EVALUATION OF CALLABLE BONDS: FINITE DIFFERENCE METHODS, STABILITY AND ACCURACY ‡

by

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Abstract:

The purpose of this paper is to evaluate numerically the semi-American callable bond by means of finite difference methods. This study implies three results. First, the numerical error is greater for the callable bond price than for the straight bond price, and too large for real applications. This phenomenon can be attributed to the discontinuity in the values of the early redemption condition. Moreover, many computed prices of the embedded call option turn out to be *negative*. Secondly, the numerical accuracy of the callable bond price computed for the relevant range of interest rates depends entirely on the finite difference scheme which is chosen for the boundary points. The paper compares the numerical error for four different boundary schemes, including the one which is extensively used in the finance literature. Thirdly, the boundary scheme which yields the smallest numerical errror with respect to the straight bond does *not* perform best with respect to the callable bond.

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I. INTRODUCTION

The callable bond is a straight (coupon) bond with the provision that allows the debtor to buy back or to 'call' the bond for a specified amount, the call price, plus the accrued interest since the last coupon date at some time, the call date(s), during the life of the bond. Three types of callable bonds can be observed in financial markets. The American callable bond may be repurchased at any time on or before the final redemption day, in contrast to the European or semi-American counterparts, which may only be called at one or several specific dates, respectively. In the case of semi-American bonds, the debtor gives the bondholder two months' notice. The callable bond can be viewed as a compound security which consists of an otherwise identical straight bond and of an embedded call option, which is not traded and the price of which is, therefore, not observable. The embedded call option, which is written on the underlying straight bond, can be viewed as being 'sold' by the initial bondholder to the issuer of the callable bond, the debtor. Hence, the price of the callable bond must be equal to the price of the underlying straight bond less the price of the embedded call option at any time.

This paper is motivated by our experience with finite difference methods as well as the wrong numerical results which have been published in two studies. Gibson-Asner (1990, Table 6, p. 666) reports the *computed* prices of two almost identical embedded call options for various points in time. These two options are identical except for the last possible redemption dates which differ by roughly three months. The prices of these two options with, e. g., approximately eight years until expiration are reported to be {15.3192, 5.3789}. Our computation indicates that these two prices differ by 0.2 only. In Leithner (1992, Fig. 5.5, p. 145), the *computed* price of the semi-American call option is less than that of the corresponding European call for small interest rates, but greater for large interest rates. These wrong results presented in Gibson-Asner and Leithner *must* be due to numerical errors.

The purpose of this paper is to evaluate numerically the semi-American callable bond by means of four finite difference methods. As an example, we use the one-factor model of the (real) term structure of interest rates proposed by Vasicek (1977). The numerical solution of the finite difference method will be compared with the analytical solution which was derived in Büttler and Waldvogel (1993a, b).

The outline of the paper is as follows. The next section describes the callable bond price model. The finite difference methods considered in this paper are explained in the third section, followed by a section which addresses the question of numerical accuracy of the finite difference methods. Conclusions are given at the end.

II. THE CALLABLE BOND PRICE MODEL

Vasicek (1977) derives the following parabolic partial differential equation to determine the price of a default-free discount bond, $P(r, \tau)$, promising to pay one unit of money on the maturity day:

$$P_\tau = \frac{1}{2}\rho^2 P_{rr} + [\alpha(\gamma - r) + \rho q] P_r - rP, \tag{1}$$

where the subscripts denote partial derivatives, r the instantaneous interest rate, τ the remaining time period until the expiration of the discount bond, and q the market price of interest-rate risk assumed to be constant. If arbitrage opportunities are ruled out, the market price of interest-rate risk must be the same for all discount bonds of different maturities. Empirically, we would expect q to be positive. The remaining parameters are those of the underlying interest rate process, namely the Ornstein-Uhlenbeck process $\mathrm{d}r = \alpha\,[\gamma - r]\,\mathrm{d}t + \rho\,\mathrm{d}z$, with $\alpha > 0$ the speed of adjustment, $\gamma > 0$ the long-run 'equilibrium' value of the instantaneous interest rate, t the calendar time, $\rho > 0$ the constant instantaneous standard deviation (volatility) of the instantaneous interest rate, and dz the Gauss-Wiener process.

On the maturity day, the price of the discount bond is equal to one: this is the 'initial' condition of (1), noting that 'time' τ is measured backwards. The boundary conditions, which lead to Vasicek's bond price formula, are given by (i) $P(r, \tau) \to 0$ as $r \to \infty$, right boundary (Brennan and Schwartz, 1977; 1979), and (ii) $P(r, \tau) = \mathbb{O}(e^{-\vartheta r})$, $\vartheta > 0$, as $r \to -\infty$, left boundary (Büttler and Waldvogel, 1993a). The right boundary condition says that the price of a discount bond tends to zero as the instantaneous interest rate grows infinitely large. The left boundary condition ensures that a particular solution is chosen which grows exponentially at most as the absolute value of the interest rate becomes very large.

The callable bond satisfies the same partial differential (1) and the same boundary conditions as the discount bond between the notice dates. On the 'initial' day, i. e., the last possible redemption date, the price of the callable bond is equal to the face value plus the last coupon. Moreover, the callable bond is subject to the *early redemption condition* prevailing on each notice day when the debtor has to decide whether or not to call the bond. The call policy is optimal if the issuer of the callable bond minimizes his outstanding debt. Therefore, he will call the bond if the price of the callable bond is greater than the time value of the call price (including the next coupon). The 'break-even' (or critical) interest rate is that interest rate which equates the price of the callable bond an instant before the notice date (looking backwards in time) and the time value of the call price. Hence, the price of the callable bond an instant after the notice day is either equal to the time value of the call price if the actual interest rate is less than the 'break-even' interest rate or equal to the callable bond price an instant before the notice date if the actual interest rate is greater than the 'break-even' interest rate. This is the early redemption condition.

III. Finite Difference Methods

Four different finite difference methods have been applied to the partial differential equation (1), namely the explicit method, the implicit method, the Crank-Nicolson method and the implicit method with a two time-level extrapolation due to Lawson and Morris (1978), henceforth the Lawson-Morris method. Brennan and Schwartz (1978) have shown that the explicit method corresponds to the simple binomial model, in contrast to the implicit method which corresponds to a multinomial model with jumps.

We use Brennan and Schwartz' (1979) transformation of variables:

$$x = \frac{1}{1 + \hat{m} r}, \quad \tau = \frac{t - t_0}{s - t_0}, \quad (0 \le x,\ \tau \le 1), \quad F(x,\ \tau) = P(r,\ t). \tag{2}$$

Here, $\hat{m}$ is a scaling factor, s the maximum time until expiration of all the bonds considered, and τ measures the *elapsed* time rather than the time to maturity (the differential equation is still solved backwards in time). The transformed differential equation then becomes (3).

$$\begin{aligned} 0 &= \frac{\partial F(x,\ \tau)}{\partial \tau} + \hat{a}(x) \frac{\partial F(x,\ \tau)}{\partial x} + \hat{b}(x) \frac{\partial^2 F(x,\ \tau)}{\partial x^2} + \hat{c}(x)\, F(x,\ \tau), \quad \text{with} \\ \hat{a}(x) &= -\left[(\alpha\gamma + \rho q)\, \hat{m}\, x^2 - \alpha\,(1 - x)\, x - \rho^2 \hat{m}^2\, x^3\right] (s - t_0) \gtreqless 0, \\ \hat{b}(x) &= \frac{1}{2}\, \rho^2 \hat{m}^2\, x^4\, (s - t_0) > 0, \quad \hat{c}(x) = -\frac{(1 - x)}{\hat{m} x}\, (s - t_0) < 0. \end{aligned} \tag{3}$$

Two comments are necessary. First, although we consider only non-negative interest rates in this paper, the transformation (2) allows for a negative interest rate range which is sufficient in practical applications, for instance, $r > -100\%$ for $\hat{m} = 1$. In fact, a calculation of the analytical price of a callable bond with twenty years to expiration and with ten call dates has shown that the smallest ‘break-even’ interest rate is –13% (Büttler and Waldvogel, 1993b). Secondly, the numerical error calculated in this paper is based on the analytical price for positive ‘break-even’ interest rates (see footnote 3 of Table 3).

Table 1: Finite Differences for Internal Mesh Points.

	Derivative	Denominator	F_{i+1}	F_i	F_{i-1}
Equal Interest	F_x	$2\,\Delta x$	1	—	-1
Rate Interval	F_{xx}	$(\Delta x)^2$	1	-2	1
Unequal Interest	F_x	$\Delta x + \Delta x_0$	1	—	-1
Rate Interval	F_{xx}	$\Delta x\, \Delta x_0\, [\Delta x + \Delta x_0]$	$2\,\Delta x_0$	$-2\,[\Delta x + \Delta x_0]$	$2\,\Delta x$

Comment: Read the second row as $F_x = [1 \cdot F_{i+1} - 1 \cdot F_{i-1}] / [2\,\Delta x]$ and similarly the other rows. Δx_0 denotes the interval length to the left of the internal mesh point in question and Δx the interval length to the right of this mesh point.

The region of definition of the transformed differential equation, $\Omega = [0,\ 1] \times [0,\ 1]$, is now divided into meshes. Since we wish to calculate the price of callable bonds for the *current* value of the instantaneous interest rate, the meshes are *not* all of the same size. Moreover, it might be desirable to have narrow meshes for small interest rates such that the ‘break-even’ interest rate can be determined with ‘high’ accuracy. Therefore, the x-axis is divided into n_1 equal intervals between x_m and 1 (equivalently $r \in [0,\ r_m]$) and n_2 equal intervals between 0 and x_m (equivalently $r \in [r_m,\ +\infty]$). If n_2 is set equal to zero, then the whole range of positive interest rates is divided into n_1 equal intervals, possibly except for an initial step at $x = 0$ ($r = \infty$). The *current* value of the instantaneous interest rate, r_0, lies always on a mesh point. The steps between two points in time of interest (e. g., a coupon date and a notice date) will be of equal length. The mesh points are counted with index i along the x-axis and with index j along the time axis.

The standard second-order finite differences are applied to (3), namely $F_\tau = [F_{j+1} - F_j] / \Delta\tau$ for the partial derivative with respect to time, and those of Table 1 for the partial derivatives with respect to the interest rate (Smith, 1985; for unequal interest-rate intervals see Schwarz, 1988). For further reference, the *mesh ratio* is defined to be $\psi = \Delta\tau / (\Delta x)^2$.

The *left boundary condition* of the transformed differential equation is $F_j^0 = 0$ for all j. The *right boundary condition* of the transformed differential equation (3) is obtained in the following way. First, note that the transformed differential equation is regular at $x = 1$, that is, F, F_x and F_{xx} are finite at $x = 1$. Hence, the fourth term of (3) vanishes. Secondly, the partial derivatives on the right boundary may be approximated by various boundary schemes (Stiefel, 1965; Schwarz, 1988). Unfortunately, these boundary schemes do *not* have the same local truncation error as the finite difference approximation at internal mesh points because no points outside of the region of definition can be taken into account. We tried the five simplest boundary schemes for $i = n$ ($x = 1$) which are shown in Table 2. The *second* boundary scheme is a complete second-order finite difference approximation. However, it constrains the second partial derivatives at the points $(n, \cdot)$ and $(n - 1, \cdot)$ to be *equal*. This restriction, which is similar to the 'not-a-knot' condition used for cubic splines (De Boor, 1978), helps to stabilize possible oscillations. The *first* boundary scheme, which is extensively used in the finance literature (see, e. g., Brennan and Schwartz, 1977, 1979; Courtadon, 1982; Duffie, 1992), imposes the same restriction with respect to the second partial derivative as the second boundary scheme. Moreover, the first partial derivative of the first boundary scheme applies to the point $(n - 1/2, \cdot)$ rather than to the point $(n, \cdot)$. The *third* boundary scheme is, in principle, equivalent to the one for the internal mesh points: the first partial derivative is of the same second order and the second partial derivatives at the points $(n, \cdot)$ and $(n - 1, \cdot)$ are *not* restricted to be equal. However, the second partial derivative of this finite difference scheme is now of the third order. The *fourth* boundary scheme is of the third order for both partial derivatives, and finally, the *fifth* boundary scheme of the fourth order. In summary, *none* of the five boundary schemes considered in Table 2 is fully compatible with the finite difference scheme for the internal mesh points. The choice of one of them is, therefore, at the user's discretion.

IV. The Accuracy of Finite Difference Methods

We look first at the stability of the boundary schemes. Numerical experiments for a discount bond indicate that the first four boundary schemes seem to be *stable* for all four finite difference methods considered in this paper, that is, the numerical error shrinks as the meshes get smaller, holding the mesh ratio constant. However, the fifth boundary scheme seems to be *unstable*: the numerical error grows infinitely large (although very slowly) as the meshes get smaller, holding the mesh ratio constant. This might be due to the fact that the local truncation error of the fifth boundary scheme is much smaller than that of the internal mesh points. The fifth boundary scheme is neglected in the following.

It is well known that the explicit mehod is unstable for ‘big’ mesh ratios (Smith, 1985). Since small mesh ratios require a great deal of time steps, the explicit method is computationally not efficient. With this respect, the other three finite difference methods considered in this paper are preferable. Although these three methods are stable for any mesh ratio, they may exhibit slowly decaying finite *oscillations* in the neighbourhood of discontinuities in the initial values or between initial values and boundary values (Smith, 1985). Indeed, our own computations indicate that oscillations occur after each coupon date of the straight bond for large mesh ratios, especially with the Crank-Nicolson method. The results to follow refer to the Lawson-Morris method which performs slightly best.

The numerical accuracy of the finite difference methods under consideration when applied to the *callable bond* is rather poor, given a number of interest-rate intervals which is both comparable with similar problems (Gourlay and Morris, 1980) and computationally feasible. Moreover, we find that many computed prices of the embedded call option turn out to be *negative*, in particular for the third and fourth boundary schemes.

What is the reason for the poor numerical accuracy or the negative prices? We explain this phenomenon by the discontinuity in the values of the early redemption condition. A closer look at the evolution of the price vector of a particular callable bond in time reveals this fact quite impressively. To bear out this assertion most clearly, we chose a European callable bond, the analytical price of which can be computed with approximate machine precision (Büttler and Waldvogel, 1993a, b). The European callable bond under consideration has a maximum life of 6.811 years until the final expiration date and bears an annual coupon of 7%. Proceeding backwards in time, we stop the calculation for the first time an instant before the notice day and look at the numerical error of the callable bond as shown in Fig. 1. The instantaneous interest rate ranges between zero and 200% in the panel (a). The same function is shown in the panel (b) in a magnified mode for interest rates between zero and 15%. One time step after the notice day, the numerical error has a spike at the ‘break-even’ interest rate as shown in Fig. 2. Although this spike broadens and spreads out over the next few time steps, it introduces slowly decaying finite oscillations which amplify the numerical error of the callable bond compared with that of the underlying straight bond as shown in Fig. 3 – 5. *We conclude that the difference in accuracy between the callable bond and its underlying straight bond is entirely due to the discontinuity in the values of the early redemption condition*. The numerical errors for a small sample of exchange-traded callable bonds are shown in Table 3.

V. CONCLUSIONS

This study implies three results. First, the numerical error is greater for the callable bond price than for the straight bond price, and too large for real applications which require a two-digit accuracy at least. This phenomenon can be attributed to the discontinuity in the values of the early redemption condition. Moreover, many computed prices of the embedded call option turn out to be *negative*. The phenomenon of negative *computed* prices of the embedded call option has also been observed, but has *not* been explained, in Gibson-Asner (1990, p. 670). In an empirical study, Longstaff (1992) finds that 'nearly two-thirds of the call values implied by a sample of recent callable bond prices are negative.' Our own computations indicate that all but two call option prices implied by the sample of Table 3 are negative. We argue that the negative computed or implied prices of the embedded call option might be due to the numerical error of the finite difference methods under consideration.

Secondly, the numerical accuracy of the callable bond price computed for the relevant range of interest rates depends entirely on the finite difference scheme which is chosen for the boundary points (see Fig. 1 – 5).

Thirdly, the boundary scheme which yields the smallest numerical errror with respect to the straight bond does *not* perform best with respect to the callable bond. The boundary scheme #4 has the smallest root mean square error with respect to the underlying straight bond for the sample of Table 3, in contrast to the boundary scheme #2 which has the smallest root mean square error with respect to the callable bond. Since, in general, you do not know the analytical solution of the partial differential equation in question, you would probably choose that boundary scheme which gives you the smallest numerical error with respect to a similar security with a known analytical solution, that is, the straight bond. However, this choice is misleading.

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Table 3: Price of Callable Bonds Belonging to 'Bond Baskets' of the SWISS NATIONAL BANK and Numerical Error of Finite Difference Methods. [1]

Number of Security	Name of Security	Years to Maturity [2]	Number of Call Dates [3]	Call Condition [4]	Price of Callable Bond		
					Analytical [5]	Finite Difference Method #1	Finite Difference Method #2
Col. 1	Col. 2	Col. 3	Col. 4	Col. 5	Col. 6	Col. 7	Col. 8
16 310	4 1/2% Bern 1986–1998	6.547	2 (1)	A	84.57	81.41	82.60
16 242	4 1/4% Bern 1987–1999	7.519	2 (1)	A	82.55	79.54	80.75
16 506	5 1/4% Graub. 89–1999	7.728	2 (2)	A	85.67	85.23	86.34
17 459	4 1/4% Waadt 1986–1998	6.269	2 (1)	A	83.66	80.38	81.58
17 360	4 1/4% Wallis 1986–1998	6.478	2 (1)	A	83.45	80.20	81.41
17 364	7% Wallis 1990–2000	8.811	2 (2)	A	95.68	91.39	93.04
17 610	6 1/2% Zürich 1991–2001	9.322	2 (2)	A	93.17	93.90	90.59
15 461	4 3/4% Eidg. 1986–2001	9.036	5 (1)	B	84.63	82.30	83.38
15 710	4 1/4% Eidg. 1986–2001	9.292	5 (1)	B	81.32	78.94	80.04
15 712	4 1/4% Eidg. 1986–2011	19.942	10 (1)	C	76.76	78.63	78.40
15 718	4 1/4% Eidg. 1987–2012	20.172	10 (1)	C	76.67	78.62	78.37
15 722	4% Eidg. 1988–1999	7.117	3 (1)	D	81.63	78.43	79.67
15 726	4 1/4% Eidg. 1989–2001	9.050	4 (1)	E	81.50	79.02	80.14
15 736	5 1/2% Eidg. 1989–1998	6.819	2 (2)	A	86.96	86.49	87.59
15 738	5 1/2% Eidg. 1990–1999	7.042	2 (2)	A	86.98	86.54	87.64
15 740	6 1/4% Eidg. 1990–2000	8.208	2 (2)	A	91.05	91.33	92.34
15 745	6 1/2% Eidg. 1990–2000	8.378	2 (2)	A	92.49	92.97	89.60
15 227	6 1/2% Eidg. 1990–1999	7.564	2 (2)	A	91.93	92.23	88.82
15 747	6 3/4% Eidg. 1991–2001	9.081	2 (2)	A	94.50	91.95	91.83
15 749	6 1/4% Eidg. 1991–2001	9.228	2 (2)	A	91.62	92.18	93.12
15 751	6 1/4% Eidg. 1991–2003	11.400	4 (2)	E	92.87	94.12	91.89
15 753	6 1/4% Eidg. 1991–2002	10.561	2 (2)	A	92.39	93.36	94.17

1 Trading day 23 December 1991. Given the current value of the instantaneous interest rate and three *observed* continuous-time discount bond yields {0.075228, 0.07753, 0.06647, 0.06298} with {0, 1.0, 7.175, 10.25} years until expiration, the parameters of Vasicek's theoretical yield curve have been estimated by means of a modified Newton-Raphson algorithm. The underlying interest-rate process is the Ornstein-Uhlenbeck process $dr = \alpha\,(\gamma - r)\,dt + \rho\,dz$, where r is the instantaneous interest rate and dz the Gauss-Wiener process. The estimated parameters of the term structure are: the speed of adjustment $\alpha = 0.4418$, the long-run 'equilibrium' value of the instantaneous interest rate $\gamma = 0.03485$ [discrete-time equivalent 3.546% p. a.], the volatility of the instantaneous interest rate $\rho = 0.1326$, and the market price of interest-rate risk $q = 0.2117$. The instantaneous interest rate has been approximated by the tomorrow-next rate; it was $r_0 = 0.075228$ [discrete-time equivalent 7.813% p. a.] on the trading day in question. The bonds in the upper panel have been issued by cantons, those in the lower panel by the Swiss confederation.

2 Maximum time period until maturity.

3 The numbers in brackets denote the numbers of positive 'break-even' interest rates. These numbers have been employed to compute the analytical prices for this Table because the finite difference methods are applied to the partial differential equation with non-negative interest rates only.

4 Call condition type A: the call price is equal to 100 at both call dates. Call condition type B: the first call price (when moving forwards in time) is equal to 101.5; annual reduction of 0.5 percentage points until 100. Call condition type C: the first call price (when moving forwards in time) is equal to 102.5; annual reduction of 0.5 percentage points until 100. Call condition type D: the first call price (when moving forwards in time) is equal to 100.5; annual reduction of 0.5 percentage points until 100. Call condition type E: the first call price (when moving forwards in time) is equal to 101; annual reduction of 0.5 percentage points until 100. There is a notice period of two months for all call conditions.

Table 3: Continued. Price of Callable Bonds Belonging to 'Bond Baskets' of the SWISS NATIONAL BANK and Numerical Error.

Number of Security	Name of Security	Price of Callable Bond with Bound. Scheme #... [6]		Percentage Error of Finite Difference Method — Boundary Scheme			
		#3	#4	#1 [7]	#2 [8]	#3 [9]	#4 [10]
Col. 1	Col. 2	Col. 9	Col. 10	Col. 11	Col. 12	Col. 13	Col. 14
16 310	4 1/2% Bern 1986–1998	92.84	91.12	− 3.7	− 2.3	+ 9.8	+ 7.7
16 242	4 1/4% Bern 1987–1999	91.27	89.43	− 3.6	− 2.2	+ 10.6	+ 8.3
16 506	5 1/4% Graub. 89–1999	95.16	93.48	− 0.5	+ 0.8	+ 11.1	+ 9.1
17 459	4 1/4% Waadt 1986–1998	92.19	90.46	− 3.9	− 2.5	+ 10.2	+ 8.1
17 360	4 1/4% Wallis 1986–1998	92.06	90.30	− 3.9	− 2.4	+ 10.3	+ 8.2
17 364	7% Wallis 1990–2000	103.10	101.74	− 4.5	− 2.8	+ 7.8	+ 6.3
17 610	6 1/2% Zürich 1991–2001	100.85	99.46	+ 0.8	− 2.8	+ 8.2	+ 6.7
15 461	4 3/4% Eidg. 1986–2001	96.79	94.35	− 2.8	− 1.5	+ 14.4	+ 11.5
15 710	4 1/4% Eidg. 1986–2001	95.33	92.71	− 2.9	− 1.6	+ 17.2	+ 14.0
15 712	4 1/4% Eidg. 1986–2011	91.08	88.62	+ 2.4	+ 2.1	+ 18.7	+ 15.4
15 718	4 1/4% Eidg. 1987–2012	90.77	88.36	+ 2.5	+ 2.2	+ 18.4	+ 15.2
15 722	4% Eidg. 1988–1999	92.91	90.72	− 3.9	− 2.4	+ 13.8	+ 11.1
15 726	4 1/4% Eidg. 1989–2001	94.32	91.74	− 3.0	− 1.7	+ 15.7	+ 12.6
15 736	5 1/2% Eidg. 1989–1998	96.21	94.61	− 0.5	+ 0.7	+ 10.6	+ 8.8
15 738	5 1/2% Eidg. 1990–1999	96.25	94.63	− 0.5	+ 0.8	+ 10.7	+ 8.8
15 740	6 1/4% Eidg. 1990–2000	99.39	97.88	+ 0.3	+ 1.4	+ 9.2	+ 7.5
15 745	6 1/2% Eidg. 1990–2000	100.54	99.08	+ 0.5	− 3.1	+ 8.7	+ 7.1
15 227	6 1/2% Eidg. 1990–1999	100.16	98.69	+ 0.3	− 3.4	+ 8.9	+ 7.3
15 747	6 3/4% Eidg. 1991–2001	102.05	100.68	− 2.7	− 2.8	+ 8.0	+ 6.5
15 749	6 1/4% Eidg. 1991–2001	99.58	98.14	+ 0.6	+ 1.6	+ 8.7	+ 7.1
15 751	6 1/4% Eidg. 1991–2003	102.09	99.99	+ 1.3	− 1.1	+ 9.9	+ 7.7
15 753	6 1/4% Eidg. 1991–2002	99.67	98.37	+ 1.1	+ 1.9	+ 7.9	+ 6.5
Root Mean Square Error for the Callable Bonds				2.6	2.1	11.8	9.6
Root Mean Square Error for the Underlying Straight Bonds [11]				1.8	1.5	1.2	0.9

[5] Price obtained from the analytical solution by means of numerical quadrature involving Green's function (Büttler and Waldvogel, 1993a, b) minus the accrued interest since the last coupon date. See footnote 3. All the digits displayed are correct (the accuracy is almost equal to the machine precision).

[6] Price obtained from the numerical solution of the partial differential equation by means of the Lawson-Morris method minus the accrued interest since the last coupon date. The parameters of the finite difference method are: $n_1 = 50$, $n_2 = 50$, $\psi = 200$, $s = 10$, $m;^\wedge = 1$, and $r_m = 0.15$. Actually, the computer program modifies slightly these parameters as described in the longer version of this paper. The computer program has been written in PASCAL (Jensen and Wirth, 1978) and runs on the APPLE® MACINTOSH™ family, the machine precision of which is 19 – 20 decimal digits (mantissa of the floating-point form). The tridiagonal matrix algorithm of Press et al. (1989) has been applied to the resulting equation system of the four finite difference methods under consideration.

[7] Percentage deviation: (col. 7 / col. 6 – 1) * 100.

[8] Percentage deviation: (col. 8 / col. 6 – 1) * 100.

[9] Percentage deviation: (col. 9 / col. 6 – 1) * 100.

[10] Percentage deviation: (col. 10 / col. 6 – 1) * 100.

[11] The analytical price of the underlying straight bond has been obtained from Vasicek's bond price model minus the accrued interest since the last coupon date. The numerical price of the underlying straight bond has been obtained from the numerical solution of the partial differential equation by means of the Lawson-Morris method minus the accrued interest since the last coupon date. The parameters of the finite difference method are the same as those given in footnote 6.

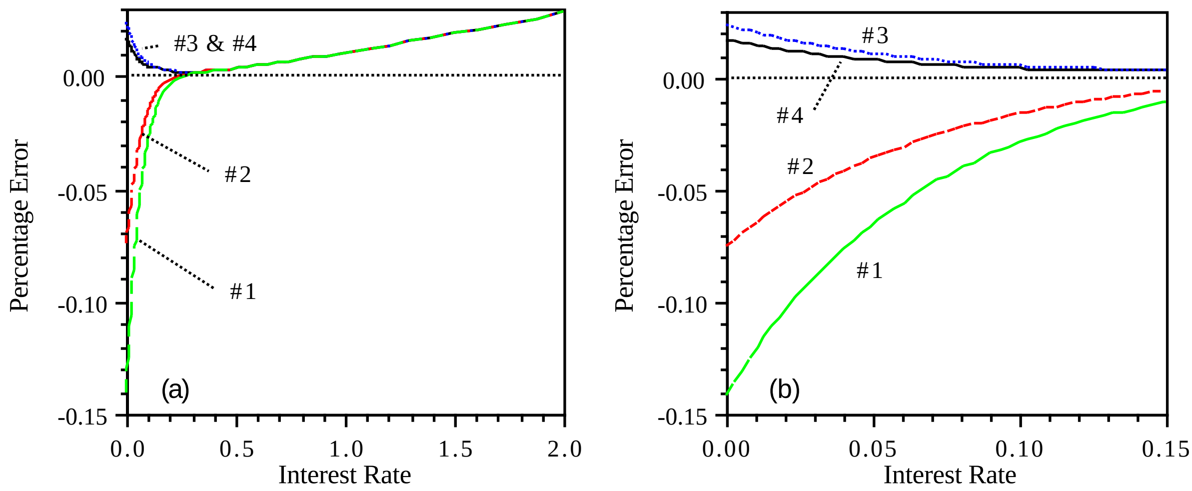

Fig. 1a & b: Percentage Error on the Notice Day. ‡

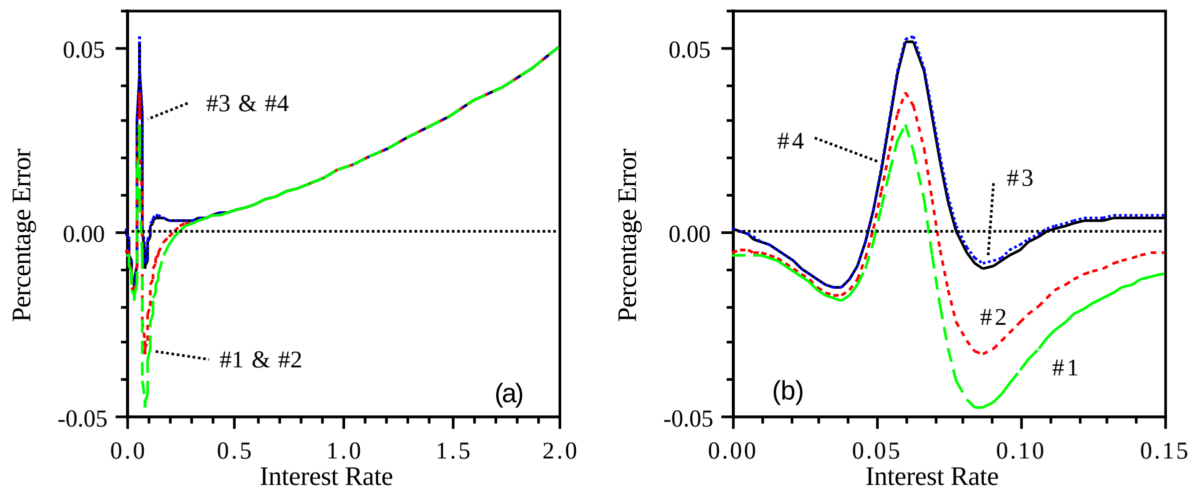

Fig. 2a & b: Percentage Error One Time Step after the Notice Day. ‡

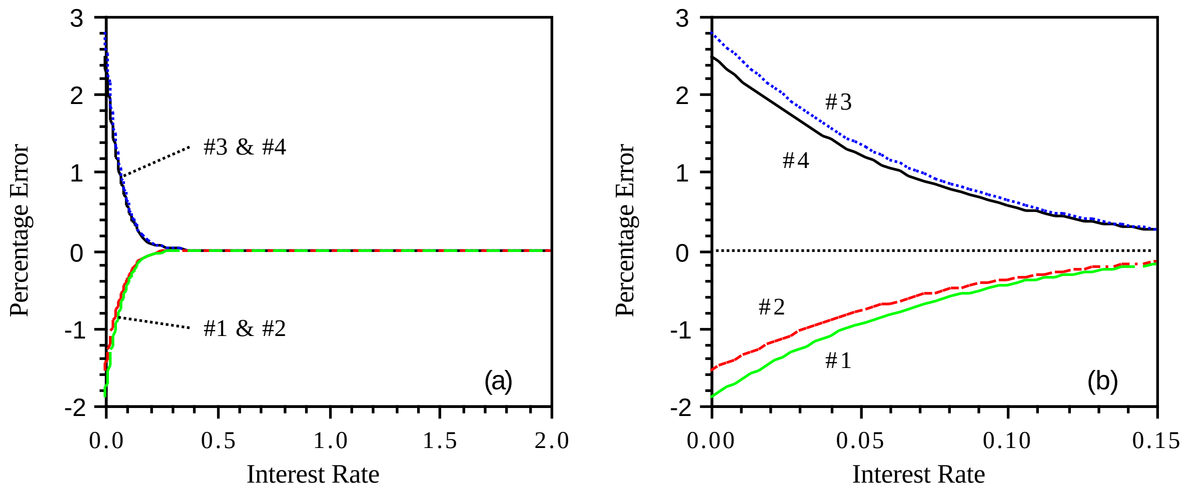

Fig. 3a & b: Percentage Error after Two Years. ‡

‡ The numbers refer to the boundary schemes of Table 2. The parameters of the Lawson-Morris method are $n_1 = 50$, $r_m = 0.15$, $n_2 = 50$, $\psi = 200$, $\Delta t = 1/74th$ of a year, $s = 10$ and m;^ = 1.

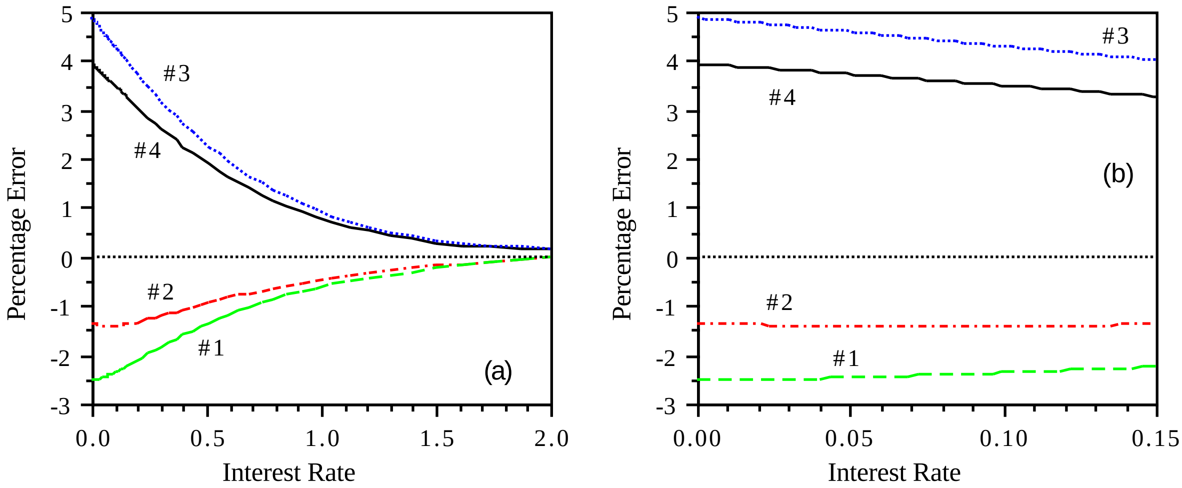

Fig. 4a & b: Percentage Error after 6.811 Years. ‡

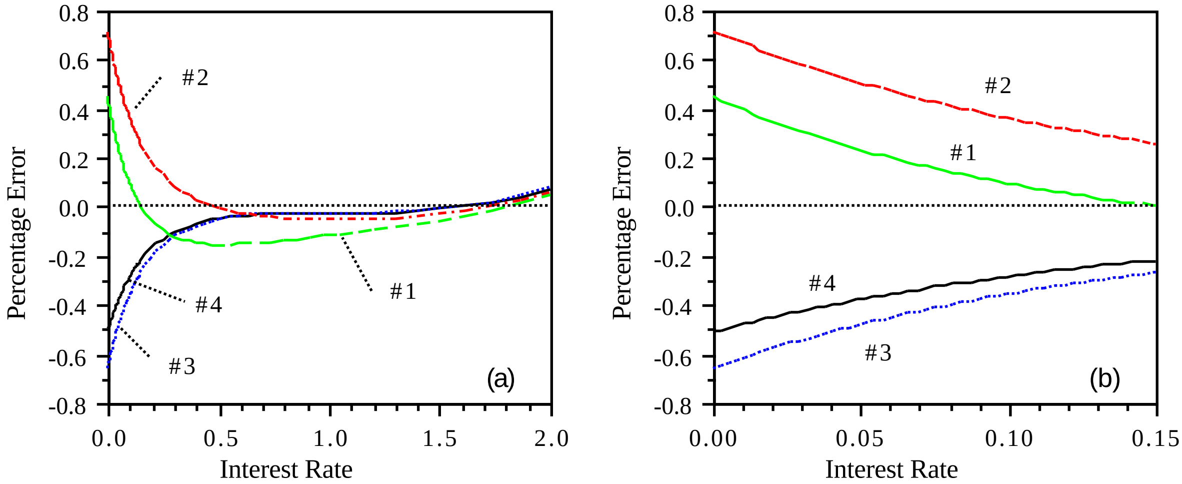

Fig. 5a & b: Percentage Error of the Underlying Straight Bond. ‡

Table 2: Five Boundary Schemes.

Scheme #	Derivative	Denominator	F_n	F_{n-1}	F_{n-2}	F_{n-3}	F_{n-4}
1	F_x	Δx	1	−1	—	—	—
	F_{xx}	$(\Delta x)^2$	1	−2	1	—	—
2	F_x	$2\,\Delta x$	3	−4	1	—	—
	F_{xx}	$(\Delta x)^2$	1	−2	1	—	—
3	F_x	$2\,\Delta x$	3	−4	1	—	—
	F_{xx}	$(\Delta x)^2$	2	−5	4	−1	—
4	F_x	$6\,\Delta x$	11	−18	9	−2	—
	F_{xx}	$(\Delta x)^2$	2	−5	4	−1	—
5	F_x	$12\,\Delta x$	25	−48	36	−16	3
	F_{xx}	$12\,(\Delta x)^2$	35	−104	114	−56	11

‡ The numbers refer to the boundary schemes of Table 2. The parameters of the Lawson-Morris method are n_1 = 50, r_m = 0.15, n_2 = 50, ψ = 200, Δt = 1/74*th* of a year, s = 10 and m;^ = 1.

Comment: Read the second row as $F_x = [1 \cdot F_n - 1 \cdot F_{n-1}] / \Delta x$ and similarly the other rows.