Evaluation of Callable Bonds

Finite Difference Methods, Stability and Accuracy ‡

by

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Abstract::

The purpose of this paper is to evaluate numerically the semi-American callable bond by means of finite difference methods. This study implies three results. First, the numerical error is greater for the callable bond price than for the straight bond price, and too large for real applications. This phenomenon can be attributed to the discontinuity in the values of the early redemption condition. Moreover, many computed prices of the embedded call option turn out to be *negative*. Secondly, the numerical accuracy of the callable bond price computed for the relevant range of interest rates depends entirely on the finite difference scheme which is chosen for the boundary points. The paper compares the numerical error for four different boundary schemes, including the one which is extensively used in the finance literature. Thirdly, the boundary scheme which yields the smallest numerical errror with respect to the straight bond does *not* perform best with respect to the callable bond.

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1. Introduction

The callable bond is a straight (coupon) bond with the provision that allows the debtor to buy back or to "call" the bond for a specified amount, the call price, plus the accrued interest since the last coupon date at some time, the call date(s), during the life of the bond (see, e. g., BRENNAN AND SCHWARTZ [1977]). Three types of callable bonds can be observed in financial markets. The American callable bond may be repurchased at any time on or before the final redemption date, in contrast to the European or semi-American counterparts, which may only be called at one or several specific dates, respectively. American callable bonds have predominantly been issued by corporations in the United States, whereas semi-American callable bonds have solely been issued by both public and private institutions in Switzerland. In the case of Swiss financial markets, the single call date of the European callable bond is the last but one coupon date, whereas the additional call dates of the semi-American callable bond are the preceding coupon dates. The number of call dates of outstanding semi-American callable bonds varies between one and ten in Switzerland. Moreover, the Swiss debtor gives the bondholder two months' notice. In general, the initial call price may be somewhat above the face value of the bond and may decline gradually over time. Moreover, the call provision will be deferred sometimes until the bond has been outstanding for some length of time.

The callable bond can be viewed as a compound security which consists of an otherwise identical straight bond and of an embedded call option, which is not traded and the price of which is, therefore, not observable. The embedded call option, which is written on the underlying straight bond, can be viewed as being "sold" by the initial bondholder to the issuer of the callable bond, the debtor. Hence, the price of the callable bond must be equal to the price of the underlying straight bond less the price of the embedded call option at any time. Or, the callable bond is worth less than the underlying straight bond.

This paper is motivated by our experience with finite difference methods as well as the wrong numerical results which have been published in two studies. GIBSON-ASNER [1990, Table 6, p. 666] reports the *computed* prices of two almost identical embedded call options for four points in time. These two options are identical except for the last possible redemption dates which differ by roughly three months. The prices of these two options with approximately eleven years until expiration are reported to be {1.1303, 2.6164}, with approximately ten years until expiration to be {0.0878, 0.2994}, with approximately nine years until expiration to be {0.0482, 0.1471}, and with approximately eight years until expiration to be {15.3192, 5.3789}. First of all, the time pattern of the call price is wrong (see section 7 below). Secondly, our computation for a similar pair of call options indicates that the price difference is in the order of magnitude of 0.2 (see, e. g., securities #15 745 and #15 227 in Table A.4). Moreover, some *computed* prices of the embedded call option turn out to be negative (GIBSON-ASNER [1990, p. 670]). Further results presented in LEITHNER [1992] also point to negative call prices or large numerical errors, respectively, although the computed prices of callable bonds are not explicitly reported.[1] For example, the *computed* price of the semi-American call option is less than that of the corresponding European call for small interest rates, but greater for large interest rates (LEITHNER [1992, Figure 5.5, p. 145]). Hence, these wrong results presented in GIBSON-ASNER and LEITHNER *must* be due to numerical errors.[2]

The purpose of this paper is to evaluate numerically the semi-American callable bond by means of four finite difference methods. As an example, the paper uses the one-factor model of the real term structure of interest rates proposed by VASICEK [1977]. The numerical solution of the finite difference method will be compared with the analytical solution derived in BüTTLER AND WALDVOGEL [1993a].

The outline of the paper is as follows. The next section surveys briefly different bond valuation models. The model which is used in this paper is explained in the third section. The model parameters are estimated in the fourth section. The finite difference methods, which are considered in this paper, are explained in the fifth section, followed by a section which addresses the question of numerical accuracy of the finite difference methods. Finally, we look at the implied ("observed") prices of the embedded call option. Conclusions are given at the end.

2. A Brief Survey on Bond Valuation Models

By and large, the different valuation models which have been proposed in the literature can be divided into three groups. The following grouping is, of course, a matter of personal taste. An extensive survey on bond option pricing models can be found in COURTADON [1990], and another classification, e. g., in HEATH ET AL. [1992].

The first group consists of models which value *directly* a derivative financial instrument (e. g., a callable bond) written on a fixed-income security (instead of using the term structure of interest rates). Examples are given in BALL AND TOROUS [1983], in MERTON [1973] as well as in SCHAEFER AND SCHWARTZ [1987]. In the first two papers, the derivative financial instrument is written on a discount bond, whereas in the last paper it is written on a straight (coupon) bond. The valuation proceeds in the usual way. First, a stochastic process is assumed for the price of the unterlying security. BALL AND TOROUS as well as MERTON assume a geometric BROWNian bridge process for the price of the underlying security, which forces the price of the underlying security to approach the given payoff function on the maturity date. SCHAEFER AND SCHWARTZ consider a geometric BROWNian motion of the price of the underlying security where the volatility of the underlying bond's price is proportional to the duration of the straight bond. The assumption of SCHAEFER AND SCHWARTZ is quite unrealistic because the price of the straight bond does not approach the given payoff function (i. e., the face value plus

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1 See pages 122 – 125, Table 4.4 on page 123, Figure 5.3 on page 143, and Figure 5.5 on page 145.

2 It may be possible that the results presented in GIBSON-ASNER are a mixture of numerical error and pricing bias due to her particular numerical algorithm.

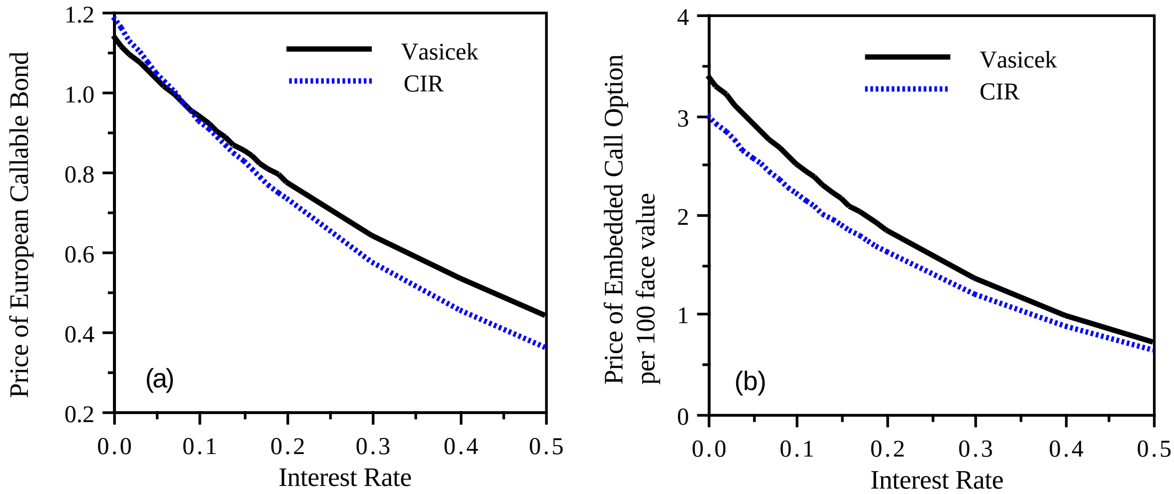
coupon) on the maturity date, nor is the time pattern of the volatility plausible. In the second step, a riskless portfolio is constructed from the underlying security and the financial instrument to be considered. The condition that this riskless portfolio earns the riskless instantaneous interest rate allows to solve for the price of the derivative financial instrument. There are two shortcomings with these valuation models. First, NELSON AND RAMASWAMY [1990] have shown that these models are *not* arbitrage-free. Secondly, the market price of interest-rate risk remains undetermined. GIBSON-ASNER [1990] uses the model of SCHAEFER AND SCHWARTZ [1987] to value Swiss callable bonds. Besides the two shortcomings mentioned above, there is a third deficiency in her model, since the price of the callable bond does not approach the given payoff function on the last possible redemption date, if the bond is never called.

The second group consists of discrete-time models which extend the popular binomial model[3] to the term structure of interest rates. Examples are BOOKSTABER, JACOB AND LANGSAM [1986], BOYLE [1988], BOYLE, EVNINE UND GIBBS [1989], DYER AND JACOB [1988], PEDERSEN, SHIU AND THORLACIUS [1989] as well as RENDLEMAN AND BARTTER [1980]. An early attempt of a multinomial model is given in LANGETIEG [1980]. However, some of these models are not arbitrage-free, or do not consider the market price of interest-rate risk, or do not explicitly specify the stochastic process of the underlying interest rate(s) within the framework of the binomial or multinomial model. The binomial model proposed by HO AND LEE [1986] overcomes two of these shortcomings, that is, it is arbitrage-free and takes the market price of interest-rate risk into account. However, it does not specify explicitly the stochastic process of the instantaneous interest rate which explains the whole term structure (in fact, it is a one-factor model). In their model, the market price of interest-rate risk must be estimated empirically by means of a multi-stage econometric procedure. Moreover, it has been shown that the (implicitly) underlying stochastic process of the instantaneous interest rate in HO AND LEE's model does not tend to a long-run equilibrium distribution (see HULL AND WHITE [1990]). To my knowledge, there is only the discrete-time model of NELSON AND RAMASWAMY [1990] which is arbitrage-free, which considers the market price of interest-rate risk, and which specifies explicitly the stochastic process of the interest rate. They show which stochastic processes can be treated within the framework of a *generalized* binomial model[4]. In view of the one-factor model to be considered in this paper, they conclude that both the ORNSTEIN-UHLENBECK process and the square-root process of the instantaneous interest rate can be treated with the generalized binomial model, but not the double square-root process of the instantaneous interest

[3] The binomial model has been developed for a geometric BROWNian process of the price of a common stock with *constant* volatility. It has been proposed by SHARPE and it has independently been derived by RENDLEMAN AND BARTTER [1979] as well as COX, ROSS AND RUBINSTEIN [1979].

[4] NELSON AND RAMASWAMY [1990] develop a generalized version of the binomial model which allows for *variable* volatilities, given some necessary conditions. The basic idea of their model is to transfrom the original stochastic process to the geometric BROWNian motion with constant volatility, i. e., the well-known binomial model. If the inverse function of this variable transformation is one-to-one, one can determine the probabilities of the up and down moves in the simple binomial model and then transform the simple binomial process back to the original process.

rate.[5] Since the binomial model is outlined in such a way as to calculate only one price today, in contrast to the finite difference (or similar[6]) methods which compute a vector of prices on all necessary dates during the life of the security, the binomial model is, in general, numerically more efficient.[7] However, NELSON AND RAMASWAMY [1990, p. 421, Table 2] report numerical results for the square-root process which are not promising, because the computed price may deviate from the theoretical price by as much as 13% for a five-year discount bond. This numerical error grows with the length of the time to maturity.



Figures 1a & b: Price of the European Callable Bond (a) and the Embedded Call Option (b) for the One-factor Models of VASICEK and CIR. The time to expiration is 6.811 years and the annual coupon is 7%. The model parameters have been estimated from the same inverse yield structure which is used in the text.

The third group consists of continuous-time models. Common to the continuous-time models and the discrete-time models mentioned above is the explanation of the term structure of interest rates by means of one or more "driving" factors of the economy. An early attempt of both a one-factor model and a two-factor model of the term structure can be found in BRENNAN AND SCHWARTZ [1977, 1979]. The two factors considered are the instantaneous interest rate and the yield of a consol discount bond. By now, there are two popular one-factor models of the term structure: the one proposed by VASICEK [1977], which is based on the ORNSTEIN-UHLENBECK process for the instantaneous interest rate, and the one proposed by COX, INGERSOLL AND ROSS [1985b], henceforth abbreviated as CIR, which is based on the square-root process for the instantaneous interest rate.[8] The instantaneous interest rate may become negative with

5 The ORNSTEIN-UHLENBECK has been considered by VASICEK [1977], the square-root process by COX, INGERSOLL AND ROSS [1985b] and the double square-root process by LONGSTAFF [1989].

6 For instance, the method of lines or the method of finite elements, respectively.

7 For a comparison of the simple binomial model with finite difference methods see, e. g., GESKE AND SHASTRI [1985].

8 VASICEK's derivation of the partial differential equation of the price of a discount bond does not rely on a specific stochastic process for the interest rate. The ORNSTEIN-UHLENBECK process was merely used as an illustration in order to derive an analytical solution of the partial differential equation. While CIR derive the interest-

the ORNSTEIN-UHLENBECK process, but remains positive with the square-root process. Keeping in mind that both one-factor models explain the *real* term structure of interest rates, the ORNSTEIN-UHLENBECK process may be preferable. In practice, however, both models are applied to the *nominal* term structure.[9] From the latter perspective, the square-root process is preferable. A comparison of the two theoretical prices of the European callable bond is shown in Figure 1. As expected, the price of the embedded call option is greater in the case of VASICEK's model than in the case of the CIR model, but the difference is rather small.[10] A third one-factor model, which is based on the double square-root process for the instantaneous interest rate, has been proposed by LONGSTAFF [1989]. However, one can show that LONGSTAFF's model may be reduced to VASICEK's.[11] Examples of the two-factor model are the one proposed by LONGSTAFF AND SCHWARTZ [1992], in which both the instantaneous interest rate and its volatility determine the whole term structure, or the one proposed by CIR [1985b], in which the instantaneous interest rate together with the inflation rate determine the nominal term structure.[12] Finally, a generalization to a multi-factor model is given in HEATH, JARROW AND MORTON [1992].[13] Their model considers stochastic movements of the whole term structure of interest rates, given an initial term structure of forward rates, and given the condition that arbitrage is excluded. While all the yields are perfectly correlated with the instantaneous interest rate (the explanatory variable) in the one-factor model, this is no longer true with the multi-factor model.

3. The Model

As an example, we consider the one-factor model of VASICEK [1977], in which the instantaneous interest rate is the only explanatory factor that determines completely the future movements of the term structure of interest rates. As mentioned before, this implies that all the yields with different maturities are perfectly correlated with the instantanesous interest rate. From an economic point of view, it seems reasonable to consider the instantaneous interest rate

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rate process endogeneously from a general equilibrium model (COX ET AL. [1985a]), VASICEK assumes such a process. Moreover, CIR derive the partial differential equation to value a discount bond from the same general equilibrium model, in constrast to VASICEK who develops his partial differential equation from the no-arbitrage condition.

9 A recent theory of the nominal term structure of interest rates can be found in CONSTANTINIDES [1992] and SUN [1992].

10 The difference is small as long as negative interest rates do not play a role in determining the optimal call policy.

11 See BüTTLER AND WALDVOGEL [1993a]. Moreover, LONGSTAFF claims that his formula for the price of a discount bond was derived for a reflecting barrier at a zero interest rate. However, BEAGLEHOLE AND TENNEY [1992] have shown that LONGSTAFF's claim is incorrect: his formula holds for interest rates on the whole real axis, only.

12 An application of the CIR two-factor model to various bonds and futures can be found in CHEN AND SCOTT [1992].

13 As was noted by HEATH ET AL., their model is a generalization of the one-factor model put forward by HO AND LEE [1986]. A discrete-time version necessary to implement numerically their continuous-time, multi-factor model is given in HEATH ET AL. [1990].

as the explanatory factor because most central banks steer their monetary course by means of daily operations on bank reserves which, in turn, determine the very short-term interest rates for interbank borrowing and lending.

Denote as $P(t, \tau)$ the price of a discount bond, promising to pay one unit of money on the expiration date, given time t and the time period to expiration τ, and denote the corresponding yield as $R(t, \tau)$. The price of the discount bond is defined to be the present value of the face value with the bond yield being the discount rate, that is, $P(t, \tau) = \exp(-R(t, \tau)\,\tau) \cdot 1$. Thus, the instantaneous interest rate today, $r(t)$, is defined to be the yield of a discount bond that matures within the next instant:

$$r(t) := R(t, 0) = \lim_{\tau \to 0} R(t, \tau) = \lim_{\tau \to 0} -\frac{\ln(P(t, \tau))}{\tau} . \tag{3.1}$$

In VASICEK's model, it is assumed that the instantaneous interest rate can be described by the following ORNSTEIN-UHLENBECK process:[14]

$$\mathrm{d}r = \alpha\,(\gamma - r)\,\mathrm{d}t + \rho\,\mathrm{d}z, \tag{3.2}$$

where $\alpha > 0$ denotes the speed of adjustment, $\gamma > 0$ the long-run "equilibrium" value of the instantaneous interest rate, t the calendar time, $\rho > 0$ the constant instantaneous standard deviation of the instantaneous interest rate, and dz the GAUSS-WIENER process. The spot interest rate may become negative, but "mean-reverting" will eventually pull it back to its long-run equilibrium value. Given the interest-rate process of equation (3.2), VASICEK derives the following partial differential equation to determine the price of a default-free discount bond, $P(r, \tau)$, promising to pay one unit of money on the maturity date:

$$P_\tau = \frac{1}{2}\rho^2 P_{rr} + \left[\alpha\,(\gamma - r) + \rho q\right] P_r - rP, \tag{3.3}$$

where the subscripts denote partial derivatives, τ the remaining time period until the expiration of the discount bond, and q the market price of interest-rate risk assumed to be constant. If arbitrage opportunities are ruled out, the market price of interest-rate risk must be the same for all discount bonds of different maturities. Empirically, we would expect q to be positive. The interval of definition for r is $r \in \mathcal{I}_1 = \mathbb{R}$, whereas we always assume $\tau \in \mathcal{I}_2 = (0, \infty)$. The region of definition of $P(r, \tau)$ is $\Omega = \mathcal{I}_1 \times \mathcal{I}_2$, i. e., the upper half plane. Equation (3.3) is nowhere singular. The analytical solution of the partial differential equation (3.3) is given in VASICEK as:

$$\begin{aligned} &P(r, \tau) = \exp(f(\tau) + g(\tau)\,r), \quad \text{with } f(\tau) := \frac{1}{\alpha} R_\infty \left[1 - \mathrm{e}^{-\alpha\tau}\right] - R_\infty \tau - \frac{\rho^2}{4\,\alpha^3}\left[1 - \mathrm{e}^{-\alpha\tau}\right]^2, \\ &g(\tau) := -\frac{1}{\alpha}\left[1 - \mathrm{e}^{-\alpha\tau}\right], \quad R_\infty := \gamma + \frac{\rho\,q}{\alpha} - \frac{1}{2}\frac{\rho^2}{\alpha^2} . \end{aligned} \tag{3.4}$$

14 The probability density function of the Ornstein-Uhlenbeck process can be found, e. g. , in COX AND MILLER [1965].

The abbreviation R_∞ denotes the yield of a discount bond with infinite time until maturity, i. e., a consol discount bond, and both "e" and "exp" denote the exponential function. In a meaningful model, the nominal yield of a consol bond should be positive. The equation (3.4) has been derived for the initial condition $P(r, 0) = 1$, but no boundary conditions have been specified.

In order to obtain a unique solution initial values $P(r, 0)$ must be given for $r \in \mathcal{I}_1$. Also, in general two boundary conditions must be given, namely one on the left and one on the right boundary of $\mathcal{I}_1$ for every $\tau \in \mathcal{I}_2$. The boundary conditions which lead to the price formula (3.4) are given by (BÜTTLER AND WALDVOGEL [1993a]):

$$\begin{aligned} &P(r,\ \tau) \to 0 \text{ as } r \to \infty, \text{ (right boundary)}, \\ &P(r,\ \tau) = \mathbb{O}\left(e^{-\vartheta r}\right),\ \vartheta > 0, \text{ as } r \to -\infty, \text{ (left boundary)}. \end{aligned} \tag{3.5}$$

The right boundary condition has been proposed by BRENNAN AND SCHWARTZ [1977, 1979]. It says that the price of a discount bond tends to zero as the instantaneous interest rate grows infinitely large. The left boundary condition ensures that a particular solution is chosen which grows exponentially at most as the absolute value of the interest rate becomes very large.

The term structure of discount bond yields follows from equation (3.4):

$$R(r,\ \tau) = -\frac{\ln(P(r,\ \tau))}{\tau} = R_\infty + \frac{\left[r(t) - R_\infty\right]\left[1 - e^{-\alpha\tau}\right]}{\alpha\,\tau} + \frac{\rho^2\left[1 - e^{-\alpha\tau}\right]^2}{4\,\alpha^3\tau}\ . \tag{3.6}$$

As already mentioned, if the time until expiration grows infinitely large, then the yield becomes $R_\infty = \lim_{\tau\to\infty} R(r,\ \tau)$, and if the time until expiration vanishes, then the yield becomes the instantaneous interest rate $r(t) = \lim_{\tau\to 0} R(r,\ \tau)$. The term structure is shown for four different values of the prevailing instantaneous interest rate in Figure 2 with $z := R(r,\ \tau)$ on the vertical axis. If the current value of the instantaneous interest rate is less than $R_\infty - \rho^2/(4\ \alpha^2)$, then the yield curve is monotone increasing (normal yield curve); if the current value of the instantaneous interest is greater than $R_\infty + \rho^2/(2\ \alpha^2)$, then the yield curve is monotone decreasing (inverse yield curve), otherwise it is a humped curve. The same three types of yield curves are obtained with the CIR model as well.

The callable bond satisfies the same partial differential equation (3.3) and the same boundary conditions (3.5) as the discount bond between the notice dates. Moreover, there is the *early redemption condition* prevailing on each notice date when the debtor has to decide whether or not to call the bond. Suppose there are ν call dates identical to the coupon dates $[t_r - j]$ for $j = 1, 2, \ldots, \nu$, with t_r the last possible redemption date if the bond is never called. The notice period is denoted as τ_n, which is, in general, two months. Let the call prices be denoted as $X_j := X(t_r - j)$ for $j = 1, 2, \ldots, \nu$, the instantaneous interest rates prevailing on the notice dates $[t_r - \tau_n - j]$ as $r_j := r(t_r - \tau_n - j)$ for $j = 1, 2, \ldots, \nu$, the instantaneous interest rate today as $r := r(t_0)$, the time periods between call dates as τ (one year), the time period between today and the last notice date (when moving backwards in time) as $\hat{\tau}$, the number of whole years of the remaining time until the last possible redemption date as μ, and the annual coupon as η.

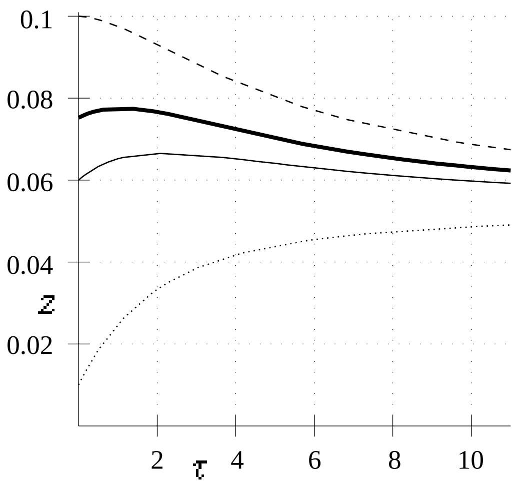

Figure 2: Term Structure.

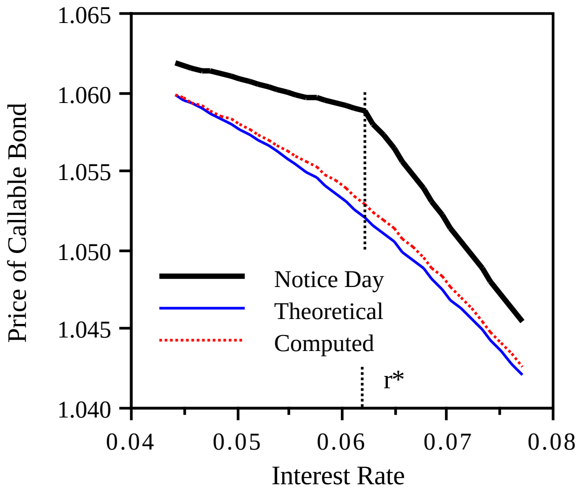

Figure 3: Early Redemption Condition.

We proceed backwards in time. If the bond is never called, the price of the callable bond will be equal to the price of the underlying straight bond on the last possible redemption date, t_r, that is, equal to the face value plus the coupon payment, $[1 + \eta]$ (the inital condition). An instant before the first notice date, the price of the callable bond is equal to the time value of the underlying straight bond, $K(r_1, \tau_n + \tau)^+ = [1 + \eta] \cdot P(r_1, \tau_n + \tau) + \eta \cdot P(r_1, \tau_n)$. The time value of the call price plus the coupon payment on the first notice date is given by $[X_1 + \eta] \cdot P(r_1, \tau_n)$. The call policy is optimal if the issuer of the callable bond minimizes his outstanding debt. Therefore, he will call the bond if the price of the callable bond is greater than the time value of the call price including the coupon payment. The “break-even” (or critical) interest rate on the first notice day, r_1^*, is that interest rate which equates the price of the callable bond an instant before the notice date and the time value of the call price. Hence, the price of the callable bond an instant after the first notice day is given by $K(r_1, \tau_n + \tau)^- = \min\,\{K(r_1, \tau_n + \tau)^+, [X_1 + \eta] \cdot P(r_1, \tau_n)\}$. This is the early redemption condition on the first notice day. The callable bond then behaves like the underlying straight bond during the period between the first and second notice day. The procedure mentioned above repeats on the second and the following notice days. Hence, the *early redemption conditons* may be written as

$$
\begin{aligned}
&K(r_j^*,\ \tau_n + j\cdot\tau)^+ = \left[X_j + \eta\right] P(r_j, \tau_n) \;\Rightarrow\; r_j^*, \quad j = 1, 2, \ldots, \nu, \\
&K(r_j,\ \tau_n + j\cdot\tau)^- = \begin{cases} \left[X_j + \eta\right] P(r_j, \tau_n), & \text{for } r_j \leqq r_j^*, \\ K(r_j,\ \tau_n + j\cdot\tau)^+, & \text{for } r_j \geqq r_j^*. \end{cases}
\end{aligned} \tag{3.7}
$$

Again, the callable bond behaves like the underlying straight bond during the period between today and the last notice day when the bond cannot be called. An early redemption condition is shown by the bold-faced kinked curve in Figure 3.

4. The Estimation of Model Parameters

There are four parameters to be estimated in VASICEK's model: the speed of adjustment α, the long-run "equilibrium" value of the instantaneous interest rate γ, the constant instantaneous standard deviation of the instantaneous interest rate ρ, and the market price of interest-rate risk q. The first three parameters could be estimated by means of a regression of the discrete-time version of the ORNSTEIN-UHLENBECK process (VASICEK [1977, p. 187]). Here, all the parameters are estimated from a cross section sample of yields for a particular risk class (e. g., cantonal bonds) on a particular trading day. This procedure —to be repeated for each risk class on each trading day— has the advantage that the most recent information is used. It is analogous to estimating implicit volatilities of stock options.

In this paper, the cross section sample consists of three observations only, that is, the yield curve is perfectly fitted to the observations. In practice, one would like to estimate the parameters with some degree of freedom. An inspection of the theoretical yield curve of equation (3.6) shows that the two pairs of partial derivatives $\{\partial R/\partial\gamma, \partial R/\partial q\}$ and $\{\partial R/\partial\gamma, \partial R/\partial r\}$ are linearly dependent. Since we want to calculate the price of the callable bond for the *current* value of the instantaneous interest rate which is observed on the trading day in question, this value is included in the sample of the observed yield curve. The long-run "equilibrium" value of the instanteneous interest rate, γ, is chosen to be exogeneous. It is estimated as the long-run mean value of the observed instantaneous interest rate over the last twenty years ($\gamma = 0.03484$). Hence, there remain the three parameters α, ρ, and q to be estimated. In the calculations to follow, we choose three exchange-traded straight bonds with $\tau_i = \{1, 7.175, 10.25\}$ years until expiration. The corresponding observed discrete-time yields, $\tilde{R}_i$, are transformed to continuous-time yields according to $\hat{R}_i = \ln(1 + \tilde{R}_i)$; they are $\hat{R}_i = \{0.07753, 0.06647, 0.06298\}$. The estimation is done by means of the following three nonlinear equations which relate the theoretical yield curve of equation (3.6) to the observed yields as follows:

$$f_i(\mathbf{x}) := R(r, \tau_i; \mathbf{x}) - \hat{R}_i = 0, \quad i = 1, 2, 3. \tag{4.1}$$

Here, $\mathbf{x}$ denotes the vector of parameters, i. e., $\mathbf{x} := [\alpha, \rho, q]'$. If both theoretical and observed yields are equal, then the three functions f_i must vanish. Given a starting value $\mathbf{x}^{(0)}$, successive increments $\Delta\mathbf{x}$ are obtained by means of the NEWTON-RAPHSON method as follows:

$$\Delta\mathbf{x} = \mathbf{A}^{-1}\,\mathbf{b}, \quad \text{with } \mathbf{A} = \begin{bmatrix} \partial f_1/\partial x_1, & \partial f_1/\partial x_2, & \partial f_1/\partial x_3 \\ \partial f_2/\partial x_1, & \partial f_2/\partial x_2, & \partial f_2/\partial x_3 \\ \partial f_3/\partial x_1, & \partial f_3/\partial x_2, & \partial f_3/\partial x_3 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} -f_1 \\ -f_2 \\ -f_3 \end{bmatrix}. \tag{4.2}$$

The JACOBIan matrix $\mathbf{A}$ is inverted by means of the LU decomposition of PRESS ET AL. [1989] and is given by the partial derivatives of the theoretical yield function of equation (3.6) as follows:

$$\begin{aligned}
\frac{\partial f_i}{\partial x_1} &= \frac{\partial R(r,\,\tau_i)}{\partial \alpha} = \left[-\frac{\rho q}{\alpha^2} + \frac{\rho^2}{\alpha^3}\right]\left[1 - \frac{1 - e^{-\alpha\tau_i}}{\alpha\tau_i}\right] + \left[r - R_\infty\right]\frac{\left[1 + \alpha\tau_i\right] e^{-\alpha\tau_i} - 1}{\alpha^2\tau_i} \\
&\quad + \frac{\rho^2}{4\alpha^3\tau_i}\left[2\,\alpha\,\tau_i\left(1 - e^{-\alpha\tau_i}\right) e^{-\alpha\tau_i} - 3\left(1 - e^{-\alpha\tau_i}\right)^2\right], \\
\frac{\partial f_i}{\partial x_2} &= \frac{\partial R(r,\,\tau_i)}{\partial \rho} = \left[\frac{q}{\alpha} - \frac{\rho}{\alpha^2}\right]\left[1 - \frac{1 - e^{-\alpha\tau_i}}{\alpha\tau_i}\right] + \frac{2\rho\left(1 - e^{-\alpha\tau_i}\right)^2}{4\alpha^3\tau_i}, \\
\frac{\partial f_i}{\partial x_3} &= \frac{\partial R(r,\,\tau_i)}{\partial q} = \frac{\rho}{\alpha}\left[1 - \frac{1 - e^{-\alpha\tau_i}}{\alpha\tau_i}\right].
\end{aligned} \tag{4.3}$$

The NEWTON-RAPHSON method (PRESS ET AL. [1989]) which is used in this paper is modified as follows. First, the first two parameters are restricted to remain positive during the search of the roots. Secondly, the iteration takes the form $\mathbf{x}^{(j)} = \mathbf{x}^{(j-1)} + g(\Delta\mathbf{x})$, where $g(\cdot)$ is a predefined function that takes the relative change into account, i. e., the smaller the relative change $\Delta\mathbf{x}/\mathbf{x}^{(j-1)}$, the greater is the increment $g(\Delta\mathbf{x})$. This helps to avoid an overshooting of the roots to be found.

Given the current value of the instantaneous interest rate, $r_0 = 0.07522$, we find for the three parameters the following values: $\alpha = 0.4418$, $\rho = 0.1326$ and $q = 0.2117$. As expected, the market price of interest-rate risk, q, turns out to be positive. The theoretical yield curve of equation (3.6) for these numerical values is shown as bold-faced curve in Figure 2. The actual yield curve was humped and inverse for longer-term bonds on the trading day in question. In Table A.1, the quoted price of several cantonal straight bonds is compared with the analytical price; the latter is obtained from the portfolio of discount bonds the prices of which are given in equation (3.4). The percentage errors are quite small (see column 7). Since there is only one straight bond issued by the Swiss confederation which belongs to the corresponding "bond basket" of the SWISS NATIONAL BANK (SNB), the theoretical yield curve cannot be estimated for this risk class. Therefore, the yield curve just found for the cantonal bonds will be used for the federal bonds as well.

5. Finite Difference Methods

Four different finite difference methods have been applied to the partial differential equation (3.3), namely the explicit method, the implicit method, the CRANK-NICOLSON method, and the implicit method with a two time-level extrapolation due to LAWSON AND MORRIS [1978], henceforth the LAWSON-MORRIS method. BRENNAN AND SCHWARTZ [1978] have shown that the explicit method corresponds to the simple binomial model, in contrast to the implicit method which corresponds to a multinomial model with jumps.

The differential equation (3.3) is transformed first. Several possibilities arise. If one is interested in the whole range of interest rates from minus infinity to plus infinity, the transformation of variable $x = \tanh(r)$ is suitable because it does not introduce singularities in the interval x

$\in$ [–1, +1].[15] If one is interested in *positive* interest rates only as we are in this paper, the logarithmic transformation $x = \ln(1 + r)$, $r \in (-1, +\infty)$, or the transformation proposed by BRENNAN AND SCHWARTZ [1979] may be used:[16]

$$x = \frac{1}{1 + \hat{m} r}, \quad \tau = \frac{t - t_0}{s - t_0}, \quad (0 \le x,\ \tau \le 1), \quad F(x,\ \tau) = P(r,\ t). \tag{5.1}$$

Here, $\hat{m}$ is a scaling factor, s the maximum time to expiration of all bonds considered, and τ measures the *elapsed* time rather than the time to maturity.[17] Two comments are necessary. First, although we consider only non-negative interest rates in this paper, the transformation (5.1) allows for a negative interest-rate range which is sufficient in practical applications, for instance, $r > -100\%$ for $\hat{m} = 1$. In fact, a calculation of the analytical price of a callable bond with twenty years to expiration and with ten call dates has shown that the smallest “break-even” interest rate is –13% (BÜTTLER AND WALDVOGEL [1993b]). Secondly, the numerical error of the finite difference method under consideration is based on the analytical price for *positive* “break-even” interest rates (see Table A.3). In Table A.4 however, the analytical price is calculated for *all* the possible “break-even” interest rates. The partial derivatives of the discount bond price become with transformation (5.1):

$$\begin{aligned} &\frac{\partial P(r,\ t)}{\partial t} = \frac{1}{s - t_0} \frac{\partial F(x,\ \tau)}{\partial \tau}, \quad \frac{\partial P(r,\ t)}{\partial r} = -\hat{m} x^2 \frac{\partial F(x,\ \tau)}{\partial x}, \\ &\frac{\partial^2 P(r,\ t)}{\partial r^2} = 2\hat{m}^2 x^3 \frac{\partial F(x,\ \tau)}{\partial x} + \hat{m}^2 x^4 \frac{\partial^2 F(x,\ \tau)}{\partial x^2}. \end{aligned} \tag{5.2}$$

Substituting equation (5.2) into equation (3.3), the transformed differential equation is obtained:

$$\begin{aligned} &0 = \frac{\partial F(x,\ \tau)}{\partial \tau} + \hat{a}(x) \frac{\partial F(x,\ \tau)}{\partial x} + \hat{b}(x) \frac{\partial^2 F(x,\ \tau)}{\partial x^2} + \hat{c}(x)\, F(x,\ \tau), \quad \text{with} \\ &\hat{a}(x) = -\left[(\alpha\gamma + \rho q)\, \hat{m}\, x^2 - \alpha\,(1 - x)\, x - \rho^2 \hat{m}^2\, x^3\right] (s - t_0) \gtreqless 0, \\ &\hat{b}(x) = \frac{1}{2} \rho^2 \hat{m}^2\, x^4\, (s - t_0) > 0, \quad \hat{c}(x) = -\frac{(1 - x)}{\hat{m} x} (s - t_0) < 0. \end{aligned} \tag{5.3}$$

The region of definition of the transformed differential equation, $\Omega = [0, 1] \times [0, 1]$, is now divided into meshes. Since we want to calculate the price of callable bonds for the *current* value of the instantaneous interest rate, the meshes are *not* all of the same size. Moreover, it might be desirable to have narrow meshes for small interest rates such that the “break-even” in-

---

15 From a computational point of view, it is not possible to use the whole range of negative interest rates because the price of the discount bond grows exponentially which, in turn, would produce overflows for large absolute values of the interest rate.

16 LEITHNER [1992] applies the square-root transformation $x = \sqrt{r}$ and $F(x, \tau) = P(r, t)$ to the differential equation of CIR which is regular singular at $r = 0$. The square-root transformation introduces a new singularity at $x = 0$. As a consequence, the first partial derivative, $\partial F/\partial x$, *has* to be restricted to zero at the origin due to the fact that the price of the discount bond is a *regular* solution with finite derivative $\partial P/\partial r$ at $r = 0$.

17 This does not mean that we are solving forwards the differential equation (3.3). As will be seen later, the differential equation is solved *backwards* in time.

terest rate can be determined with "high" accuracy. Therefore, the x-axis is divided into n_1 equal intervals between x_m and 1 (equivalently between $r = r_m$ and $r = 0$) and n_2 equal intervals between 0 and x_m (equivalently between $r = \infty$ and $r = r_m$). If n_2 is set equal to zero, then the whole range of positive interest rates is divided into n_1 equal intervals, possibly except for an initial step at $x = 0$ ($r = \infty$). The current value of the instantaneous interest rate, r_0, lies always on a mesh point. The steps between two points in time of interest (e. g., a coupon date and a notice date) will be of equal length. The mesh points are counted with index i along the x-axis, henceforth the locational axis, and with index j along the time axis.

Denote the price of a bond $F(x, \tau)$ at mesh point (i, j) as F^i_j. The partial derivatives of equation (5.3) are approximated by the standard second-order finite differences if these occur at an *internal* mesh point (SMITH [1985]):

$$\begin{aligned} &\frac{\partial F_{i,j+\frac{1}{2}}}{\partial \tau} \sim \frac{1}{\Delta \tau}\left[F_{i,j+1} - F_{i,j}\right], \quad \frac{\partial F_{i,j}}{\partial x} \sim \frac{1}{2\Delta x}\left[F_{i+1,j} - F_{i-1,j}\right], \\ &\frac{\partial^2 F_{i,j}}{\partial x^2} \sim \frac{1}{(\Delta x)^2}\left[F_{i+1,j} - 2F_{i,j} + F_{i-1,j}\right]. \end{aligned} \tag{5.4}$$

The CRANK-NICOLSON method refers to the mesh point $\{\theta \cdot (i, j) + (1 - \theta) \cdot (i, j + 1)\}$, where θ is one-half (SMITH [1985]). The *explicit* method holds for $\theta = 0$, and the *implicit* method for $\theta = 1$. Substitute equation (5.4) into the transformed differential equation (5.3), the difference equation for an *internal* mesh point (i, j) with *equidistant* intervals to the left and to the right becomes:

$$\begin{aligned} &a_i F_{i-1,j} + b_i F_{i,j} + c_i F_{i+1,j} = e_i F_{i-1,j+1} + f_i F_{i,j+1} + g_i F_{i+1,j+1}, \quad \text{where} \\ &a_i = \left[\frac{\hat{a}_i}{2}\varphi - \widehat{b}_i \psi\right]\theta, \qquad e_i = -\left[\frac{\hat{a}_i}{2}\varphi - \widehat{b}_i \psi\right](1-\theta), \\ &b_i = 1 + \left[2\widehat{b}_i \psi - \hat{c}_i \Delta\tau\right]\theta, \qquad f_i = 1 - \left[2\widehat{b}_i \psi - \hat{c}_i \Delta\tau\right](1-\theta), \\ &c_i = -\left[\frac{\hat{a}_i}{2}\varphi + \widehat{b}_i \psi\right]\theta, \qquad g_i = \left[\frac{\hat{a}_i}{2}\varphi + \widehat{b}_i \psi\right](1-\theta), \\ &\varphi = \frac{\Delta\tau}{\Delta x}, \quad \psi = \frac{\Delta \tau}{(\Delta x)^2}. \end{aligned} \tag{5.5}$$

Here ψ denotes the *mesh ratio* (SMITH [1985]). Both the numer of intervals of the locational variable and the mesh ratio to be chosen determine the size of the time step. If the step size to the left, Δx_0, is not the same as that to the right, Δx, of a mesh point $(\imath, j)$, the following finite difference approximation to the partial derivatives is used (STIEFEL [1965], SCHWARZ [1988]):

$$\begin{aligned} &\frac{\partial F_{\imath,j+\frac{1}{2}}}{\partial \tau} \sim \frac{1}{\Delta \tau}\left[F_{\imath,j+1} - F_{\imath,j}\right], \quad \frac{\partial F_{\imath,j}}{\partial x} \sim \frac{1}{\Delta x + \Delta x_0}\left[F_{\imath+1,j} - F_{\imath-1,j}\right], \\ &\frac{\partial^2 F_{\imath,j}}{\partial x^2} \sim \frac{2}{\Delta x \Delta x_0\left[\Delta x + \Delta x_0\right]}\left[\Delta x_0\left(F_{\imath+1,j} - F_{\imath,j}\right) - \Delta x\left(F_{\imath,j} - F_{\imath-1,j}\right)\right]. \end{aligned} \tag{5.6}$$

Substitute equation (5.6) into the transformed differential equation (5.3), the finite difference equation for the *internal* mesh point $(\imath, j)$ with *unequal* intervals to the left and to the right becomes:

$$
\begin{aligned}
&a_\imath F_{\imath-1,j} + b_\imath F_{\imath,j} + c_\imath F_{\imath+1,j} = e_\imath F_{\imath-1,j+1} + f_\imath F_{\imath,j+1} + g_\imath F_{\imath+1,j+1}, \quad \text{with} \\
&a_\imath = \left[\hat{a}_\imath \varphi_\imath - \hat{b}_\imath \psi_l\right] \theta, \qquad\qquad e_\imath = -\left[\hat{a}_\imath \varphi_\imath - \hat{b}_\imath \psi_l\right] (1-\theta), \\
&b_\imath = 1 + \left[\hat{b}_\imath(\psi_l + \psi_r) - \hat{c}_\imath \Delta\tau\right] \theta, \qquad f_\imath = 1 - \left[\hat{b}_\imath(\psi_l + \psi_r) - \hat{c}_\imath \Delta\tau\right] (1-\theta), \\
&c_\imath = -\left[\hat{a}_\imath \varphi_\imath + \hat{b}_\imath \psi_r\right] \theta, \qquad\qquad g_\imath = \left[\hat{a}_\imath \varphi_\imath + \hat{b}_\imath \psi_r\right] (1-\theta), \\
&\varphi_\imath = \frac{\Delta\tau}{\Delta x + \Delta x_0}, \; \psi_l = \frac{2\Delta\tau}{\Delta x_0\left[\Delta x + \Delta x_0\right]}, \; \psi_r = \frac{2\Delta\tau}{\Delta x\left[\Delta x + \Delta x_0\right]}.
\end{aligned}
\tag{5.7}
$$

The *left boundary condition* of the transformed differential equation is $F_j^0 = 0$ for all j. Hence, we set a_1 and e_1 equal to zero in equation (5.5) for $i = 1$ or in equation (5.7) for $\imath = 1$, respectively.

The *right boundary condition* of the transformed differential equation (5.3) is obtained in the following way. First, note that the transformed differential equation is regular at $x = 1$ (or the differential equation (3.3) at $r = 0$), that is, the partial derivatives F_x and F_{xx} are finite at $x = 1$ $(r = 0)$. As a consequence, the partial differential equation reduces to

$$
\begin{aligned}
&0 = \frac{\partial F(x,\tau)}{\partial \tau} + \hat{a}(1)\frac{\partial F(x,\tau)}{\partial x} + \hat{b}(1)\frac{\partial^2 F(x,\tau)}{\partial x^2}, \quad \text{with} \\
&\hat{a}(1) = -\left[(\alpha\gamma + \rho q)\,\hat{m} - \rho^2 \hat{m}^2\right](s - t_0) \gtreqqless 0, \quad \hat{b}(1) = \frac{1}{2}\rho^2 \hat{m}^2 (s - t_0) > 0.
\end{aligned}
\tag{5.8}
$$

Secondly, the partial derivatives on the right boundary may be approximated by various boundary schemes (STIEFEL [1965], SCHWARZ [1988]). Unfortunately, these boundary schemes do *not* have the same local truncation error as the finite difference approximation at internal mesh points because no points outside of the region of definition can be taken into account. We tried the five simplest boundary schemes for $i = n$ $(x = 1)$ which are shown in Table 5.1. The *second* boundary scheme is a complete second-order finite difference approximation. However, it constrains the second partial derivatives at the points $(n, \cdot)$ and $(n-1, \cdot)$ to be *equal*. This restriction, which is similar to the "not-a-knot" condition used for cubic splines (DE BOOR [1978]), helps to stabilize possible oscillations. The *first* boundary scheme, which is extensively used in the finance literature (see, e. g., BRENNAN AND SCHWARTZ [1977, 1979], COURTADON [1982], or DUFFIE [1992]), imposes the same restriction with respect to the second partial derivative as the second boundary scheme. Moreover, the first partial derivative of the first boundary scheme applies to the point $(n - 1/2, \cdot)$ rather than to the point $(n, \cdot)$. The *third* boundary scheme is, in principle, equivalent to the one for the internal mesh points: the first partial derivative is of the same second order and the second partial derivatives at the points $(n, \cdot)$ and $(n-1, \cdot)$ are *not* restricted to be equal. However, the second partial derivative of this finite difference scheme is now of the third order. The *fourth* boundary scheme is of the third order for both partial derivatives, and finally, the *fifth* boundary scheme of the fourth order. In summary, *none* of the five boundary schemes considered in Table 5.1 corresponds exactly with the finite difference scheme for the internal mesh points. The choice of one of them is, therefore, at the user's discretion.

Table 5.1: Five Boundary Schemes.

Scheme #	Derivative	Denominator	F_n	F_{n-1}	F_{n-2}	F_{n-3}	F_{n-4}
1	F_x	Δx	1	−1	–	–	–
	F_{xx}	$(\Delta x)^2$	1	−2	1	–	–
2	F_x	$2\,\Delta x$	3	−4	1	–	–
	F_{xx}	$(\Delta x)^2$	1	−2	1	–	–
3	F_x	$2\,\Delta x$	3	−4	1	–	–
	F_{xx}	$(\Delta x)^2$	2	−5	4	−1	–
4	F_x	$6\,\Delta x$	11	−18	9	−2	–
	F_{xx}	$(\Delta x)^2$	2	−5	4	−1	–
5	F_x	$12\,\Delta x$	25	−48	36	−16	3
	F_{xx}	$12\,(\Delta x)^2$	35	−104	114	−56	11

Comment: Read the second row as $F_x = [1 \cdot F_n - 1 \cdot F_{n-1}] / \Delta x$ and similarly the other rows.

The finite difference equation on the right boundary is obtained from substituting the various boundary schemes into the reduced differential equation (5.8). In the case of the second boundary scheme this equation becomes:

$$\widetilde{A}_{n,\,n-2} F_{n-2,j} + \tilde{a}_n F_{n-1,j} + \tilde{b}_n F_{n,j} = \widetilde{B}_{n,\,n-2} F_{n-2,j+1} + \tilde{e}_n F_{n-1,j+1} + \tilde{f}_n F_{n,j+1}\,, \quad \text{with}$$

$$\begin{aligned} \widetilde{A}_{n,\,n-2} &= -\left[\tfrac{1}{2}\hat{a}_n\varphi + \hat{b}_n\psi\right]\theta, & \widetilde{B}_{n,\,n-2} &= \left[\tfrac{1}{2}\hat{a}_n\varphi + \hat{b}_n\psi\right](1-\theta), \\ \tilde{a}_n &= 2\left[\hat{a}_n\varphi + \hat{b}_n\psi\right]\theta, & \tilde{e}_n &= -\left[\hat{a}_n\varphi + \hat{b}_n\psi\right](1-\theta), \\ \tilde{b}_n &= 1 - \left[\tfrac{3}{2}\hat{a}_n\varphi + \hat{b}_n\psi\right]\theta, & \tilde{f}_n &= 1 + \left[\tfrac{3}{2}\hat{a}_n\varphi + \hat{b}_n\psi\right](1-\theta). \end{aligned} \tag{5.9}$$

Here, the parameters φ and ψ have been defined in equation (5.5). Collecting all the finite-difference equations shows that the resulting linear equation system is tridiagonal except for the last equation on the right boundary, irrespective of the boundary scheme considered in this paper. Since the inversion of a tridiagonal matrix is computationally much more efficient than the inversion of a sparse, non-tridiagonal matrix, we reduce the "boundary equation" (5.9) to tridiagonal form by means of equations (5.5) for the internal mesh points. In the case of the second boundary scheme, the resulting "boundary equation" becomes:[18]

[18] The reduction to tridiagonal form is not necessary in the case of the explicit method because no matrix inversion has to be done.

$$a_n F_{n-1,j} + b_n F_{n,j} = B_{n,\,n-2} F_{n-2,j+1} + e_n F_{n-1,j+1} + f_n F_{n,\,j+1}\,, \quad \text{with}$$

$$\begin{array}{ll} A_{n,\,n-2} = 0, & B_{n,\,n-2} = a_{n-1}\widetilde{B}_{n,\,n-2} - \widetilde{A}_{n,\,n-2} e_{n-1}, \\ a_n = a_{n-1}\tilde{a}_n - \widetilde{A}_{n,\,n-2} b_{n-1}, & e_n = a_{n-1}\tilde{e}_n - \widetilde{A}_{n,\,n-2}\, f_{n-1}, \\ b_n = a_{n-1}\tilde{b}_n - \widetilde{A}_{n,\,n-2} c_{n-1}, & f_n = a_{n-1}\tilde{f}_n - \widetilde{A}_{n,\,n-2} g_{n-1}, \end{array} \tag{5.10}$$

Finally, the linear equation system consisting of equations (5.5), (5.7), and (5.10) may be written compactly for the explicit, implicit and CRANK-NICOLSON methods as follows:

$$\mathbf{A}\,\mathbf{F}_j = \mathbf{B}\,\mathbf{F}_{j+1}, \quad \text{or written out} \tag{5.11a}$$

$$\begin{bmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & \\ & & \ddots & & \\ & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & a_n & b_n \end{bmatrix} \begin{bmatrix} F_{1,\,j} \\ \vdots \\ F_{n,\,j} \end{bmatrix} = \begin{bmatrix} f_1 & g_1 & & & & & \\ e_2 & f_2 & g_2 & & & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & e_{n-1} & f_{n-1} & g_{n-1} \\ & & [B_{n,\,n-4}] & [B_{n,\,n-3}] & B_{n,\,n-2} & e_n & f_n \end{bmatrix} \begin{bmatrix} F_{1,\,j+1} \\ \vdots \\ F_{n,\,j+1} \end{bmatrix} \tag{5.11b}$$

The elements in square brackets are non-zero in the case of boundary schemes #3 – #5. Matrix **A** is now a tridiagonal matrix, but not matrix **B**. Proceeding backwards in time, the price vector at time j is obtained from the price vector at time $j + 1$ by the operation $\mathbf{F}_j = \mathbf{A}^{-1}\,\mathbf{B}\,\mathbf{F}_{j+1}$. In the case of the explicit method, matrix **A** is the identity matrix, that is, no inversion is necessary as in the simple binomial model. In the case of the implicit method, matrix **B** is the identity matrix except for the last row. Matrix **A** is inverted by the tridiagonal matrix algorithm of PRESS ET AL. [1989].

The LAWSON-MORRIS method [1978] has among the various "methods of lines" the advantage that its two linear equation systems are tridiagonal if the problem at hand is symmetric with respect to the boundary conditions. In the case of the partial differential equation (5.3), the boundary conditions are not sammetric but the tridiagonal matrix structure can be restored by means of the GAUSSian elimination in the same way as before. The LAWSON-MORRIS method improves the accuracy in time, and proceeds in two steps. In the first step, the price vector at time-level j is calculated from a two-time step, (1, 0) PADé approximation to the analytical solution of an *ordinary* differential equation system in time, the latter being obtained from a transformation of the partial differential equation (5.3). Denote the first price vector at time-level j as $\mathbf{F}_{(1);j}$ and the price vector at time-level $j + 2$ (the starting value) as $\mathbf{F}_{j+2}$, that is, two time steps earlier (when moving backwards in time), then one can write for the first price vector:

$$\widetilde{\mathbf{A}}^{(1)}\,\mathbf{F}_j^{(1)} = \mathbf{F}_{j+2}, \quad \text{with } \widetilde{\mathbf{A}}^{(1)} := \left[\mathbf{I} - 2\Delta t\,\widetilde{\mathbf{D}}\right], \quad \widetilde{\mathbf{D}} := \frac{1}{\Delta t}\left[\mathbf{I} - \widetilde{\mathbf{A}}\right], \tag{5.12}$$

where Δt denotes the size of the time step, **I** the identity matrix and $\widetilde{\mathbf{A}}$ the non-tridiagonal matrix obtained from equations (5.5), (5.7) and (5.9). The linear equation system (5.12) can be transformed into a tridiagonal linear equation system as in equation (5.11) before. Using a superscript to denote the matrices that apply to the first price vector $\mathbf{F}^{(1)}$, this linear equation system may be written as follows:

$$\mathbf{A}^{(1)}\,\mathbf{F}_j^{(1)} = \mathbf{B}^{(1)}\,\mathbf{F}_{j+2}, \quad \text{or } \mathbf{F}_j^{(1)} = \left[\mathbf{A}^{(1)}\right]^{-1}\mathbf{B}^{(1)}\,\mathbf{F}_{j+2}\,. \tag{5.13}$$

Matrix $\mathbf{A}^{(1)}$ is now tridiagonal and Matrix $\mathbf{B}^{(1)}$ is the identity matrix except for the last line. In the second step, a second price vector denoted as $\mathbf{F}^{(2)}$ is calculated by means of equation system (5.11) for $\theta = 1$, that is, for the implicit method. Applying the implicit method to two consecutive time steps, the second price vector becomes at time-level *j*:

$$\mathbf{F}_j^{(2)} = \left[\mathbf{A}^{-1}\,\mathbf{B}\right]^2\mathbf{F}_{j+2}\,. \tag{5.14}$$

Finally, the extrapolated price vector $\mathbf{F}_j$ at time-level *j* is a linear combination of the two price vectors of equations (5.13) and (5.14):

$$\mathbf{F}_j = 2\,\mathbf{F}_j^{(2)} - \mathbf{F}_j^{(1)}. \tag{5.15}$$

The extrapolated price vector, $\mathbf{F}_j$, serves as initial price vector for the next two time steps.

6. The Accuracy of Finite Difference Methods

The computer program has been written in PASCAL (JENSEN AND WIRTH [1978]) and runs on the APPLE® MACINTOSH™ family, the machine precision of which is 19 – 20 decimal digits (the range of real numbers is from $1.9 \cdot 10^{-4951}$ to $1.1 \cdot 10^{4932}$).

We look first at the *stability* of the five boundary schemes by computing the numerical error for a *discount bond*. Numerical experiments indicate that the first four boundary schemes seem to be *stable* for all four finite difference methods considered in this paper, that is, the numerical error shrinks as the meshes get smaller, holding the mesh ratio constant. However, the fifth boundary scheme seems to be *unstable*: the numerical error grows infinitely large (although very slowly) as the meshes get smaller, holding the mesh ratio constant. This might be due to the fact that the local truncation error of the fifth boundary scheme is much smaller than that of the internal mesh points, that is, it is much more accurate than necessary, compared with the finite difference approximation at internal mesh points. While the bond price is nailed down to zero on the left boundary ($r = \infty$) for every time-level, this is not the case on the right boundary ($r = 0$). In the latter case, the numerical error propagates forwards to the next time-level without any readjustment. The fifth boundary scheme is disregarded in the sequel.

It is well known that the explicit mehod is unstable for “big” mesh ratios (SMITH [1985]). Since small mesh ratios require a great deal of time steps, the explicit method is computationally not efficient. With this respect, the other three finite difference methods considered in this paper are preferable. Although the implicit method, the CRANK-NICOLSON method, and the LAWSON-MORRIS method are stable for any mesh ratio, these (and the explicit) methods may exhibit slowly decaying finite *oscillations*. This phenomenon might be due to two facts. First, the larger the number of interest-rate intervals, the smaller the local truncation error, but at the same time, the larger is the number and range of eigenvalues of the PADé approximant $\mathbf{A}^{-1}\,\mathbf{B}$ of equation (5.11). As a consequence, the analytical solution of the difference equations may contain a large number of components with widely varying rates of decay. Equations giving rise to this phenomenon are said to be *stiff* (SMITH [1985]). Secondly, numerical studies indicate that very slowly decaying finite oscillations can occur in the neighbourhood of discontinuities in the initial values or between initial values and boundary values. An example for the parabolic partial differential equation of the heat conduction in a rod solved with the CRANK-NICOLSON method can be found in SMITH [1985, pp. 122 – 124]. The parabolic partial differential equation (3.4) or (5.3), respectively, has a singularity between the initial values, which are equal to one in the case of the discount bond, and the boundary values, which are zero on the right boundary. Therefore, oscillations may occur in the neighbourhood of the expiration date or coupon date. Indeed, our own numerical experiments indicate that oscillations occur after each coupon date of the straight bond for large mesh ratios, especially with the CRANK-NICOLSON method. Finite oscillations, however, are not restricted to the CRANK-NICOLSON method, nor to singularities, as was demonstrated by GOURLAY AND MORRIS [1980] for a variety of multiple time-level finite difference methods with extrapolation.

We compare the performance of the finite difference methods under consideration for the case of the *discount bond*. The discount bond has only one discontinuity in the neighbourhood of the maturity date, in contrast to the coupon bond which has additional discontinuities on each coupon date, and in contrast to the callable bond which has additional discontinuities on each notice day. We find that the LAWSON-MORRIS method, when optimized with respect to the parameters of the finite difference method, performs slightly best. Hence, the results to follow refer to the LAWSON-MORRIS method.

The numerical accuracy of the finite difference methods under consideration when applied to the *callable bond* is rather poor, given a number of interest-rate intervals which is both comparable with similar problems (see GOURLAY AND MORRIS [1980]) and computationally feasible.[19] In fact, the numerical accuracy is, by and large, *one* significant decimal digit only (see, e. g., Table A.3). Hence, the finite difference methods under consideration are not useful for real applications. Moreover, we find that many computed prices of the embedded call option turn out to be *negative*, in particular for the third and fourth boundary scheme.

What is the reason for this poor numerical accuracy or the negative prices? We explain this phenomenon by the discontinuity in the values of the early redemption condition which prevails on each notice day. A closer look at the evolution of the price vector of a particular callable bond in time reveals this fact quite impressively. To bear out this assertion most clearly, we chose an European callable bond, the analytical price of which can be computed with approximate

[19] The size of the *stack memory* restricts the length of vectors to about 250 elements due to the 32-kilobytes limit. This limitation can be circumvented with PASCAL if vectors are declared as pointers to arrays which are dynamically allocated variables of the *heap memory*. The size of the heap memory is limited by the size of the *random access memory* of the PC at hand. Besides the size of various memories, there is a limitation of *computation time* in real applications. The computation time grows, by and large, with the square of the number of interest-rate intervals, given a tridiagonal equation system.

machine precision (BÜTTLER AND WALDVOGEL [1993a & b]). The European callable bond under consideration has a maximum life of 6.811 years until the final redemption date, bears an annual coupon of 7%, and the single call date is the last but one coupon date. Moreover, there is a notice period of two months ahead of the call date. The bold-faced yield curve shown in Figure 2 is used. Note that the annual coupon of 7% lies above the yields with terms greater than about four years; hence the price of the underlying straight bond first declines and then grows as a function of the time period until expiration. The parameters of the LAWSON-MORRIS method have been chosen for this and all the other examples to be: $n_1 = 50$, $r_m = 0.15$, $n_2 = 50$, $\psi = 200$, $\Delta t = 1/74th$ of a year, $s = 10$ and $m;^{\wedge} = 1$.[20] Proceeding backwards in time, we stop the calculation for the first time an instant before the notice day and look at the numerical error of the callable bond, which is shown in the Figures A.1. The panel (a) refers to interest rates between zero and 200%, whereas the panel (b) refers to the relevant interest-rate range between zero and 15%. The numerical error is very small so far. The early redemption condition on the notice day is shown as kinked curve in Figure 3. The numerical error as a function of the instantaneous interest rate has a spike at the "break-even" interest rate one time step after the notice day. The computed and theoretical prices of the callable bond one time step after the notice day are shown in Figure 3 and the percentage error in the Figures A.2. Although this spike broadens and spreads out over the next few time steps, it introduces slowly decaying finite oscillations which amplify the numerical error of the callable bond compared with that of the underlying straight bond as can be seen from the Figures A.3 – A.5. On the final redemption date, the numerical error of the callable bond is in absolute value greater than 1.5% in the relevant range of interest rates (see Figure A.4b), while the numerical error of the underlying straight bond remains in absolute values less than 0.7% (see Figure A.5). *We conclude that the difference in accuracy between the callable bond and its underlying straight bond is entirely due to the discontinuity in the values of the early redemption condition. Moreover, the accuracy in the relevant range of interest rates depends entirely on the boundary scheme under consideration (see Figures A.1 – A.5)*. In this example, the computed price of the embedded call option turns out to be negative for the boundary schemes #3 and #4.

Which boundary scheme should be chosen? Since, in general, you do not know the analytical solution of the partial differential equation in question, you would probably choose that boundary scheme which gives you the smallest numerical error with respect to a similar security

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[20] As noted in the fifth section, the number of time steps between two points in time of interest is always chosen to be an *integer* number: this requires an adequate adjustment of the mesh ratio which was originally chosen. The same applies to the number of interest-rate intervals originally chosen, n_1, since the current value of the instantaneous interest rate, r_0, lies always on a mesh point. The actual locational steps are $\Delta x = 0.0025912869$ for $r \in [0, r_m]$ and $\Delta x_0 = 0.0174087131$ for $r \in [r_m, +\infty]$. Proceeding backwards in time, the actual number of time steps is 74 for a whole year ($\Delta\tau = 0.0013513514$), 12 between a coupon date and the following notice date ($\Delta\tau = 0.0013888889$), 62 between a notice date and the following coupon date ($\Delta\tau = 0.0013440860$), and 60 between the last coupon date and the present date ($\Delta\tau = 0.0013518519$). The actual mesh ratios with respect to the time step of a whole year are $\psi = 201.25$ for $r \in [0, r_m]$ and $\psi_0 \equiv \Delta\tau / [\Delta x_0]^2 = 4.46$ for $r \in [r_m, +\infty]$. Recall that a year is equal to $1/s = 0.1$.

with a known analytical solution, that is, the straight bond.[21] A sample of underlying straight bonds is shown in Table A.2. The boundary scheme #4 has the smallest root mean square error for this sample. This choice, however, is misleading, given the same sample of corresponding callable bonds of Table A.3. Now, the boundary scheme #2 has the smallest root mean square error. *We conclude that the boundary scheme which yields the smallest numerical error with respect to the underlying straight bond does not perform best with respect to the callable bond.*

Why? Note that all the four boundary schemes considered in this paper show the same type of oscillation due to the early redemption condition (see Figure A.2b), but with respect to the underlying straight bond, the boundary schemes #3 and #4 show no oscillation while the boundary schemes #1 and #2 show very slowly decaying oscillations (see Figure A.5a). Moreover, we find that the numerical error is greater for the first two boundary schemes than for the remaining two boundary schemes on the notice day (see Figure A.1b), but after the notice day this ordering is reversed (see also Tables A.2 and A.3). *We conclude that the "not-a-knot" condition of the boundary schemes #1 and #2 helps to stabilize the oscillations which occur due to the discontinuity in the values of the early redemption condition*. This, in turn, improves the performance of the first two boundary schemes relative to that of the remaining two boundary schemes.

7. Are Negative Option Prices Possible?

In an empirical study, LONGSTAFF [1992] finds that "nearly two-thirds of the call values implied by a sample of recent callable bonds are negative." Notice that the prices of the embedded call option were computed entirely from *quoted* prices of both the callable bond and the underlying straight bond, that is, from pairs of traded and otherwise identical bonds. This empirical evidence found by LONGSTAFF might have two reasons. First of all, markets for callable bonds might be inefficient, but high transaction costs prevent riskless arbitrage. Secondly, it is conceivable that the pricing of callable bonds is inaccurate if done by numerical methods at all.

Table A.4 compares the quoted price of each callable bond with the analytical price for the cantonal and federal bonds belonging to the corresponding "bond baskets" of the SWISS NATIONAL BANK. Since the quoted price of each underlying straight bond is not available, we use the analytical price of VASICEK's model. We assume, thereby, that the analytical price of each straight bond would be equal to its quoted price if the straight bond were traded. This assumption seems to be reasonable given the results of Table A.1 which show that quoted and analytical prices of each straight bond were indeed quite close to each other on the trading day in question.

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[21] If you knew the analytical solution of a partial differential equation, then you would probably not wish to apply a finite difference method to this differential equation.

Two facts from Table A.4 are worth mentioning. First, the "observed" (implied) price of each embedded call option is *negative* except for two callable bonds. The two exceptions (security numbers 15 712 and 15 718) refer to federal bonds with a maximum time period until expiration of 19.942 and 20.172 years, respectively, and with ten call dates. The implied price of each of these two embedded call options is approximately half the analytical price. Secondly, the analytical price of each embedded call option shown in column 7 of Table A.4 is positive. Moreover, it is accurate for the decimal digits displayed. Finally, some pairs of callable bonds (security numbers {16 242, 17 459, 17 360}, {15 712, 15 718}, {15 736, 15 738}, {15 745, 15 227}, {15 740, 15 749, 15 753}) are identical except for the time period until the last possible redemption date. During the period the bond cannot be called, the semi-American call behaves like the European call with respect to time: the call price first grows, then declines and finally approaches zero as the time period grows infinitely large. The maximum call value is reached after four to six years for the coupons under consideration. Since the time period until expiration is greater than six years for each of the above mentioned pairs of bonds, all the prices of the embedded call option decline with increasing time.

8. Conclusions

This study implies four results. First, the numerical error is greater for the callable bond than for the straight bond. This phenomenon can be attributed to the discontinuity in the values of the early redemption condition. Given a number of interest-rate intervals which is both comparable with similar problems and computationally feasible, the numerical accuracy is, by and large, *one* significant decimal digit only. Hence, the finite difference methods considered in this paper are not useful for real applications. Moreover, many computed prices of the embedded call option turn out to be *negative*. The phenomenon of negative *computed* prices of the embedded call option has also been observed, but has *not* been explained, in a study by GIBSON-ASNER [1990, p. 670].[22] In an empirical paper, LONGSTAFF [1992] finds that "nearly two-thirds of the call values implied by a sample of recent callable bond prices are negative." We argue that the negative computed or implied prices of the embedded call option might be due to the numerical error of the finite difference methods.

Secondly, the numerical accuracy of the callable bond price computed for the relevant range of interest rates between zero and, say, fifteen percent depends entirely on the finite difference scheme which is chosen for the boundary points. We compare the numerical error for four different boundary schemes, including the one which is extensively used in the finance literature (see, e. g., BRENNAN AND SCHWARTZ [1977, 1979], COURTADON [1982], or DUFFIE [1992]). We show that *none* of the four boundary schemes considered in this paper corre-

22 Although many results in LEITHNER [1992] point to negative computed prices of the embedded call option, he did neither observe this possibility nor mention any numerical peculiarity.

sponds exactly with the finite difference scheme for the internal mesh points. The choice of one of them is, therefore, at the user's discretion.

Thirdly, the boundary scheme which yields the smallest numerical errror with respect to the (underlying) straight bond does *not* perform best with respect to the callable bond. The boundary scheme #4 has the smallest root mean square error with respect to the underlying straight bond for the sample of Table A.3, in contrast to the boundary scheme #2 which has the smallest root mean square error with respect to the callable bond. This phenomenon can be explained by the "not-a-knot" condition which the boundary schemes #1 and #2 impose on the second partial derivatives. Since, in general, you do not know the analytical solution of the partial differential equation in question, you would probably choose that boundary scheme which gives you the smallest numerical error with respect to a similar security with a known analytical solution, that is, the straight bond. However, this choice is misleading.

Fourthly, we find a similar result to that of LONGSTAFF [1992], namely that all but two call option prices implied by the sample of Table A.4 are negative. This empirical evidence might have two reasons. First of all, markets for callable bonds might be inefficient, but high transaction costs prevent riskless arbitrage. Secondly, it is conceivable that the pricing of callable bonds is inaccurate if done by numerical methods at all.

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Appendix: Tables A.1 – A.5 and Figures A.1 – A.5

Table A.1: Price of Straight Bonds Belonging to "Bond Baskets" of the SWISS NATIONAL BANK and Numerical Error of Finite Difference Method. [1]

Number of Security	Name of Security	Years to Maturity	Price of Straight Bond			Percentage Error	
			Observed [3]	Analytical [4]	Finite Diff. M. [5]	Analytical [6]	Finite Diff. M. [7]
Col. 1	Col. 2	Col. 3	Col. 4	Col. 5	Col. 6	Col. 7	Col. 8
15 851	4 1/4% Aargau 1988–1998	6.894	86.25	85.40	85.00	– 1.0	– 0.5
16 136	6 3/4% Basel-Stadt 91–2001	9.203	100.25	99.94	99.13	– 0.3	– 0.8
16 209	4 1/4% Bern 1988–1999 [2]	7.175	85.50	85.13	84.69	– 0.4	– 0.5
16 452	6 3/4% Bern 1991–2001	9.144	100.25	99.92	99.11	– 0.3	– 0.8
16 440	4 1/4% Genf 1987–2002 [2]	10.256	83.50	82.75	81.82	– 0.9	– 1.1
16 380	4% Genf 1988–1999	7.464	84.00	83.46	82.97	– 0.6	– 0.6
16 382	4 3/8% Genf 1989–2001	9.047	83.00	84.43	83.69	+ 1.7	– 0.9
17 566	4% Zürich 1988–1998	6.247	86.50	84.82	84.52	– 1.9	– 0.4
15 720	4% Eidg. 1988–1998	6.053	88.25	85.08	84.81	– 3.6	– 0.3

[1] Trading day 23 December 1991. VASICEK's theoretical term structure of interest rates has been estimated from the yields of the reference bonds of footnote 2. The underlying interest-rate process is the ORNSTEIN-UHLENBECK process $dr = \alpha(\gamma - r)\,dt + \rho\,dz$, where r is the instantaneous interest rate and dz the GAUSS-WIENER process. The estimated parameters of the term structure are: $\alpha = 0.4418$, $\gamma = 0.03485$ [discrete-time equivalent 3.546% p. a.], $\rho = 0.1326$ and $q = 0.2117$. The instantaneous interest rate has been approximated by the tomorrow-next rate; it was $r_0 = 0.075228$ [discrete-time equivalent 7.813% p. a.] on the trading day in question. The bonds in the upper panel have been issued by cantons, the last bond has been issued by the Swiss confederation.

[2] Reference bonds. They have been used together with the current value of the instantaneous interest rate and the twelve-month Eurofranc interest rate to fit the theoretical term structure. Since the *yields to maturity* of the two reference bonds rather than the yields of the underlying discount bonds have been used as a first approximation, quoted and analytical prices do not match exactly for these two reference bonds.

[3] Quoted price of the exchange-traded straight bonds.

[4] Analytical price according to VASICEK's bond price model minus the accrued interest since the last coupon date.

[5] Price obtained from the numerical solution of the partial differential equation minus the accrued interest since the last coupon date. The LAWSON-MORRIS method with $m;^\wedge = 1$ has been employed to solve numerically the partial differential equation.

[6] Percentage deviation of the analytical from the quoted price: (col. 5 / col. 4 – 1) * 100.

[7] Percentage deviation of the numerical from the analytical price: (col. 6 / col. 5 – 1) * 100.

Table A.2: Price of Underlying Straight Bonds and Numerical Error of Finite Difference Methods. [1]

Number of Security	Name of Security	Years to Maturity [2]	Price of Underlying Straight Bond				
			Analytical [3]	Finite Difference Method with Boundary Scheme #... [4]			
				#1	#2	#3	#4
Col. 1	Col. 2	Col. 3	Col. 4	Col. 5	Col. 6	Col. 7	Col. 8
16310	4 1/2% Bern 1986–1998	6.547	86.97	87.08	87.34	86.62	86.69
16242	4 1/4% Bern 1987–1999	7.519	84.81	85.17	85.38	84.28	84.40
16506	5 1/4% Graub. 89–1999	7.728	90.39	90.81	91.02	89.81	89.94
17459	4 1/4% Waadt 1986–1998	6.269	86.01	86.05	86.31	85.70	85.76
17360	4 1/4% Wallis 1986–1998	6.478	85.78	85.88	86.13	85.45	85.51
17364	7% Wallis 1990–2000	8.811	101.29	102.07	102.22	100.46	100.67
17610	6 1/2% Zürich 1991–2001	9.322	98.34	99.30	99.39	97.41	97.65
15461	4 3/4% Eidg. 1986–2001	9.036	86.87	87.72	87.82	86.04	86.25
15710	4 1/4% Eidg. 1986–2001	9.292	83.41	84.33	84.40	82.54	82.76
15712	4 1/4% Eidg. 1986–2011	19.942	77.96	82.09	81.08	75.44	76.19
15718	4 1/4% Eidg. 1987–2012	20.172	77.86	82.04	81.01	75.31	76.08
15722	4% Eidg. 1988–1999	7.117	83.84	84.10	84.33	83.39	83.49
15726	4 1/4% Eidg. 1989–2001	9.050	83.61	84.46	84.55	82.79	83.00
15736	5 1/2% Eidg. 1989–1998	6.819	91.98	92.14	92.40	91.57	91.65
15738	5 1/2% Eidg. 1990–1999	7.042	91.96	92.19	92.43	91.51	91.61
15740	6 1/4% Eidg. 1990–2000	8.208	96.31	96.88	97.07	95.62	95.78
15745	6 1/2% Eidg. 1990–2000	8.378	97.88	98.51	98.69	97.15	97.33
15227	6 1/2% Eidg. 1990–1999	7.564	97.49	97.86	98.10	96.93	97.06
15747	6 3/4% Eidg. 1991–2001	9.081	99.89	100.76	100.88	99.01	99.23
15749	6 1/4% Eidg. 1991–2001	9.228	96.66	97.58	97.68	95.75	95.98
15751	6 1/4% Eidg. 1991–2003	11.400	97.41	99.12	99.00	96.05	96.42
15753	6 1/4% Eidg. 1991–2002	10.561	97.11	98.51	98.48	95.92	96.24

1 Trading day 23 December 1991. VASICEK's theoretical term structure of interest rates has been estimated from observed yields (see footnote 2 of Table A.1). The underlying interest-rate process is the ORNSTEIN-UHLENBECK process $dr = \alpha(\gamma - r)\,dt + \rho\,dz$, where r is the instantaneous interest rate and dz the GAUSS-WIENER process. The estimated parameters of the term structure are: $\alpha = 0.4418$, $\gamma = 0.03485$ [discrete-time equivalent 3.546% p. a.], $\rho = 0.1326$ and $q = 0.2117$. The instantaneous interest rate has been approximated by the tomorrow-next rate; it was $r_0 = 0.075228$ [discrete-time equivalent 7.813% p. a.] on the trading day in question. The bonds in the upper panel have been issued by cantons, those in the lower panel by the Swiss confederation.

2 Maximum time period until maturity of the callable bonds described in Table A.3.

3 Analytical price according to VASICEK's bond price model minus the accrued interest since the last coupon date.

4 Price obtained from the numerical solution of the partial differential equation by means of the LAWSON-MORRIS method minus the accrued interest since the last coupon date. The parameters of the finite difference method are: $n_1 = 50$, $n_2 = 50$, $\psi = 200$, $s = 10$, $m;^\wedge = 1$, and $r_m = 0.15$.

Table A.2: Continued. Price of Underlying Straight Bonds and Numerical Error of Finite Difference Methods.

Number of Security	Name of Security	Years to Maturity [2]	Percentage Error of Finite Difference Method — Boundary Scheme #1 [5]	#2 [6]	#3 [7]	#4 [8]
Col. 1	Col. 2	Col. 3	Col. 9	Col. 10	Col. 11	Col. 12
16310	4 1/2% Bern 1986–1998	6.547	+ 0.1	+ 0.4	− 0.4	− 0.3
16242	4 1/4% Bern 1987–1999	7.519	+ 0.4	+ 0.7	− 0.6	− 0.5
16506	5 1/4% Graub. 89–1999	7.728	+ 0.5	+ 0.7	− 0.6	− 0.5
17459	4 1/4% Waadt 1986–1998	6.269	+ 0.0	+ 0.4	− 0.4	− 0.3
17360	4 1/4% Wallis 1986–1998	6.478	+ 0.1	+ 0.4	− 0.4	− 0.3
17364	7% Wallis 1990–2000	8.811	+ 0.8	+ 0.9	− 0.8	− 0.6
17610	6 1/2% Zürich 1991–2001	9.322	+ 1.0	+ 1.1	− 0.9	− 0.7
15461	4 3/4% Eidg. 1986–2001	9.036	+ 1.0	+ 1.1	− 1.0	− 0.7
15710	4 1/4% Eidg. 1986–2001	9.292	+ 1.1	+ 1.2	− 1.0	− 0.8
15712	4 1/4% Eidg. 1986–2011	19.942	+ 5.3	+ 4.0	− 3.2	− 2.3
15718	4 1/4% Eidg. 1987–2012	20.172	+ 5.4	+ 4.1	− 3.3	− 2.3
15722	4% Eidg. 1988–1999	7.117	+ 0.3	+ 0.6	− 0.5	− 0.4
15726	4 1/4% Eidg. 1989–2001	9.050	+ 1.0	+ 1.1	− 1.0	− 0.7
15736	5 1/2% Eidg. 1989–1998	6.819	+ 0.2	+ 0.5	− 0.4	− 0.4
15738	5 1/2% Eidg. 1990–1999	7.042	+ 0.2	+ 0.5	− 0.5	− 0.4
15740	6 1/4% Eidg. 1990–2000	8.208	+ 0.6	+ 0.8	− 0.7	− 0.5
15745	6 1/2% Eidg. 1990–2000	8.378	+ 0.6	+ 0.8	− 0.7	− 0.6
15227	6 1/2% Eidg. 1990–1999	7.564	+ 0.4	+ 0.6	− 0.6	− 0.4
15747	6 3/4% Eidg. 1991–2001	9.081	+ 0.9	+ 1.0	− 0.9	− 0.7
15749	6 1/4% Eidg. 1991–2001	9.228	+ 1.0	+ 1.1	− 0.9	− 0.7
15751	6 1/4% Eidg. 1991–2003	11.400	+ 1.8	+ 1.6	− 1.4	− 1.0
15753	6 1/4% Eidg. 1991–2002	10.561	+ 1.4	+ 1.4	− 1.2	− 0.9
Root Mean Square Error			1.8	1.5	1.2	0.9

[5] Percentage deviation: (col. 5 / col. 4 – 1) * 100.
[6] Percentage deviation: (col. 6 / col. 4 – 1) * 100.
[7] Percentage deviation: (col. 7 / col. 4 – 1) * 100.
[8] Percentage deviation: (col. 8 / col. 4 – 1) * 100.

Table A.3: Price of Callable Bonds Belonging to “Bond Baskets” of the SWISS NATIONAL BANK and Numerical Error of Finite Difference Methods. [1]

Number of Security	Name of Security	Years to Maturity [2]	Number of Call Dates [3]	Call Condition [4]	Price of Callable Bond		
					Analytical [5]	Finite Difference Method	
						#1	#2
Col. 1	Col. 2	Col. 3	Col. 4	Col. 5	Col. 6	Col. 7	Col. 8
16 310	4 1/2% Bern 1986–1998	6.547	2 (1)	A	84.57	81.41	82.60
16 242	4 1/4% Bern 1987–1999	7.519	2 (1)	A	82.55	79.54	80.75
16 506	5 1/4% Graub. 89–1999	7.728	2 (2)	A	85.67	85.23	86.34
17 459	4 1/4% Waadt 1986–1998	6.269	2 (1)	A	83.66	80.38	81.58
17 360	4 1/4% Wallis 1986–1998	6.478	2 (1)	A	83.45	80.20	81.41
17 364	7% Wallis 1990–2000	8.811	2 (2)	A	95.68	91.39	93.04
17 610	6 1/2% Zürich 1991–2001	9.322	2 (2)	A	93.17	93.90	90.59
15 461	4 3/4% Eidg. 1986–2001	9.036	5 (1)	B	84.63	82.30	83.38
15 710	4 1/4% Eidg. 1986–2001	9.292	5 (1)	B	81.32	78.94	80.04
15 712	4 1/4% Eidg. 1986–2011	19.942	10 (1)	C	76.76	78.63	78.40
15 718	4 1/4% Eidg. 1987–2012	20.172	10 (1)	C	76.67	78.62	78.37
15 722	4% Eidg. 1988–1999	7.117	3 (1)	D	81.63	78.43	79.67
15 726	4 1/4% Eidg. 1989–2001	9.050	4 (1)	E	81.50	79.02	80.14
15 736	5 1/2% Eidg. 1989–1998	6.819	2 (2)	A	86.96	86.49	87.59
15 738	5 1/2% Eidg. 1990–1999	7.042	2 (2)	A	86.98	86.54	87.64
15 740	6 1/4% Eidg. 1990–2000	8.208	2 (2)	A	91.05	91.33	92.34
15 745	6 1/2% Eidg. 1990–2000	8.378	2 (2)	A	92.49	92.97	89.60
15 227	6 1/2% Eidg. 1990–1999	7.564	2 (2)	A	91.93	92.23	88.82
15 747	6 3/4% Eidg. 1991–2001	9.081	2 (2)	A	94.50	91.95	91.83
15 749	6 1/4% Eidg. 1991–2001	9.228	2 (2)	A	91.62	92.18	93.12
15 751	6 1/4% Eidg. 1991–2003	11.400	4 (2)	E	92.87	94.12	91.89
15 753	6 1/4% Eidg. 1991–2002	10.561	2 (2)	A	92.39	93.36	94.17

1 Trading day 23 December 1991. VASICEK’s theoretical term structure of interest rates has been estimated from observed yields (see footnote 2 of Table A.1). The underlying interest-rate process is the ORNSTEIN-UHLENBECK process $dr = \alpha(\gamma - r)\,dt + \rho\,dz$, where r is the instantaneous interest rate and dz the GAUSS-WIENER process. The estimated parameters of the term structure are: $\alpha = 0.4418$, $\gamma = 0.03485$ [discrete-time equivalent 3.546% p. a.], $\rho = 0.1326$ and $q = 0.2117$. The instantaneous interest rate has been approximated by the tomorrow-next rate; it was $r_0 = 0.075228$ [discrete-time equivalent 7.813% p. a.] on the trading day in question. The bonds in the upper panel have been issued by cantons, those in the lower panel by the Swiss confederation.

2 Maximum time period until maturity.

3 The numbers in brackets denote the numbers of positive “break-even” interest rates. These numbers have been employed to compute the analytical prices for this Table because the finite difference methods are applied to the partial differential equation with non-negative interest rates only.

4 Call condition type A: the call price is equal to 100 on both call dates. Call condition type B: the first call price (when moving forwards in time) is equal to 101.5; annual reduction of 0.5 percentage points until 100. Call condition type C: the first call price (when moving forwards in time) is equal to 102.5; annual reduction of 0.5 percentage points until 100. Call condition type D: the first call price (when moving forwards in time) is equal to 100.5; annual reduction of 0.5 percentage points until 100. Call condition type E: the first call price (when moving forwards in time) is equal to 101; annual reduction of 0.5 percentage points until 100. There is a notice period of two months for all call conditions.

Table A.3: Continued. Price of Callable Bonds Belonging to "Bond Baskets" of the SWISS NATIONAL BANK and Numerical Error.

Number of Security	Name of Security	Price of Callable Bond with Bound. Scheme #... [6]		Percentage Error of Finite Difference Method Boundary Scheme			
		#3	#4	#1 [7]	#2 [8]	#3 [9]	#4 [10]
Col. 1	Col. 2	Col. 9	Col. 10	Col. 11	Col. 12	Col. 13	Col. 14
16310	4 1/4% Bern 1986–1998	92.84	91.12	– 3.7	– 2.3	+ 9.8	+ 7.7
16242	4 1/4% Bern 1987–1999	91.27	89.43	– 3.6	– 2.2	+ 10.6	+ 8.3
16506	5 1/4% Graub. 89–1999	95.16	93.48	– 0.5	+ 0.8	+ 11.1	+ 9.1
17459	4 1/4% Waadt 1986–1998	92.19	90.46	– 3.9	– 2.5	+ 10.2	+ 8.1
17360	4 1/4% Wallis 1986–1998	92.06	90.30	– 3.9	– 2.4	+ 10.3	+ 8.2
17364	7% Wallis 1990–2000	103.10	101.74	– 4.5	– 2.8	+ 7.8	+ 6.3
17610	6 1/2% Zürich 1991–2001	100.85	99.46	+ 0.8	– 2.8	+ 8.2	+ 6.7
15461	4 3/4% Eidg. 1986–2001	96.79	94.35	– 2.8	– 1.5	+ 14.4	+ 11.5
15710	4 1/4% Eidg. 1986–2001	95.33	92.71	– 2.9	– 1.6	+ 17.2	+ 14.0
15712	4 1/4% Eidg. 1986–2011	91.08	88.62	+ 2.4	+ 2.1	+ 18.7	+ 15.4
15718	4 1/4% Eidg. 1987–2012	90.77	88.36	+ 2.5	+ 2.2	+ 18.4	+ 15.2
15722	4% Eidg. 1988–1999	92.91	90.72	– 3.9	– 2.4	+ 13.8	+ 11.1
15726	4 1/4% Eidg. 1989–2001	94.32	91.74	– 3.0	– 1.7	+ 15.7	+ 12.6
15736	5 1/2% Eidg. 1989–1998	96.21	94.61	– 0.5	+ 0.7	+ 10.6	+ 8.8
15738	5 1/2% Eidg. 1990–1999	96.25	94.63	– 0.5	+ 0.8	+ 10.7	+ 8.8
15740	6 1/4% Eidg. 1990–2000	99.39	97.88	+ 0.3	+ 1.4	+ 9.2	+ 7.5
15745	6 1/2% Eidg. 1990–2000	100.54	99.08	+ 0.5	– 3.1	+ 8.7	+ 7.1
15227	6 1/2% Eidg. 1990–1999	100.16	98.69	+ 0.3	– 3.4	+ 8.9	+ 7.3
15747	6 3/4% Eidg. 1991–2001	102.05	100.68	– 2.7	– 2.8	+ 8.0	+ 6.5
15749	6 1/4% Eidg. 1991–2001	99.58	98.14	+ 0.6	+ 1.6	+ 8.7	+ 7.1
15751	6 1/4% Eidg. 1991–2003	102.09	99.99	+ 1.3	– 1.1	+ 9.9	+ 7.7
15753	6 1/4% Eidg. 1991–2002	99.67	98.37	+ 1.1	+ 1.9	+ 7.9	+ 6.5
Root Mean Square Error				2.6	2.1	11.8	9.6

5 Price obtained from the analytical solution by means of numerical quadrature involving GREEN's function (BüTTLER AND WALDVOGEL [1993a & b]) minus the accrued interest since the last coupon date. See footnote 3. *All* the digits displayed are significant (the accuracy is almost equal to the machine precision).

6 Price obtained from the numerical solution of the partial differential equation by means of the LAWSON-MORRIS method minus the accrued interest since the last coupon date. The parameters of the finite difference method are: $n_1 = 50$, $n_2 = 50$, $\psi = 200$, $s = 10$, $m;^\wedge = 1$, and $r_m = 0.15$.

7 Percentage deviation: (col. 7 / col. 6 – 1) * 100.

8 Percentage deviation: (col. 8 / col. 6 – 1) * 100.

9 Percentage deviation: (col. 9 / col. 6 – 1) * 100.

10 Percentage deviation: (col. 10 / col. 6 – 1) * 100.

Table A.4: Price of Callable Bonds Belonging to "Bond Baskets" of the SWISS NATIONAL BANK. [1]

Number of Security	Name of Security	Quoted or Computed Price of				
		Underlying	Callable Bond [3]		Embedded Call Option	
		Straight Bond [2]	Quoted [4]	Analytical [5]	Implied [6]	Analytical [7]
Col. 1	Col. 2	Col. 3	Col. 4	Col. 5	Col. 6	Col. 7
16310	4 1/2% Bern 1986–1998	86.97	89.75	82.57	– 2.78	4.40
16242	4 1/4% Bern 1987–1999	84.81	85.00	80.66	– 0.19	4.15
16506	5 1/4% Graubünden 89–1999	90.39	91.00	85.67	– 0.61	4.72
17459	4 1/4% Waadt 1986–1998	86.01	87.00	81.73	– 0.99	4.28
17360	4 1/4% Wallis 1986–1998	85.79	86.00	81.52	– 0.21	4.27
17364	7% Wallis 1990–2000	101.29	101.50	95.68	– 0.21	5.61
17610	6 1/2% Zürich 1991–2001	98.34	99.25	93.17	– 0.91	5.17
15461	4 3/4% Eidg. 1986–2001	86.87	88.60	78.27	– 1.73	8.60
15710	4 1/4% Eidg. 1986–2001	83.41	85.00	75.59	– 1.59	7.82
15712	4 1/4% Eidg. 1986–2011	77.96	75.00	71.71	2.96	6.25
15718	4 1/4% Eidg. 1987–2012	77.86	74.50	71.69	3.36	6.17
15722	4% Eidg. 1988–1999	83.84	87.10	78.13	– 3.26	5.71
15726	4 1/4% Eidg. 1989–2001	83.61	85.25	76.73	– 1.64	6.88
15736	5 1/2% Eidg. 1989–1998	91.98	94.75	86.97	– 2.77	5.01
15738	5 1/2% Eidg. 1990–1999	91.96	95.10	86.98	– 3.14	4.98
15740	6 1/4% Eidg. 1990–2000	96.31	99.25	91.05	– 2.94	5.26
15745	6 1/2% Eidg. 1990–2000	97.88	100.25	92.49	– 2.37	5.39
15227	6 1/2% Eidg. 1990–1999	97.49	100.25	91.93	– 2.76	5.56
15747	6 3/4% Eidg. 1991–2001	99.89	102.25	94.50	– 2.36	5.39
15749	6 1/4% Eidg. 1991–2001	96.66	99.35	91.62	– 2.69	5.04
15751	6 1/4% Eidg. 1991–2003	97.41	99.00	89.07	– 1.59	8.34
15753	6 1/4% Eidg. 1991–2002	97.11	99.25	92.39	– 2.14	4.72

[1] Trading day 23 December 1991. VASICEK's theoretical term structure of interest rates has been estimated from observed yields (see footnote 2 of Table A.1). The underlying interest-rate process is the ORNSTEIN-UHLENBECK process $\mathrm{d}r = \alpha(\gamma - r)\,\mathrm{d}t + \rho\,\mathrm{d}z$, where r is the instantaneous interest rate and dz the GAUSS-WIENER process. The estimated parameters of the term structure are: $\alpha = 0.4418$, $\gamma = 0.03485$ [discrete-time equivalent 3.546% p. a.], $\rho = 0.1326$ and $q = 0.2117$. The instantaneous interest rate has been approximated by the tomorrow-next rate; it was $r_0 = 0.075228$ [discrete-time equivalent 7.813% p. a.] on the trading day in question. The bonds in the upper panel have been issued by cantons, those in the lower panel by the Swiss confederation.

[2] Analytical price according to VASICEK's bond price model minus the accrued interest since the last coupon date.

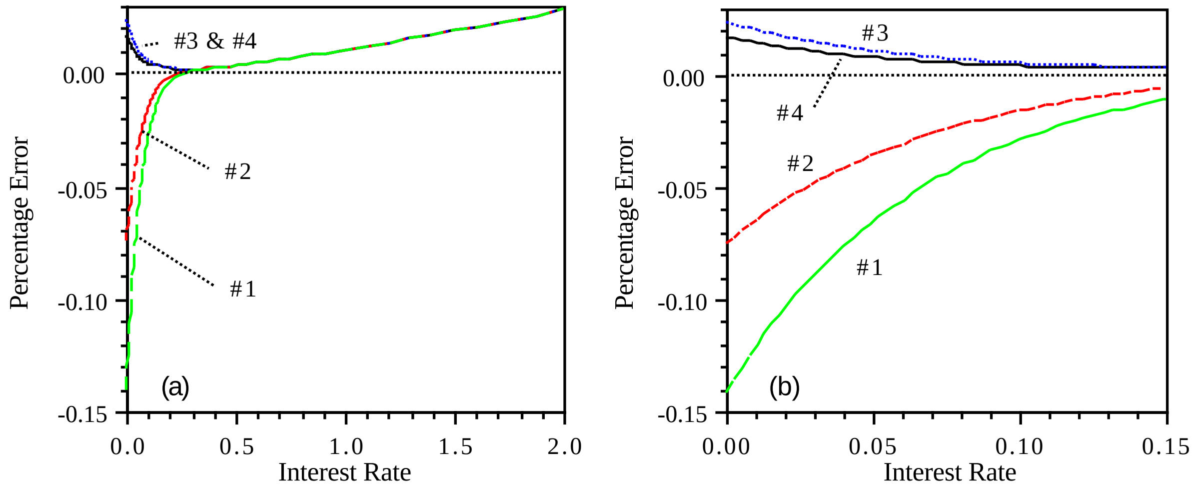
[3] The time periods until maturity, the number of call dates and the call conditions are shown in Table A.3.

[4] Quoted price on December 23, 1991.

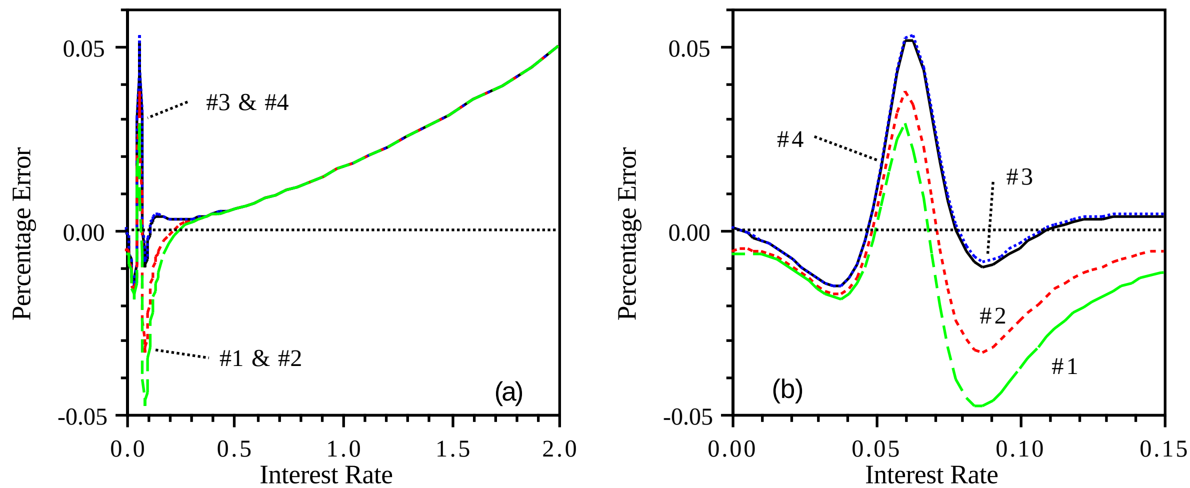
[5] Price obtained from the analytical solution by means of numerical quadrature involving GREEN's function (BüTTLER AND WALDVOGEL [1993a & b]) minus the accrued interest since the last coupon date. The analytical price has been computed for all the possible call dates given in Table A.3. *All* the digits displayed are significant.

[6] Observed price of the embedded call option in the sense that the analytical price of the straight bond would be equal to its quoted price if the underlying straight bond were traded: col. 3 – col. 4.

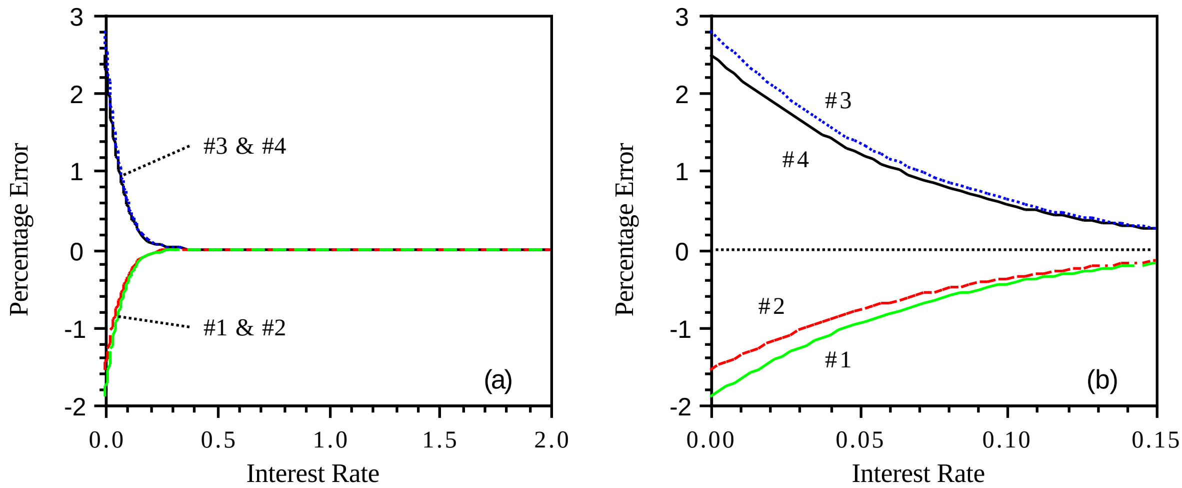
[7] Analytical price of the embedded call option: col. 3 – col. 5.



Figures A.1a & b: Percentage Error on the Notice Day. ‡

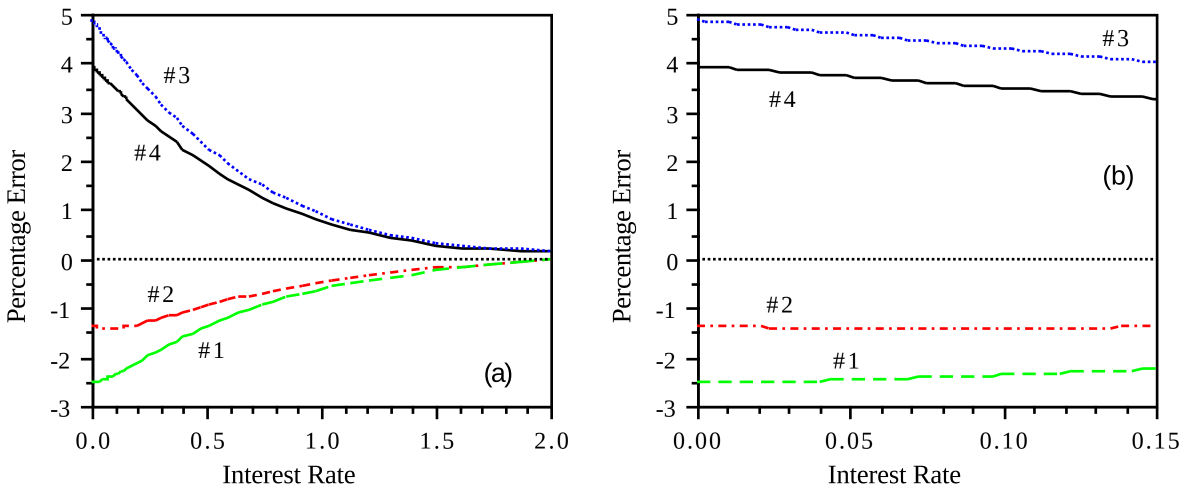


Figures A.2a & b: Percentage Error One Time Step after the Notice Day. ‡

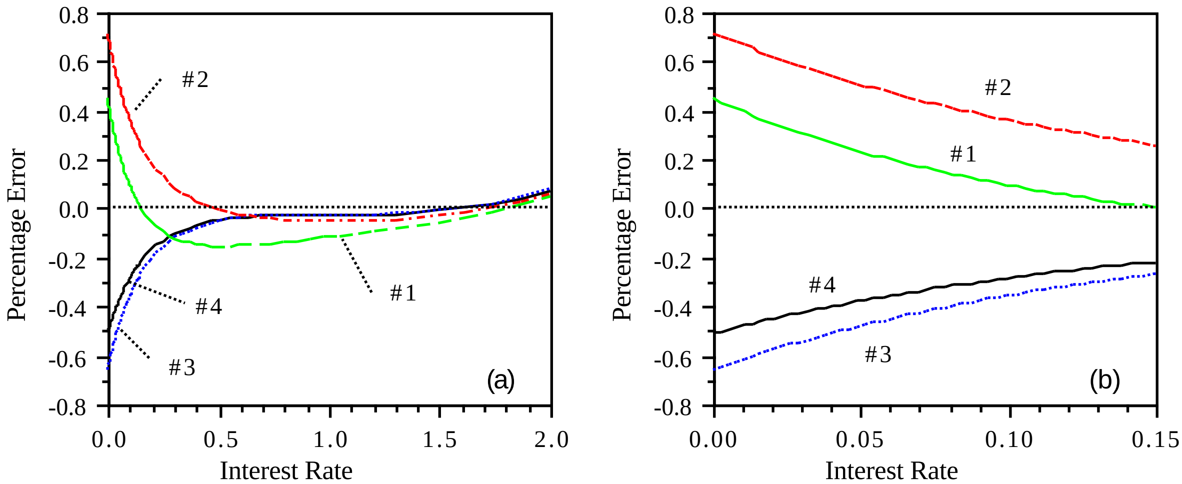


Figures A.3a & b: Percentage Error after Two Years. ‡

‡ The Parameters of the LAWSON-MORRIS method are $n_1 = 50$, $r_m = 0.15$, $n_2 = 50$, $\psi = 200$, $\Delta t = 1/74th$ of a year, $s = 10$ and m;^ = 1.



Figures A.4a & b: Percentage Error after 6.811 Years. ‡



Figures A.5a & b: Percentage Error of the Underlying Straight Bond. ‡

‡ The Parameters of the LAWSON-MORRIS method are $n_1 = 50$, $r_m = 0.15$, $n_2 = 50$, $\psi = 200$, $\Delta t = 1/74th$ of a year, $s = 10$ and m;^ = 1.