -n$ and $b = \alpha + 1$ is a linear combination of the (generally) independent solutions

$$w_1(x) = M(-n, \alpha+1; x), \quad w_2(x) = U(-n, \alpha+1; x), \quad n \in \mathbb{R}. \tag{3.10b}$$

Since $w_2'(x) = n\,U(-n+1, \alpha+2; x)$ is unbounded for $b = \alpha + 2 > 1$ as $x \to 0$ (ABRAMOWITZ AND STEGUN [1965, equ. (13.4.21) and (13.5.6) – (13.5.8)]), the second solution does not satisfy the left boundary condition. Since $w_1'(x)$ is finite as $x \to 0$, it satisfies the left boundary condition. From the asymptotic theory of confluent hypergeometric functions we obtain for the first solution (ABRAMOWITZ AND STEGUN [1965, equ. (13.1.4)])

$$w_1(x) = \frac{\Gamma(\alpha+1)}{\Gamma(-n)} e^x x^{-[n+\alpha+1]} [1 + \mathbb{O}(x^{-1})] \text{ as } x \to \infty. \tag{3.11b}$$

Therefore, $w_1(x)$ grows exponentially as $x \to \infty$ except in the cases $n = 0, 1, 2, \ldots$, where $w_1(x)$ is a polynomial and thus satisfies both boundary conditions. In fact, $w_1(x)$ reduces to the generalized LAGUERRE polynomial (ABRAMOWITZ AND STEGUN [1965, equ. (13.6.9)])

$$w_1(x) = \frac{n!}{(\alpha+1)_n} L_n^{(\alpha)}(x), \quad n \in \mathbb{N}_0 . \tag{3.12b}$$

for every $n \in \mathbb{N}_0$. Other linear combinations of $w_1(x)$ and $w_2(x)$ do not produce new eigen-functions.

3.3 Solution

In both cases, the separation parameter ζ becomes (see equations (3.7a & b))

$$\zeta = \zeta_n = nA - C, \quad n \in \mathbb{N}_0 . \tag{3.16}$$

Then the solution of equation (3.2) for $T(\tau)$ is given by

$$T(\tau) = c_n \, e^{[C-nA]\,\tau}, \quad n \in \mathbb{N}_0 . \tag{3.17}$$

where equation (3.16) has been used and c_n is an integration constant. Collecting the results of equations (2.2), (3.1), (3.3), (3.6a & b), (3.9a & b), and (3.17), the solution of (1.2a & b) becomes

$$P(r, \tau) = e^{br} \sum_{n=0}^{\infty} c_n \, e^{[C-nA]\,\tau} \cdot \begin{cases} H_n\!\left(\dfrac{Ar-B}{\sigma\sqrt{A}}\right), \text{ (first case)}, \\ L_n^{(\alpha)}\!\left(\dfrac{2Ar}{\sigma^2}\right), \text{ (second case)}. \end{cases} \tag{3.18}$$

In the next section, the constant c_n will be determined by means of the initial condition.

4. Initial Condition

The constant c_n of equation (3.18) may be determined from the initial condition, $P(r, 0)$, by means of the orthogonality relationship of the two ploynomials $H_n(x)$ or $L_n^{(\alpha)}(x)$, respectively.

4.1 First Case

The orthogonality relationship of the HERMITE polynomial is given by (ABRAMOWITZ AND STEGUN [1965, equ. (22.2.14)])

$$\int_{-\infty}^{+\infty} e^{-x^2} H_n(x)\, H_m(x)\, dx = \begin{cases} 0 & \text{if } n \neq m, \\ 2^n\, n!\, \sqrt{\pi} & \text{if } n = m. \end{cases} \tag{4.1a}$$

Set $\tau = 0$ in equation (3.18), multiply both sides of equation (3.18) by both the weight function of the orthogonality relationship, $\exp(-x^2)$, and $H_m(x)$, integrate and use the orthogonality relationship, then the “FOURIER” coefficient c_n can written as

$$c_m = \frac{2^{-m}}{\sqrt{\pi}\, m!} \int_{-\infty}^{+\infty} e^{-x^2} e^{-br(x)} P(r(x), 0)\, H_m(x)\, dx, \quad r(x) := r = \frac{\sigma \sqrt{A}\, x + B}{A} . \tag{4.2a}$$

Since we want to show the price formula (1.3a), set $P(r, 0) = 1$. Integrate by parts equation (4.2a) repeatedly using the relationship (ABRAMOWITZ AND STEGUN [1965, equ. (22.13.15)])

$$\int e^{-\xi^2} H_m(\xi)\, d\xi = -\, e^{-x^2} H_{m-1}(x), \tag{4.3a}$$

until $H_m(x)$ has been reduced to $H_0(x) = 1$, leaving you with the moment-generating function of the normal probability density function. Then the constant c_m becomes

$$c_m = \frac{(-1)^m\, 2^{-m}}{m!} \left[\frac{b\sigma}{\sqrt{A}} \right]^m \exp\!\left(-\frac{bB}{A} + \frac{b^2\sigma^2}{4A} \right). \tag{4.4a}$$

Substitute c_m into equation (3.18) to get

$$P(r, \tau) = \exp\!\left(br + C\tau - \frac{bB}{A} + \frac{b^2\sigma^2}{4A} \right) \sum_{n=0}^{\infty} \frac{1}{n!} \left[-\frac{b\sigma\, e^{-A\tau}}{2\sqrt{A}} \right]^n H_n\!\left(\frac{Ar - B}{\sigma\sqrt{A}} \right). \tag{4.5a}$$

By means of the generating function (ABRAMOWITZ AND STEGUN [1965, equ. (22.9.17)])

$$\tilde{G}(x, z) := \sum_{n=0}^{\infty} \frac{z^n}{n!} H_n(x) = e^{2xz - z^2} \tag{4.6a}$$

equation (4.5a) may be written as

$$P(r, \tau) = \exp(f(\tau) + g(\tau)\, r), \text{ where} \tag{4.7a}$$

$$f(\tau) := C\tau - \frac{bB}{A}\left[1 - e^{-A\tau}\right] + \frac{b^2\sigma^2}{4A}\left[1 - e^{-2A\tau}\right] = \frac{1}{\kappa} R_\infty \left[1 - e^{-\kappa\tau}\right] - R_\infty\, \tau - \frac{\sigma^2}{4\,\kappa^3}\left[1 - e^{-\kappa\tau}\right]^2 ,$$

$$g(\tau) := b\left[1 - e^{-A\tau}\right] = -\frac{1}{\kappa}\left[1 - e^{-\kappa\tau}\right],$$

$$R_\infty := \theta + \frac{\sigma q}{\kappa} - \frac{1}{2}\frac{\sigma^2}{\kappa^2},$$

where A, B, C and b of equation (2.5a) have been used. This is formula (1.3a) for the price of a discount bond with underlying ORNSTEIN-UHLENBECK process for the instantaneous interest rate.

4.2 Second Case

The orthogonality relationship of the generalized LAGUERRE polynomial is given by (ABRAMOWITZ AND STEGUN [1965, equ. (22.2.12)])

$$\int_0^\infty x^\alpha \,\mathrm{e}^{-x}\, L_n^{(\alpha)}(x)\, L_m^{(\alpha)}(x)\,\mathrm{d}x = \begin{cases} 0 & \text{if } n \neq m, \\ \dfrac{(\alpha+n)!}{n!} & \text{if } n = m. \end{cases} \tag{4.1b}$$

Set $\tau = 0$ in equation (3.18), multiply both sides of equation (3.18) by both the weight function of the orthogonality relationship, $x^\alpha \,\mathrm{e}^{-x}$, and $L_m^{(\alpha)}(x)$, integrate and use the orthogonality relationship, then the “FOURIER” coefficient c_n can written as

$$c_m = \frac{m!}{(\alpha+m)!}\int_0^\infty x^\alpha\,\mathrm{e}^{-\eth x}\, P\!\left(\frac{\sigma^2 x}{2A},\, 0\right) L_m^{(\alpha)}(x)\,\mathrm{d}x, \quad 0 < \eth := 1 + \frac{\sigma^2 b}{2A} < 1. \tag{4.2b}$$

Again, since we want to show the price formula (1.3b), set $P(r,\, 0) = 1$. Integrate equation (4.2b) using the LAPLACE transform (ERDéLYI ET AL. [1954, vol. 1, equ. (28), p. 174])

$$\int_0^\infty \mathrm{e}^{-pt}\, t^\alpha\, L_n^{(\alpha)}(t)\,\mathrm{d}t = \frac{\Gamma(\alpha+n+1)}{n!}\,\frac{[p-1]^n}{p^{\alpha+n+1}}, \quad \Re\alpha > -1,\ \Re p > 0, \tag{4.3b}$$

then the constant c_m becomes

$$c_m = \eth^{-\alpha-1}\left[1 - \frac{1}{\eth}\right]^m. \tag{4.4b}$$

Substitute c_m into equation (3.18) to get

$$P(r,\,\tau) = \exp(br + C\tau)\,\eth^{-\alpha-1}\sum_{n=0}^{\infty}\left[\frac{\eth-1}{\eth}\,\mathrm{e}^{-A\tau}\right]^n L_n^{(\alpha)}\!\left(\frac{2Ar}{\sigma^2}\right). \tag{4.5b}$$

By means of the generating function (ABRAMOWITZ AND STEGUN [1965, equ. (22.9.15)])

$$\widetilde{G}(x,\,z) := \sum_{n=0}^{\infty} z^n\, L_n^{(\alpha)}(x) = [1-z]^{-\alpha-1}\exp\!\left(\frac{xz}{z-1}\right),\ |z| < 1 \tag{4.6b}$$

equation (4.5b) may be written as

$$P(r,\,\tau) = f(\tau)\exp(g(\tau)\,r), \text{ where} \tag{4.7b}$$

$$f(\tau) := \mathrm{e}^{C\tau}\left[\eth - [\eth-1]\,\mathrm{e}^{-A\tau}\right]^{-\frac{2B}{\sigma^2}} = \left[\frac{2\gamma\,\mathrm{e}^{\frac{\kappa+\lambda+\gamma}{2}\tau}}{2\gamma + [\kappa+\lambda+\gamma]\,[\mathrm{e}^{\gamma\tau}-1]}\right]^{\frac{2\kappa\theta}{\sigma^2}},$$

$$g(\tau) := \frac{b\,\eth\,[1-\mathrm{e}^{-A\tau}]}{\eth - [\eth-1]\,\mathrm{e}^{-A\tau}} = -\frac{2\,[\mathrm{e}^{\gamma\tau}-1]}{2\gamma + [\kappa+\lambda+\gamma]\,[\mathrm{e}^{\gamma\tau}-1]},$$

$$\gamma := +\sqrt{[\kappa+\lambda]^2 + 2\sigma^2},$$

where A, B, C and b of equation (2.5b), and $\eth$ of equation (4.2b) have been used. This is formula (1.3b) for the price of a discount bond with underlying square-root process for the instantaneous interest rate.

5. GREEN's Function

In order to deal with the early redemption condition of the callable bond analytically, we derive GREEN's function in this section. The starting point are equations (4.2a & b), the integrals of which are transformed back from the auxiliary variable x to the original variable r.

5.1 First Case

In the case of the ORNSTEIN-UHLENBECK process for the instantaneous interest rate the transformation of variable is

$$x = \frac{A\,r - B}{\sigma\sqrt{A}} \tag{5.1a}$$

and the "FOURIER" coefficient can be written as

$$c_m = \frac{2^{-m}}{\sqrt{\pi}\,m!}\,\frac{\sqrt{A}}{\sigma}\int_{-\infty}^{+\infty}\exp\left(-\left[\frac{A\,r - B}{\sigma\sqrt{A}}\right]^2\right)\mathrm{e}^{-br}\,P(r,\,0)\,H_m\!\left(\frac{A\,r - B}{\sigma\sqrt{A}}\right)\mathrm{d}r\,. \tag{5.2a}$$

The initial data is set equal to the DIRAC delta function $\delta(r-\rho)$. Hence, equation (5.2a) becomes (5.3a).

$$c_m = \frac{2^{-m}}{\sqrt{\pi}\,m!}\,\frac{\sqrt{A}}{\sigma}\exp\left(-\left[\frac{A\,\rho - B}{\sigma\sqrt{A}}\right]^2\right)\mathrm{e}^{-b\rho}\,H_m\!\left(\frac{A\,\rho - B}{\sigma\sqrt{A}}\right). \tag{5.3a}$$

Substituting equation (5.3a) into equation (3.18) leads to GREEN's function $G(r,\,\tau,\,\rho)$:

$$\begin{aligned} G(r,\,\tau,\,\rho) = {} & \frac{\sqrt{A}}{\sigma\sqrt{\pi}}\exp\left(b\,r + C\,\tau - \left[\frac{A\,\rho - B}{\sigma\sqrt{A}}\right]^2 - b\,\rho\right) \\ & \cdot\sum_{n=0}^{\infty}\frac{\left[\frac{\mathrm{e}^{-A\,\tau}}{2}\right]^n}{n!}\,H_n\!\left(\frac{A\,\rho - B}{\sigma\sqrt{A}}\right)H_n\!\left(\frac{A\,r - B}{\sigma\sqrt{A}}\right). \end{aligned} \tag{5.4a}$$

The series in the above equation can be solved by means of the following generating function (ERDéLYI ET AL. [1953, vol. 2, equ. (22), p. 194]):

$$\tilde{G}(x,\,y,\,z) := \sum_{n=0}^{\infty}\frac{z^n}{n!}\,H_n(x)\,H_n(y) = \frac{1}{\sqrt{1-4z^2}}\exp\left(\frac{4xyz - 4z^2\left[x^2+y^2\right]}{1-4z^2}\right), \quad |z| < 1. \tag{5.5a}$$

Substituting the above equation into (5.4a) yields GREEN's function in the case of the ORNSTEIN-UHLENBECK process after some algebraic manipulations:

$$G(r,\tau,\rho) = P(r,\tau)\left\{\frac{1}{s(\tau)\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left[\frac{\rho-\mu-s(\tau)^2\,t(r,\tau)}{s(\tau)}\right]^2\right)\right\}, \text{ where}$$

$$t(r,\tau) := \frac{2\,[A\,r-B]}{\sigma^2}\,\frac{e^{-A\,\tau}}{1-e^{-2A\,\tau}} - b = \frac{2\,[\kappa r-B]}{\sigma^2}\,\frac{e^{-\kappa\tau}}{1-e^{-2\kappa\tau}} + \frac{1}{\kappa}, \tag{5.6a}$$

$$\mu := \frac{B}{A} = R_\infty - \frac{1}{2}\,\frac{\sigma^2}{\kappa^2},$$

$$s(\tau)^2 := \frac{1}{2}\,\frac{\sigma^2}{A}\left[1-e^{-2A\,\tau}\right] = \frac{1}{2}\,\frac{\sigma^2}{\kappa}\left[1-e^{-2\kappa\tau}\right].$$

$P(r,\tau)$ and R_∞ denote the price of a discount bond and the yield of a discount bond with infinite time to maturity, respectively, as given in equation (1.3a). Hence, GREEN's function is the discounted value of the normal probability density function of the instantaneous interest rate on the initial date, conditional on the current value of the instantaneous interest rate.

5.2 Second Case

In the case of the square-root process for the instantaneous interest rate the transformation of variable is

$$x = \frac{2A}{\sigma^2}\,r \tag{5.1b}$$

and the “FOURIER” coefficient can be written as

$$c_m = \frac{m!}{(\alpha+m)!}\,\frac{2A}{\sigma^2}\int_0^\infty \left[\frac{2A}{\sigma^2}\,r\right]^\alpha \exp\left(-\,\delta\,\frac{2A}{\sigma^2}\,r\right) P(r,0)\,L_m^{(\alpha)}\left(\frac{2A}{\sigma^2}\,r\right) dr\,. \tag{5.2b}$$

Again, the initial data is set equal to the DIRAC delta function $\delta(r-\rho)$. Hence, equation (5.2b) becomes (5.3b).

$$c_m = \frac{m!}{(\alpha+m)!}\left[\frac{2A}{\sigma^2}\right]^{\alpha+1} \rho^\alpha \exp\left(-\left[\frac{2A}{\sigma^2}+b\right]\rho\right) L_m^{(\alpha)}\left(\frac{2A}{\sigma^2}\,\rho\right). \tag{5.3b}$$

Substituting equation (5.3b) into equation (3.18) leads to GREEN's function $G(r,\tau,\rho)$:

$$\begin{aligned} G(r,\tau,\rho) = {} & \left[\frac{2A}{\sigma^2}\right]^{\alpha+1} \exp\left(b r + C\tau - \left[\frac{2A}{\sigma^2}+b\right]\rho\right)\rho^\alpha \\ & \cdot \sum_{n=0}^\infty \frac{n!}{(\alpha+n)!}\left[e^{-A\,\tau}\right]^n L_n^{(\alpha)}\left(\frac{2A}{\sigma^2}\,\rho\right) L_n^{(\alpha)}\left(\frac{2A}{\sigma^2}\,r\right). \end{aligned} \tag{5.4b}$$

The series in the above equation can be solved by means of the following generating function (ERDéLYI ET AL. [1953, vol. 2, equ. (20), p. 189]):

$$\widetilde{G}(x, y, z) := \sum_{n=0}^{\infty} \frac{n!\ \alpha!}{(\alpha + n)!} z^n L_n^{(\alpha)}(x)\, L_n^{(\alpha)}(y) = \alpha!\, [xyz]^{-\alpha/2} [1 - z]^{-1} \exp\left(\frac{[x + y]\, z}{z - 1}\right) I_\alpha\left(\frac{2\sqrt{xyz}}{1 - z}\right), \quad |z| < 1, \tag{5.5b}$$

where $I_\alpha(\cdot)$ denotes the modified BESSEL function of the first kind of order α. Substituting the above equation into (5.4b) yields GREEN's function in the case of the square-root process as follows:

$$G(r, \tau, \rho) = P(r, \tau)\, 2p(\tau)\, H(2p(\tau)\rho \,|\, 2\alpha + 2, \lambda(r, \tau)), \text{ where}$$

$$p(\tau) := \frac{\left[2A/\sigma^2\right] + b\left[1 - e^{-A\tau}\right]}{1 - e^{-A\tau}} = \frac{\kappa + \lambda + \gamma - [\kappa + \lambda - \gamma]\, e^{-\gamma\tau}}{\sigma^2\left[1 - e^{-\gamma\tau}\right]} > 0, \tag{5.6b}$$

$$\lambda(r, \tau) := \frac{2}{p(\tau)} \frac{\left[2A/\sigma^2\right]^2 e^{-A\tau} r}{\left[1 - e^{-A\tau}\right]^2} = \frac{8\gamma^2 e^{-\gamma\tau} r}{\sigma^2\left\{2\gamma + [\kappa + \lambda - \gamma]\left[1 - e^{-\gamma\tau}\right]\right\}\left[1 - e^{-\gamma\tau}\right]} \geqq 0,$$

$$H(x|\nu, \lambda) := \frac{1}{2}\left[\frac{x}{\lambda}\right]^{\frac{1}{4}(\nu-2)} e^{-\frac{1}{2}(\lambda + x)} I_{\frac{1}{2}(\nu-2)}\left(\sqrt{\lambda x}\right) = \sum_{n=0}^{\infty} \frac{\left[\lambda/2\right]^n e^{-\frac{1}{2}(\lambda + x)}\, x^{\frac{\nu}{2} + n - 1}}{2^{\frac{\nu}{2} + n}\, n!\, \Gamma\left(\frac{\nu}{2} + n\right)}.$$

GREEN's function is equal to the price of a discount bond, $P(r, \tau)$, as given in equation (1.3b), muliplied by the conditional probability density function of the instantaneous interest rate on the initial date. In this case, the probability density function is the non-central chi-square distribution, $2\, p(\tau)\, H(x \mid \nu, \lambda)$ for $x, \lambda \geqq 0$ and $\nu > 0$, with ν degrees of freedom and noncentrality parameter λ, see e. g. JOHNSON AND KOTZ [1970, vol. 2, equ. (28.3) and (28.5)].

5.3 The Price of a Discount Bond

The price of a discount bond with remaining time to expiration τ as a function of the instantaneous interest rate today, r, can be obtained from integrating GREEN's function over the initial data:

$$P(r, \tau) = \int_{-\infty}^{+\infty} G(r, \tau, \rho)\, P(\rho, 0)\, d\rho\ . \tag{5.7}$$

In particular, if the initial data is equal to the face value, $P(r, 0) = 1$, then integration of the above equation yields the discount bond prices of equations (1.3a & b).

6. The European Callable Bond

Suppose the single call date is the last but one coupon date and that there is a notice period τ_n, which is, in general, two months (see Figure 1). Let $K(r, \hat{\tau} + \tau_n + \tau)$ denote the price of the callable bond with remaining time until the last possible redemption date of length $[\hat{\tau} + \tau_n + \tau]$

as a function of the instantaneous interest rate today, $r := r(t_0)$. Let X denote the call price per face value, η the annual coupon payment per face value, and $\rho := \rho(t_n)$ the instantaneous interest rate on the notice day, t_n, when the debtor has to make the choice of whether or not to call the bond.

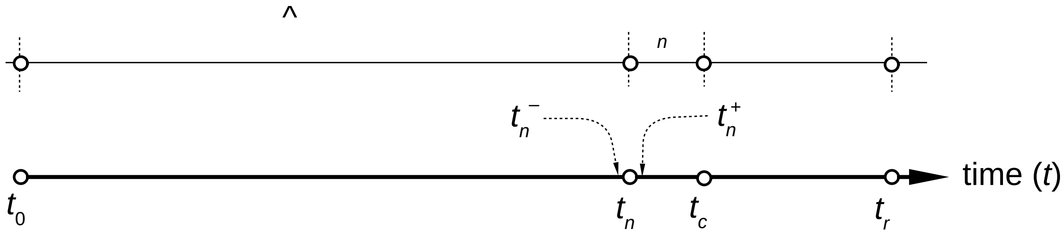

Figure 1: European Callable Bond. t_0 is the present date, t_n the notice date, t_c the call date, and t_r the last possible redemption date. The call date is the last but one coupon date, i. e., τ is equal to one year. t_n^- and t_n^+ denote an instant before and after the notice date, respectively.

If the bond is not called, the price of the callable bond will be equal to the price of the underlying straight bond on the last possible redemption date, t_r, that is, equal to the face value plus the coupon payment, $[1 + \eta]$. An instant before the notice date when moving backwards in time, the price of the callable bond is equal to the time value of the underlying straight bond:

$$K(\rho,\ \tau_n + \tau)^+ = [1 + \eta]\ P(\rho,\ \tau_n + \tau) + \eta\ P(\rho,\ \tau_n) . \tag{6.1}$$

The time value of the call price plus the coupon payment on the last but one coupon date is given by $[X + \eta]{\cdot}P(\rho,\ \tau_n)$. The optimal call policy requires that the issuer of the callable bond minimizes his outstanding debt. Therefore, he will call the bond if the price of the callable bond is greater than the time value of the call price including the coupon payment. Hence, the price of the callable bond an instant after the notice day (when moving backwards in time) is given by

$$K(\rho,\ \tau_n + \tau)^- = \begin{cases} [X + \eta]\ P(\rho,\ \tau_n), & \text{for } \ \rho \leqq \rho^*, \\ K(\rho,\ \tau_n + \tau)^+, & \text{for } \ \rho \geqq \rho^*, \end{cases} \tag{6.2}$$

where the "break-even" interest rate ρ^* is determined from setting the price of the callable bond an instant before the notice date equal to the time value of the call price:

$$[X + \eta]\ P(\rho^*,\ \tau_n) = K(\rho^*,\ \tau_n + \tau)^+ . \tag{6.3}$$

Using the price of the callable bond an instant after the notice date as the initial data, the price of the callable bond today can be obtained from GREEN's function as:

$$K(r,\ \hat{\tau} + \tau_n + \tau) = \int_{-\infty}^{+\infty} G(r,\ \hat{\tau},\ \rho)\ K(\rho,\ \tau_n + \tau)^-\, \mathrm{d}\rho + \eta \sum_{j=2}^{m} P(r,\ t_r - j - t_0) . \tag{6.4}$$

The last term in the above equation is the present value of the coupon stream after the notice date (when moving backwards in time), where m denotes the number of entire years of the remaining time until the last possible redemption date (see Figure 2).

6.1 First Case

We first determine the "break-even" interest rate on the notice day. Substituting equations (1.3a) and (6.1) into equation (6.3), one obtains:

$$\rho^* = -\frac{\ln\left(X/[1+\eta]\right) - [f(\tau_n+\tau) - f(\tau_n)]}{[g(\tau_n) - g(\tau_n+\tau)]}\,, \tag{6.5a}$$

where the functions $f(\cdot)$ and $g(\cdot)$ are given in equation (1.3a). In this and the next section, we will use the following lemma which simplifies the integration to be performed with GREEN's function.

LEMMA 1a: *Suppose the length of the remaining time until an intermediary date, which serves as "initial" date, is τ_2, where the initial condition is given by the price of a discount bond with remaining time until expiration of length τ_1, then*

$$\begin{aligned}
&\int_a^b G(r,\,\tau_2,\,\rho)\,P(\rho,\,\tau_1)\,\mathrm{d}\rho \\
&\quad = P(r,\,\tau_2)\exp\!\left(f(\tau_1) + \mu\,g(\tau_1) + s(\tau_2)^2\,\ell(r,\,\tau_2)\,g(\tau_1) + \tfrac{1}{2}\,s(\tau_2)^2\,g(\tau_1)^2\right)\mathfrak{N} \\
&\quad = P(r,\,\tau_1+\tau_2)\,\mathfrak{N}\,,\ \text{where} \\
&\mathfrak{N} := \frac{1}{s(\tau_2)\sqrt{2\pi}}\int_a^b \exp\!\left(-\frac{1}{2}\left[\frac{\rho - \mu - s(\tau_2)^2\,[\ell(r,\,\tau_2) + g(\tau_1)]}{s(\tau_2)}\right]^2\right)\mathrm{d}\rho\,.
\end{aligned} \tag{6.6a}$$

Proof: Substituting equations (5.6a) and (1.3a) into equation (6.6a) yields the first equality. Let $a \to -\infty$ and $b \to +\infty$, then $\mathfrak{N}$ is equal to one due to the normal probability density function. Since $\int_{-\infty}^{+\infty} G(r,\,\tau_2,\,\rho)\,P(\rho,\,\tau_1)\,\mathrm{d}\rho = P(r,\,\tau_1+\tau_2)$, the so-called semi-group property discussed in FELLER [1952] which can be verified by straightforward algebraic manipulations, the second equality of the lemma follows. □

Substitute the price of the callable bond an instant after the notice date (when moving backwards in time) as given in equations (6.2) and (1.3a) into equation (6.4), and apply lemma 1a, then the price of the European callable bond today with remaining time until the last possible redemption date of length $[\hat{\tau} + \tau_n + \tau]$ becomes

$$
\begin{aligned}
K(r,\,\hat{\tau}+\tau_n+\tau) &= X\,P(r,\,\hat{\tau}+\tau_n)\,\mathcal{N}(d_1) + [1+\eta]\,P(r,\,\hat{\tau}+\tau_n+\tau)\,\mathcal{N}(d_2) + \eta\sum_{j=1}^{m} P(r,\,t_r-j-t_0)\,,\\
\mathcal{N}(d) &:= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d}\exp\!\left(-\frac{1}{2}x^2\right)\mathrm{d}x\,,\ \text{(standard normal probability distribution)}\\
d_1 &:= \frac{\rho^* - \mu - s(\hat{\tau})^2\left[t(r,\,\hat{\tau}) + g(\tau_n)\right]}{s(\hat{\tau})}\,,\\
d_2 &:= -\,d_1 + s(\hat{\tau})\left[\,g(\tau+\tau_n) - g(\tau_n)\right].
\end{aligned}
\tag{6.7a}
$$

The first term represents the present value of the call price weighted by the probability that the bond will be called early, the second term represents the present value of the callable bond's payoff on the last possible redemption date weighted by the probability that the bond will not be called early, and the third term represents the present value of the coupon stream during the time the bond cannot be called. The last but one coupon will be received in any case whether or not the bond will be called early.

6.2 Second Case

Substituting equations (1.3b) and (6.1) into equation (6.3), one obtains for the "break-even" interest rate:

$$
\rho^* = -\,\frac{\ln\!\left(\dfrac{X\,f(\tau_n)}{[1+\eta]\,f(\tau_n+\tau)}\right)}{[g(\tau_n) - g(\tau_n+\tau)]}\,,
\tag{6.5b}
$$

where the functions $f(\cdot)$ and $g(\cdot)$ are given in equation (1.3b). In the following we assume that the "break-even" interest rate is positive. Similar to the first case, the following lemma simplifies the integration to be performed with GREEN's function in this and the next section.

LEMMA 1b: *Suppose the length of the remaining time until an intermediary date, which serves as "initial" date, is τ_2, where the initial condition is given by the price of a discount bond with remaining time until expiration of length τ_1, then*

$$
\begin{aligned}
\int_a^b G(r,\,\tau_2,\,\rho)\,P(\rho,\,\tau_1)\,\mathrm{d}\rho &= P(r,\,\tau_2)\,f(\tau_1)\,\Pi(\tau_1,\,\tau_2)^{\alpha+1}\exp\!\left(-\frac{1}{2}\left[\lambda(r,\,\tau_2)\,\{1-\Pi(\tau_1,\,\tau_2)\}\right]\right)\mathfrak{H}\\
&= P(r,\,\tau_1+\tau_2)\,\mathfrak{H}\,,\ \text{where}\\
\mathfrak{H} &:= 2\,[p(\tau_2) - g(\tau_1)]\int_a^b H\big(2\,[p(\tau_2)-g(\tau_1)]\,\rho\,|\,2\alpha+2,\,\Pi(\tau_1,\,\tau_2)\,\lambda(r,\,\tau_2)\big)\,\mathrm{d}\rho\,,\\
\Pi(\tau_1,\,\tau_2) &:= \frac{p(\tau_2)}{p(\tau_2) - g(\tau_1)}\,.
\end{aligned}
\tag{6.6b}
$$

The proof follows the same lines as in that of lemma 1a. The price of the European callable bond today with remaining time until the last possible redemption date of length $[\hat{\tau} + \tau_n + \tau]$ follows from substituting equations (6.2) and (1.3b) into equation (6.4) and from applying lemma 1b:

$$
\begin{aligned}
K(r,\hat{\tau}+\tau_n+\tau) &= X\,P(r,\hat{\tau}+\tau_n)\,\mathcal{H}\left(d_1 \mid 2\alpha+2,\,\Lambda(r,\hat{\tau},\tau_n)\right) + \left[1+\eta\right] P(r,\hat{\tau}+\tau_n+\tau) \\
&\quad \cdot\left[1-\mathcal{H}\left(d_2 \mid 2\alpha+2,\,\Lambda(r,\hat{\tau},\tau_n+\tau)\right)\right] + \eta \sum_{j=1}^{m} P(r,t_r-j-t_0)\ \text{, where} \\
\mathcal{H}(d \mid \nu,\lambda) &:= \int_0^d H(x \mid \nu,\lambda)\,\mathrm{d}x\ \text{, (non-central chi-square probability distribution)} \\
d_1 &:= \rho^*\,2\left[p(\hat{\tau})-g(\tau_n)\right], \\
d_2 &:= \rho^*\,2\left[p(\hat{\tau})-g(\tau_n+\tau)\right], \\
\Lambda(r,\tau_1,\tau_2) &:= \frac{p(\tau_1)}{p(\tau_1)-g(\tau_2)}\,\lambda(r,\tau_1)\,.
\end{aligned}
\tag{6.7b}
$$

The interpretation of the different terms is the same as in the first case.

7. The Semi-American Callable Bond

Suppose there are n call dates identical to the coupon dates $[t_r-j]$ for $j = 1, 2, \ldots, n$ (see Figure 2). For the sake of simplicity, the notice period is disregarded. Let the call prices be denoted as $X_j := X(t_r-j)$ for $j = 1, 2, \ldots, n$, the instantaneous interest rates prevailing on the call dates $[t_r-j]$ as $r_j := r(t_r-j)$ for $j = 1, 2, \ldots, n$, the instantaneous interest rate today as $r := r(t_0)$, the time periods between call dates as τ (one year), the time period between today and the last call date (when moving backwards in time) as $\hat{\tau}$, and the number of whole years of the remaining time until the last possible redemption date as m.

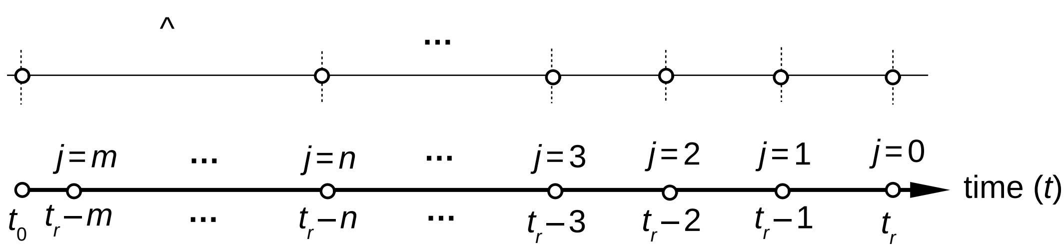

Figure 2: Semi-American Callable Bond. t_0 is the present date and t_r the last possible redemption date. The call dates are the n coupon dates t_r-j, for $j = 1, 2, \ldots, n$. There is no notice period. The number of whole years of the remaining time until the last possible redemption date of length $[t_r-t_0]$ is m.

As for the European callable bond, we proceed backwards in time. On each call date, the price of the callable bond an instant before the call date, $K(r_j, j\cdot\tau)^+$, is determined from the previous call date by applying GREEN's function. The "break-even" interest rate, r_j^*, equates the price of the callable bond, $K(r_j, j\cdot\tau)^+$, and the call price, X_j. Again, the optimal call policy requires the price of the callable bond an instant after the call date to be the minimum of these two prices, including the additional coupon payment on that call date. In summary, the procedure goes as follows:

$$\begin{aligned}
&K(r_j, j\cdot\tau)^+ = \int_{-\infty}^{+\infty} G(r_j, \tau, r_{j-1})\, K\big(r_{j-1}, [j-1]\,\tau\big)^-\, \mathrm{d}r_{j-1}\,, \quad j = 1, 2, \ldots, n,\\
&K(r_j^*, j\cdot\tau)^+ = X_j \;\Rightarrow\; r_j^*, \qquad\qquad (7.1)\\
&K(r_j, j\cdot\tau)^- = \begin{cases} X_j + \eta, & \text{for } r_j \leqq r_j^*,\\ K(r_j, j\cdot\tau)^+ + \eta, & \text{for } r_j \geqq r_j^*. \end{cases}
\end{aligned}$$

On the last possible redemption date, t_r, the price of the callable bond is equal to its face value plus the last coupon payment, that is, $K(r_0, 0) = [1 + \eta]$ with $r_0 := r(t_r)$. On the last call date, $t_r - n$, the final time step is taken with $\hat{\tau}$ rather than τ in equation (7.1).

7.1 First Case

We first determine the price of the callable bond on each call date, which is needed to determine the “break-even” interest rate prevailing on each call date, r_j^* for $j = 1, 2, \ldots, n$.

LEMMA 2a: *The price of the Semi-American callable bond an instant before each call date, when moving backwards in time, denoted as $K(r_k, k\cdot\tau)^+$ for $k = 1, 2, \ldots, n$, is given by*

$$\begin{aligned}
K(r_k, k\cdot\tau)^+ = &\sum_{j=1}^{k-1} X_{k-j}\, P(r_k, j\cdot\tau)\, \mathfrak{N}^+(r_{k-1}^*, r_{k-j}^*) + P(r_k, k\cdot\tau)\, \mathfrak{N}(r_{k-1}^*, r_1^*) \qquad (7.2a)\\
&+ \eta \sum_{j=1}^{k} P(r_k, j\cdot\tau)\, \mathfrak{N}(r_{k-1}^*, r_{k+1-j}^*)\,, \text{ where}
\end{aligned}$$

$$\mathfrak{N}^+(r_{k-1}^*, r_{k-j}^*) := \frac{1}{\big[s(\tau)\sqrt{2\pi}\big]^j} \int_{r_{k-1}^*}^{\infty} \cdots \int_{r_{k+1-j}^*}^{\infty} \int_{-\infty}^{r_{k-j}^*} \exp\!\left(-\frac{1}{2}\sum_{i=1}^{j} \varphi(k, j, i)^2\right) \mathrm{d}r_{k-j} \ldots \mathrm{d}r_{k-1},$$

$$\mathfrak{N}(r_a^*, r_d^*) := \frac{1}{\big[s(\tau)\sqrt{2\pi}\big]^{a-d+1}} \int_{r_a^*}^{\infty} \cdots \int_{r_d^*}^{\infty} \exp\!\left(-\frac{1}{2}\sum_{i=1}^{a-d+1} \varphi(k, a-d+2, i)^2\right) \mathrm{d}r_d \ldots \mathrm{d}r_a,$$

$$\varphi(k, j, i) := \frac{r_{k-i} - \mu - s(\tau)^2\big[f(r_{k+1-i}, \tau) + g([j-i]\,\tau)\big]}{s(\tau)}\,, \quad g(0) = 0, \quad r_{n+1} := r,$$

$$\mathfrak{N}^+(r_a^*, r_d^*) = [\ldots]\int_{-\infty}^{r_d^*} \exp(\cdot)\,\mathrm{d}r_d \text{ if } d+1 > a, \quad \mathfrak{N}(r_a^*, r_d^*) = 1 \text{ if } d > a.$$

Proof: by induction. If $k = 1$, then the equation above reduces to $K(r_1, \tau)^+ = [1 + \eta]\, P(r_1, \tau)$. If $k = 2$, then the above equation reduces to the price of the European callable bond as given in equation (6.7a) with $\hat{\tau} = \tau$, $r = r_2$ and $m = 1$. Suppose that the above equation were correct for $k = n$. Substitute the expression for $K(r_n, n\cdot\tau)^+$ into the first and last lines of equation (7.1) in order to obtain the price of the callable bond for the date $t_r - [n + 1]$, apply lemma 1a term by term, and observe that $\varphi(n, j, i) = \varphi(n + 1, j + 1, i + 1)$, then the resulting expression is, after a suitable shift of the summation indices, equal to that of the above equation for $k = n + 1$. □

The price of the Semi-American callable bond today is obtained by means of the last line of equation (7.1) for $K(r_n, n\cdot\tau)^+$ as given in equation (7.2a), and for the final time step $\hat{\tau}$.

$$K(r,\hat{\tau}+n\cdot\tau) = \sum_{j=1}^{n} X_{n+1-j}\, P(r,\, \hat{\tau}+[j-1]\,\tau)\; \tilde{\mathfrak{N}}^{+}(r_n^*,\, r_{n+1-j}^*) + P(r,\, \hat{\tau}+n\cdot\tau)\; \tilde{\mathfrak{N}}(r_n^*,\, r_1^*)$$
$$+\, \eta \sum_{j=1}^{n} P(r,\, \hat{\tau}+j\cdot\tau)\; \tilde{\mathfrak{N}}(r_n^*,\, r_{n+1-j}^*) + \eta \sum_{j=n}^{m} P(r,\, t_r-j-t_0)\;, \text{ where} \tag{7.3a}$$

$$\tilde{\mathfrak{N}}^{+}(r_n^*,\, r_{n+1-j}^*) := \frac{1}{s(\hat{\tau})\, s(\tau)^{j-1} \left[\sqrt{2\pi}\right]^{j}}$$
$$\cdot \int_{r_n^*}^{\infty} \cdots \int_{r_{n+2-j}^*}^{\infty} \int_{-\infty}^{r_{n+1-j}^*} \exp\!\left(-\frac{1}{2}\sum_{i=1}^{j} \tilde{\varphi}(n+1,\, j,\, i)^2\right) \mathrm{d}r_{n+1-j} \ldots \mathrm{d}r_n\,,$$

$$\tilde{\mathfrak{N}}(r_a^*,\, r_d^*) := \frac{1}{s(\hat{\tau})\, s(\tau)^{a-d} \left[\sqrt{2\pi}\right]^{a-d+1}}$$
$$\cdot \int_{r_a^*}^{\infty} \cdots \int_{r_d^*}^{\infty} \exp\!\left(-\frac{1}{2}\sum_{i=1}^{a-d+1} \tilde{\varphi}(n+1,\, a-d+2,\, i)^2\right) \mathrm{d}r_d \ldots \mathrm{d}r_a\,,$$

$$\tilde{\varphi}(n,\, j,\, i) := \begin{cases} \varphi(n,\, j,\, i) := \dfrac{r_{n-i} - \mu - s(\tau)^2 \left[\, t(r_{n+1-i},\, \tau) + g([j-i]\,\tau)\right]}{s(\tau)}\,, & (i > 1), \\[2ex] \hat{\varphi}(n,\, j,\, i) := \dfrac{r_{n-i} - \mu - s(\hat{\tau})^2 \left[\, t(r_{n+1-i},\, \hat{\tau}) + g([j-i]\,\tau)\right]}{s(\hat{\tau})}\,, & (i = 1), \end{cases}$$

$$\tilde{\mathfrak{N}}^{+}(r_a^*,\, r_d^*) = [\ldots] \int_{-\infty}^{r_d^*} \exp(\cdot)\, \mathrm{d}r_d \text{ if } d+1 > a,\; \tilde{\mathfrak{N}}(r_a^*,\, r_d^*) = 1 \text{ if } d > a,\; g(0) = 0,\; r_{n+1} := r\,.$$

The first term represents the sum of the present values of call prices weighted by the probabilities that the bond will be called early. The second term represents the present value of the face value weighted by the probabilities that the bond will not be called early. The third term represents the sum of the present values of coupon payments during the call period weighted by the probabilities that the bond will not be called. Finally, the last term represents the present values of the coupon stream that will be received by the investor with certainty, that is, during the period the bond cannot be called. If there is a single call date, then equation (7.3a) reduces to that of the European callable bond as given in equation (6.7a).

7.2 Second Case

The price of the callable bond on each call date will be derived first. It is needed to determine the "break-even" interest rate prevailing on each call date, r_j^* for $j = 1, 2, \ldots, n$.

LEMMA 2b: *The price of the Semi-American callable bond an instant before each call date, when moving backwards in time, denoted as* $K(r_k, k\cdot\tau)^+$ *for* $k = 1, 2, \ldots, n$, *is given by*

$$K(r_k,\, k\cdot\tau)^+ = \sum_{j=1}^{k-1} X_{k-j}\, P(r_k,\, j\cdot\tau)\; \mathfrak{S}^{+}(r_{k-1}^*,\, r_{k-j}^*) + P(r_k,\, k\cdot\tau)\; \mathfrak{S}(r_{k-1}^*,\, r_1^*)$$
$$+\, \eta \sum_{j=1}^{k} P(r_k,\, j\cdot\tau)\; \mathfrak{S}(r_{k-1}^*,\, r_{k+1-j}^*)\;, \text{ where} \tag{7.2b}$$

$$\mathfrak{H}^{+}(r^*_{k-1}, r^*_{k-j}) := \left[\prod_{i=j}^{1} \omega(j,i)\right]$$
$$\cdot \int_{r^*_{k-1}}^{\infty} \ldots \int_{r^*_{k+1-j}}^{\infty} \int_{-\infty}^{r^*_{k-j}} \left[\prod_{i=1}^{j} H\Big(\omega(j,i)\, r_{k-i} \,|\, 2\alpha+2,\, \hat{\lambda}(k-1,j,i)\Big)\right] \mathrm{d}r_{k-j} \ldots \mathrm{d}r_{k-1}\,,$$
$$\mathfrak{H}(r^*_a, r^*_d) := \left[\prod_{i=a-d+1}^{1} \omega(a-d+2,i)\right]$$
$$\cdot \int_{r^*_a}^{\infty} \ldots \int_{r^*_d}^{\infty} \left[\prod_{i=1}^{a-d+1} H\Big(\omega(a-d+2,i)\, r_{k-i} \,|\, 2\alpha+2,\, \hat{\lambda}(k-1,a-d+2,i)\Big)\right] \mathrm{d}r_d \ldots \mathrm{d}r_a\,,$$
$$\omega(j,i) := 2\,[p(\tau) - g([j-i]\,\tau)], \quad g(0)=0, \quad \hat{\lambda}(n,j,i) := \frac{p(\tau)}{p(\tau)-g([j-i]\,\tau)}\,\lambda(r_{n+2-i},\,\tau),$$
$$\mathfrak{H}^{+}(r^*_a, r^*_d) = [\,\cdot\,] \int_{-\infty}^{r^*_d} H(\cdot)\,\mathrm{d}r_d \text{ if } d+1 > a, \quad \mathfrak{H}(r^*_a, r^*_d) = 1 \text{ if } d > a.$$

The non-central chi-square probability density function $H(\cdot)$ is given in equation (5.6b). The proof of lemma 2b proceeds by induction in the same way as in lemma 2a.

The price of the Semi-American callable bond today is obtained by means of the last line of equation (7.1) for $K(r_n, n\cdot\tau)^+$ as given in equation (7.2b), and for the final time step $\hat{\tau}$.

$$K(r,\hat{\tau}+n\cdot\tau) = \sum_{j=1}^{n} X_{n+1-j}\, P(r,\hat{\tau}+[j-1]\,\tau)\, \tilde{\mathfrak{H}}^{+}(r^*_n, r^*_{n+1-j}) + P(r,\hat{\tau}+n\cdot\tau)\, \tilde{\mathfrak{H}}(r^*_n, r^*_1)$$
$$+\, \eta \sum_{j=1}^{n} P(r,\hat{\tau}+j\,\tau)\, \tilde{\mathfrak{H}}(r^*_n, r^*_{n+1-j}) + \eta \sum_{j=n}^{m} P(r, t_r - j - t_0)\,, \text{ where} \tag{7.3b}$$
$$\tilde{\mathfrak{H}}^{+}(r^*_n, r^*_{n+1-j}) := \left[\prod_{i=j}^{1} \tilde{\omega}(j,i)\right]$$
$$\cdot \int_{r^*_n}^{\infty} \ldots \int_{r^*_{n+2-j}}^{\infty} \int_{-\infty}^{r^*_{n+1-j}} \left[\prod_{i=1}^{j} H\Big(\tilde{\omega}(j,i)\, r_{n+1-i} \,|\, 2\alpha+2,\, \tilde{\lambda}(n,j,i)\Big)\right] \mathrm{d}r_{n+1-j} \ldots \mathrm{d}r_n\,,$$
$$\tilde{\mathfrak{H}}(r^*_a, r^*_d) := \left[\prod_{i=a-d+1}^{1} \tilde{\omega}(a-d+2,i)\right]$$
$$\cdot \int_{r^*_a}^{\infty} \ldots \int_{r^*_d}^{\infty} \left[\prod_{i=1}^{a-d+1} H\Big(\tilde{\omega}(a-d+2,i)\, r_{n+1-i} \,|\, 2\alpha+2,\, \tilde{\lambda}(n,a-d+2,i)\Big)\right] \mathrm{d}r_d \ldots \mathrm{d}r_a\,,$$
$$\tilde{\omega}(j,i) := \begin{cases} \omega(j,i) := 2\,[p(\tau)-g([j-i]\,\tau)]\,,\ (i>1), \\ \hat{\omega}(j,i) := 2\,[p(\hat{\tau})-g([j-i]\,\tau)]\,,\ (i=1), \end{cases} \quad, g(0)=0,$$
$$\tilde{\lambda}(n,j,i) := \begin{cases} \hat{\lambda}(n,j,i) := \dfrac{p(\tau)}{p(\tau)-g([j-i]\,\tau)}\,\lambda(r_{n+2-i},\,\tau)\,,\ (i>1), \\ \hat{\hat{\lambda}}(n,j,i) := \dfrac{p(\hat{\tau})}{p(\hat{\tau})-g([j-i]\,\tau)}\,\lambda(r_{n+2-i},\,\hat{\tau})\,,\ (i=1), \end{cases} \quad, r_{n+1} := r\,,$$
$$\tilde{\mathfrak{H}}^{+}(r^*_a, r^*_d) = [\,\cdot\,] \int_{-\infty}^{r^*_d} H(\cdot)\,\mathrm{d}r_d \text{ if } d+1 > a, \quad \tilde{\mathfrak{H}}(r^*_a, r^*_d) = 1 \text{ if } d > a,$$

The interpretation of the above result is the same as that of equation (7.3a). Again, if there is a single call date, then equation (7.3b) reduces to that of the European callable bond as given in equation (6.7b).

January, 1993

References

ABRAMOWITZ, MILTON AND IRENE A. STEGUN (eds.) [1965]: *Handbook of Mathematical Functions*, New York: Dover Publications, Inc., ninth edition, 1972.

BRENNAN, MICHAEL J. AND EDUARDO S. SCHWARTZ [1977]: "Savings Bonds, Retractable Bonds and Callable Bonds", *Journal of Financial Economics*, 5: 67 – 88.

BRENNAN, MICHAEL J. AND EDUARDO S. SCHWARTZ [1979]: "A Continuous Time Approach to the Pricing of Bonds", *Journal of Banking and Finance*, 3: 133 – 155.

COX, D. R. AND H. D. MILLER [1965]: *The Theory of Stochastic Processes*, London: Chapman and Hall Ltd.

COX, JOHN C., JONATHAN E. INGERSOLL, JR., AND STEPHEN A. ROSS [1985a]: "An Intertemporal General Equilibrium Model of Asset Prices", *Econometrica*, 53: 363 – 384.

COX, JOHN C., JONATHAN E. INGERSOLL, JR., AND STEPHEN A. ROSS [1985b]: "A Theory of the Term Structure of Interest Rates", *Econometrica*, 53: 385 – 407.

ERDéLYI, ARTHUR, WILHELM MAGNUS, FRITZ OBERHETTINGER, AND FRANCESO G. TRICOMI [1953]: *Higher Transcendental Functions*, New York: McGraw-Hill Book Company, Inc., volumes 1 – 3.

ERDéLYI, ARTHUR, WILHELM MAGNUS, FRITZ OBERHETTINGER, AND FRANCESO G. TRICOMI [1954]: *Tables of Integral Transforms*, New York: McGraw-Hill Book Company, Inc., volumes 1 & 2.

FELLER, WILLIAM [1951]: "Two Singular Diffusion Problems", *Annals of Mathematics*, 54: 173 – 182.

FELLLER, WILLIAM [1952]: "The Parabolic Differential Equations and the Associated Semigroups of Transformations", *Annals of Mathematics*, 55: 468 – 519.

GIBSON-ASNER, RAJNA [1990]: "Valuing Swiss Default-free Callable Bonds", *Journal of Banking and Finance*, 14: 649 – 672.

JOHNSON, NORMAN L. AND SAMUEL KOTZ (eds.) [1970]: *Continuous Univariate Distributions*, New York: John Wiley and Sons, volumes 1 & 2.

LONGSTAFF, FRANCIS [1989]: "A Nonlinear General Equilibrium Model of the Term Structure of Interest Rates", *Journal of Financial Economics*, 23: 195 – 224.

VASICEK, OLDRICH [1977]: "An Equilibrium Characterization of the Term Structure", *Journal of Financial Economics*, 5: 177 – 188.