Pricing Callable Bonds by Means of Green's Function ☆

by

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Abstract:

This paper derives a closed-form solution for the price of the European and semi-American callable bond for two popular one-factor models of the term structure of interest rates which have been proposed by VASICEK as well as COX, INGERSOLL AND ROSS. The price is derived by means of repeated use of GREEN's function which, in turn, is derived from a series solution of the partial differential equation to value a discount bond. The boundary conditions which lead to the well-known formulae for the price of a discount bond are also identified. The algorithm to implement the explicit solution relies on numerical quadrature involving GREEN's function. It offers both higher accuracy and higher speed of computation than finite difference methods which suffer from numerical instabilities due to discontinuous boundary values. For suitably small time steps, the proposed algorithm can also be applied to American callable bonds, or to any American-type option with GREEN's function being explicitly known.

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KEYWORDS: Callable bond, Green's function, series solution of the partial differential equation to value a discount bond, numerical quadrature, orthogonal polynomial interpolation.

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1 Introduction and Summary

The callable bond is a straight bond (or coupon bond) with the provision that allows the debtor to buy back or to "call" the bond for a specified amount, the call price, plus the accrued interest since the last coupon date at some time, the call date(s), during the life of the bond (see, e. g., BRENNAN AND SCHWARTZ [1977]). As regards the call dates, three types of callable bonds can be observed in financial markets. The American callable bond may be redeemed at any time before the final redemption date, in contrast to the European or semi-American counterparts which may only be called at one or several specific dates, respectively. For instance, American callable bonds have predominantly been issued by corporations in the United States, whereas semi-American callable bonds have solely been issued by both public and private institutions in Switzerland. In the case of Swiss financial markets, the single call date of the European callable bond is the last but one coupon date, whereas the additional call dates of the semi-American callable bond are the preceding coupon dates. The number of call dates of outstanding bonds varies between one and ten in Switzerland. Moreover, the Swiss debtor has to consider a notice period of two months ahead of the call date. For any type of the callable bond, the initial call price may be somewhat above the face value of the bond and may decline gradually over time. In general, the call provision will be deferred sometimes until the bond has been outstanding for some length of time.

The callable bond can be viewed as a compound security, which consists of an otherwise identical straight bond and of an embedded call option, which is not traded and the price of which is, therefore, not observable. The embedded call option, which is written on the underlying straight bond, can be viewed as being "sold" by the initial bondholder to the issuer of the callable bond, the debtor. Hence, the price of the callable bond must be equal to the price of the underlying straight bond less the price of the embedded call option at any time. Or, the callable bond is worth less than the underlying straight bond.

This paper is motivated by the fact that a finite-difference solution of the partial differential equation for the price of a callable bond may exhibit slowly decaying finite oscillations after each call date due to the discontinuity in the values of the early redemption condition (see, e. g., SMITH [1985] for an explanation). These oscillations, in turn, are responsible for an *insufficient* numerical accuracy of finite difference methods in real applications (BÜTTLER [1993]). Some numerical results may even be contradictory to theory. For instance, GIBSON-ASNER [1990, p. 670] reports some *negative* computed prices of the embedded call option and LEITHNER [1992, Figure 5.5, p. 145] obtains computed prices for the semi-American option which are *smaller* than for the European option when the spot rate is less than six percent.

The purpose of this paper is to derive, and to numerically implement, the explicit price of the European and semi-American callable bond for two different one-factor models of the term structure of interest rates. The two models used in this paper are the one proposed by VASICEK

[1977] and the one by COX, INGERSOLL AND ROSS [1985a & b], henceforth CIR. The price of the callable bond is expressed in terms of a multiple integral of the conditional, forward-risk-adjusted probability density function of the underlying instantaneous spot interest rate. Using the separation-of-variables technique, we express the price of a discount bond as a series of orthogonal polynomials, also to be used in the numerical algorithm, and identify the boundary conditions which lead to the well-known formulae for the price of a discount bond (see the appendix A for details).[1] The algorithm to implement the explicit solution relies on numerical quadrature involving GREEN's function. Since the boundary conditions, including that for an early redemption of the callable bond, are explicitly built into the algorithm, oscillations do not occur. The proposed algorithm offers both higher accuracy and higher speed of computation than finite difference methods. For suitably small time steps, the proposed algorithm can also be applied to American callable bonds, or to any American-type option with GREEN's function being explicitly known.

An explicit solution of the semi-American callable bond within the framework of the GAUSSian interest rate model has been derived by JAMSHIDIAN [1991a].[2] In that paper, the price of the semi-American callable bond with n call dates is expressed as a series of callable bonds, which are written on zero-coupon bonds on the one hand and which have a number of call dates ranging between one and $n - 1$ on the other hand. The numerical algorithm proposed by JAMSHIDIAN in the same paper refers to the path-independent GAUSSian model which, in turn, reduces the partial differential equation for bonds to the partial differential equation of the heat conduction in a rod with *constant* coefficients. In contrast, the partial differential equations for the models of VASICEK [1977] or COX, INGERSOLL AND ROSS [1985b] can *not* be reduced to the standard heat equation. Rather, we will show that these partial differential equations are of the form of KUMMER's (or confluent hypergeometric) differential equation.

The outline of the paper is as follows. We will deal with the CIR model in the main text and collect the corresponding equations for the VASICEK model in the appendix A. In the next section, we will explain briefly the series solution for the CIR differential equation, which will also be used for the interpolation of the price function in the numerical algorithm, and state GREEN's function, while some derivations are given in the appendix A. GREEN's function will be applied to the European and semi-American callable bond in sections 3 and 4. The numerical algorithm is explained in the fifth section, while leaving details to the appendix B. Finally, a numerical example for an exchange-traded callable bond with ten call dates and a maximum life of more than twenty years is provided in the last section, again leaving some details to the appendix C.

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1 It is well known that the parabolic partial differential equation for the price of a discount bond can be solved by means of either classical analysis or the modern martingale approach. In the latter approach, discounted prices of zero-coupon bonds have to be a martingale for a risk-neutral probability; see, e. g., ARTZNER AND DELBAEN [1989] for a thorough discussion of the martingale approach to the term structure of interest rates. In this paper, we follow the classical approach because it provides a solution with numerically desirable properties (series of orthogonal polynomials as well as boundary conditions to be used for a finite difference method, say).

2 We thank Freddy Delbaen for drawing our attention to this result.

2 The Series Solution and Green's Function

In the CIR model, the instantaneous interest rate, r, serves as the single factor which describes the whole term structure of insterest rates. The process for the instananeous interest rate is the square-root process:[3]

$$dr = \kappa(\theta - r)\,dt + \sigma\sqrt{r}\,dz, \tag{2.1}$$

where $\kappa > 0$ denotes the speed of adjustment, $\theta > 0$ the long-run "equilibrium" value of the instantaneous interest rate, t the calendar time, $\sigma > 0$ the instantaneous standard deviation (volatility) of the instantaneous interest rate, and z the GAUSS-WIENER process. The spot interest rate cannot become negative. CIR derive the following partial differential equation (PDE) to determine the price of a discount bond, $P(r, \tau)$, promising to pay one unit of money on the maturity day.

$$P_\tau = \frac{1}{2}\sigma^2 r P_{rr} + [\kappa\theta - \zeta r] P_r - rP. \tag{2.2}$$

The subscripts denote partial derivatives, τ the period of time remaining until the maturity date, and ζ the sum of the risk premium of the single factor which drives the economy and the speed of adjustment. (In the notation of CIR, $\zeta = \lambda + \kappa$). Empirically, we would expect the risk premium, $\zeta - \kappa$, to be negative (CHAN ET AL. [1992]).

The interval of definition for the instantaneous interest rate is (without additional parameter restrictions) $r \in \mathcal{I} = [0, \infty)$ and for the remaining time $\tau \in [0, \infty)$, that is, the region of definition of $P(r, \tau)$ is the first quadrant $\Omega = \mathcal{I} \times \mathcal{I}$. The linear parabolic partial differential equation (2.2) is singular both at $r = 0$ (regularly singular) and $r = \infty$ (irregularly singular). In order to obtain a unique solution, initial values $P(r, 0)$ must be given for $r \in \mathcal{I}$. Also, in general two boundary conditions must be given, namely one on the left and one on the right boundary of $\mathcal{I}$ for every $\tau \in \mathcal{I}$. As a principle for obtaining boundary conditions we postulate that for every fixed $\tau > 0$ the quantities $P(r, \tau)$, $P_r(r, \tau)$, $r P_{rr}(r, \tau)$ remain bounded in any compact interval and tend to zero as $r \to \infty$. It is easily seen that this is a consequence of

$$\begin{aligned} &P(r, \tau) \to 0 \text{ as } r \to \infty, \quad \text{(right boundary)}, \\ &P_r(r, \tau) \text{ finite as } r \to 0, \quad \text{(left boundary)}, \end{aligned} \tag{2.3}$$

for every $\tau \in \mathcal{I}$. The left boundary condition admits regular solutions only. The right boundary condition has been proposed by BRENNAN AND SCHWARTZ [1977, 1979]. Note that CIR do not specify boundary conditions because they apply the risk-neutral expectations approach, which corresponds to the PDE approach for appropriate boundary conditions.

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[3] The probability density function of the square-root process was first derived by FELLER [1951]. Both the square-root process and the ORNSTEIN-UHLENBECK process are special cases of the BESSEL process, which has been thoroughly studied in PITMAN AND YOR [1982].

Using the separation-of-variables technique, we will show in the appendix A that the unique solution of the partial differential equation (2.2) subject to the boundary conditions (2.3) is given by an infinite series of generalized LAGUERRE polynomials, $L_n^{(\alpha)}(\cdot)$, in terms of the instantaneous interest rate, multiplied by an exponential function of the remaining period of time:

$$P(r,\,\tau) = \mathrm{e}^{br} \sum_{n=0}^{\infty} c_n \,\mathrm{e}^{[C-n\gamma]\,\tau}\, L_n^{(\alpha)}\!\left(\frac{2\gamma r}{\sigma^2}\right),$$
$$\text{with } \alpha + 1 := \frac{2\kappa\theta}{\sigma^2} > 0, \quad b := \frac{\zeta-\gamma}{\sigma^2} < 0, \quad \gamma := +\sqrt{\zeta^2 + 2\sigma^2}, \quad C := \kappa\theta\,\frac{\zeta-\gamma}{\sigma^2}\,. \tag{2.4}$$

The constants c_n must be determined from the initial condition. We will show in the appendix A that the price formula for the discount bond given by CIR is, indeed, obtained from the equation (2.4) subject to the initial condition $P(r, 0) = 1$. For further reference, we repeat here the price formula for the discount bond (with a notation slightly different from that in CIR):

$$P(r,\,\tau) = f(\tau)\,\exp\!\big(g(\tau)\,r\big), \text{ where}$$
$$f(\tau) := \left[\frac{2\gamma\,\mathrm{e}^{\frac{\zeta+\gamma}{2}\tau}}{2\gamma + [\zeta+\gamma]\,[\mathrm{e}^{\gamma\tau}-1]}\right]^{\frac{2\kappa\,\theta}{\sigma^2}}, \quad g(\tau) := -\,\frac{2\,[\mathrm{e}^{\gamma\tau}-1]}{2\gamma + [\zeta+\gamma]\,[\mathrm{e}^{\gamma\tau}-1]}\,. \tag{2.5}$$

The symbol γ has already been defined in the equation (2.4).

GREEN's function, $G(r(t_0),\, t_0,\, \rho(t_1),\, t_1)$, or the fundamental solution of the partial differential equation (2.2) subject to the boundary conditions (2.3) represents the price of a primitive security at time t_0, given an instantaneous interest rate $r(t_0)$, and which promises to pay one unit of money at the maturity date t_1 if the instantaneous interest rate $r(t_1)$ will be equal to a particular value $\rho(t_1)$, but nothing in all other events. Obviously, the initial condition is now the DIRAC delta function, $\delta(r-\rho)$, which determines the constants c_n of the series solution (2.4). We will derive in the appendix A the explicit form of GREEN's function for the CIR model to be

$$G(r,\,\tau,\,\rho) = P(r,\,\tau)\;2p(\tau)\;H\big(2p(\tau)\rho\,|\,2\alpha+2,\,\lambda(r,\,\tau)\big)\,, \text{ with} \tag{2.6}$$
$$p(\tau) := \frac{\zeta+\gamma-[\zeta-\gamma]\,\mathrm{e}^{-\gamma\tau}}{\sigma^2\,[1-\mathrm{e}^{-\gamma\tau}]} > 0, \quad \lambda(r,\,\tau) := \frac{8\gamma^2\,\mathrm{e}^{-\gamma\tau}\,r}{\sigma^2\,\big\{2\gamma + [\zeta-\gamma]\,[1-\mathrm{e}^{-\gamma\tau}]\big\}\,[1-\mathrm{e}^{-\gamma\tau}]} \geqq 0,$$
$$H(x\,|\,\nu,\,\lambda) := \frac{1}{2}\left[\frac{x}{\lambda}\right]^{(\nu-2)/4} \mathrm{e}^{-(\lambda+x)/2}\, I_{(\nu-2)/2}\big(\sqrt{\lambda x}\big) = \sum_{n=0}^{\infty} \frac{[\lambda/2]^n\,\mathrm{e}^{-(\lambda+x)/2}\,x^{\nu/2+n-1}}{2^{\nu/2+n}\,n!\,\Gamma(\nu/2+n)},$$

where $I_\alpha(\cdot)$ denotes the modified BESSEL function of the first kind of order α and $\Gamma(\cdot)$ EULER's gamma function. Following JAMSHIDIAN [1991, theorem 1, p. 143, and corollary 2, p. 145], the price of a primitive security is equal to the price of a discount bond times the "forward-risk-adjusted" expectation of the DIRAC delta function. Hence, GREEN's function is equal to the price of a discount bond, $P(r,\,\tau)$, as given in the equation (2.5), multiplied by the forward-risk-adjusted probability density function of the instantaneous interest rate prevailing at the terminal date (when looking forwards), conditional on the current value of the instantaneous interest

rate.[4] The forward-risk adjustment, applied to the drift rate of the underlying interest rate process, consists of the usual risk adjustment and an additional adjustment due to the fact that the discount bond price has been factored out. In the CIR model, the forward-risk adjustment is given by the expression $-[\zeta - \kappa]\, r + \sigma^2\, r\, g(\tau)$, and the probability density function by the non-central chi-square distribution, $2\, p(\tau)\, H(x \mid \nu, \lambda)$ for $x, \lambda \geqq 0$, $\nu > 0$ and with $x := 2\, p(\tau)\, \rho$, $\nu = 2\, \alpha + 2$ degrees of freedom and noncentrality parameter $\lambda = \lambda(r, \tau)$, see e. g. JOHNSON AND KOTZ [1970, vol. 2, equ. (28.3) and (28.5)].[5] If a security with the pay-off function $\phi(\rho(t_1))$ satisfies the same partial differential equation and boundary conditions as GREEN's function, then its price at time t_0 is equal to $\int_{-\infty}^{+\infty} G(r, t_0, \rho, t_1)\, \phi(\rho)\, d\rho$. Since the callable bond satisfies the same partial differential equation and boundary conditions as the discount bond except for the notice days, we can use GREEN's function (2.6) to determine explicitly the price of the callable bond. The structure of GREEN's function (2.6) becomes apparent from integrating it over the initial data of a discount bond, $\phi(\rho(t_1)) := P(r, 0) = 1$. Then we must have $P(r, \tau) = \int_{-\infty}^{+\infty} G(r, t_0, \rho, t_1)\, \phi(\rho)\, d\rho = 2\, p(\tau)\, P(r, \tau) \int_{\rho=0}^{\rho=+\infty} H(2\, p(\tau)\, \rho \mid 2\, \alpha + 2, \lambda(r, \tau))\, d\rho = P(r, \tau) \int_{x=0}^{x=+\infty} H(x \mid 2\, \alpha + 2, \lambda(r, \tau))\, dx = P(r, \tau)$.

GREEN's function has been derived by JAMSHIDIAN [1989, 1991a & b] for the GAUSSian interest rate model and by BEAGLEHOLE AND TENNEY [1991] for the CIR model. JAMSHIDIAN recognized first that GREEN's function is the discounted value of the conditional, forward-risk-adjusted probability density function of the underlying interest rate process. The same approach has been taken in CHEN AND SCOTT [1992] for the two-factor CIR model (their forward-risk adjustment is the same as the one given above).[6]

3 The European Callable Bond

Suppose the single call date is the last but one coupon date and that there is a notice period τ_n, which is, in general, two months (see Figure 1). Let $K(r, \hat{\tau} + \tau_n + \tau)$ denote the price of the callable bond with the remaining time until the last possible redemption date of length $[\hat{\tau} + \tau_n + \tau]$ as a function of the instantaneous interest rate today, $r := r(t_0)$. Let X denote the call (or strike) price per face value, η the annual coupon payment per face value, and $\rho := \rho(t_n)$ the instantaneous interest rate on the notice day, t_n, when the debtor has to make the choice of whether or not to call the bond. Note that all the following formulae for the European and semi-American callable bond may easily be re-interpreted if the stream of coupon payments has another periodicity than a year.

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4 We are indebted to an anonymous referee who corrected our false interpretation in an earlier version. The notion "forward-risk-adjusted probability" is due to JAMSHIDIAN.

5 Note that the argument of the BESSEL function is misprinted in JOHNSON AND KOTZ.

6 We are indebted to Freddy Delbaen and an anonymous referee for bringing these results to our knowledge.

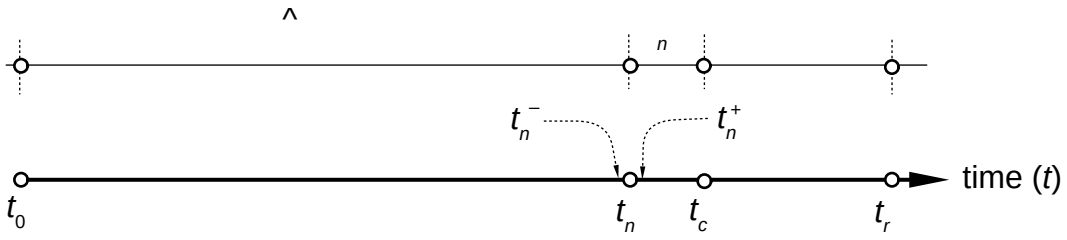


Figure 1: European Callable Bond. t_0 is the present date, t_n the notice date, t_c the call date, and t_r the last possible redemption date. The call date is the last but one coupon date, i. e., τ is equal to one year. t_n^- and t_n^+ denote an instant before and after the notice date, respectively.

If the bond is not called, the price of the callable bond will be equal to the price of the underlying straight bond on the last possible redemption date, t_r, that is, equal to the face value plus the coupon payment, $1 + \eta$. An instant before the notice date when proceeding backwards in time, the price of the callable bond is equal to the time value of the underlying straight bond:

$$K(\rho,\ \tau_n+\tau)^+ = [1+\eta]\ P(\rho,\ \tau_n+\tau) + \eta\ P(\rho,\ \tau_n). \tag{3.1}$$

The time value of the call price plus the coupon payment on the last but one coupon date is given by $[X + \eta]\cdot P(\rho, \tau_n)$. The optimal call policy requires that the issuer of the callable bond minimizes his outstanding debt. Therefore, he will call the bond if the price of the callable bond is greater than the time value of the call price including the coupon payment. Hence, the price of the callable bond an instant after the notice day (when proceeding backwards in time) is given by

$$K(\rho,\ \tau_n+\tau)^- = \begin{cases} [X+\eta]\ P(\rho,\ \tau_n), & \text{for } \rho \leqq \rho^*, \\ K(\rho,\ \tau_n+\tau)^+, & \text{for } \rho \geqq \rho^*, \end{cases} \tag{3.2}$$

where the "break-even" (or critical) interest rate ρ^* is determined from setting the price of the callable bond an instant before the notice date equal to the time value of the call price:

$$[X+\eta]\ P(\rho^*,\ \tau_n) = K(\rho^*,\ \tau_n+\tau)^+. \tag{3.3}$$

Using the price of the callable bond an instant after the notice date as the initial data, the price of the callable bond today can be obtained from GREEN's function as follows:

$$K(r,\ \widehat{\tau}+\tau_n+\tau) = \int_{-\infty}^{+\infty} G(r,\ \widehat{\tau},\ \rho)\ K(\rho,\ \tau_n+\tau)^-\,\mathrm{d}\rho + \eta \sum_{j=2}^{m} P(r,\ t_r - j\tau - t_0)\ . \tag{3.4}$$

The last term in the above equation is the present value of the coupon stream after the notice date (when proceeding backwards in time), where m denotes the number of entire years of the remaining time until the last possible redemption date (see Figure 2).

Substituting the equations (2.5) and (3.1) into the equation (3.3), one obtains for the "break-even" interest rate:

$$\rho^* = -\frac{1}{g(\tau_n) - g(\tau_n + \tau)} \ln\left(\frac{X\, f(\tau_n)}{[1+\eta]\, f(\tau_n + \tau)}\right), \tag{3.5}$$

where the functions $f(\cdot)$ and $g(\cdot)$ are given in the equation (2.5). The following lemma simplifies the integration to be performed with GREEN's function in the sequel.

LEMMA 1: *Suppose the length of the remaining time until an intermediary date, which serves as "initial" date, is τ_2, where the initial condition is given by the price of a discount bond with remaining time until maturity of length τ_1, then*

$$\begin{aligned}\int_a^b G(r, \tau_2, \rho)\, P(\rho, \tau_1)\, d\rho &= P(r, \tau_2)\, f(\tau_1)\, \Pi(\tau_1, \tau_2)^{\alpha+1} \exp\left(-\frac{1}{2}\left[\lambda(r, \tau_2)\,\{1 - \Pi(\tau_1, \tau_2)\}\right]\right) \mathfrak{H} \\ &= P(r, \tau_1 + \tau_2)\, \mathfrak{H}, \text{ where} \\ \mathfrak{H} &:= 2\left[p(\tau_2) - g(\tau_1)\right] \int_a^b H\left(2\left[p(\tau_2) - g(\tau_1)\right]\rho \mid 2\alpha+2,\ \Pi(\tau_1, \tau_2)\, \lambda(r, \tau_2)\right) d\rho, \\ \Pi(\tau_1, \tau_2) &:= \frac{p(\tau_2)}{p(\tau_2) - g(\tau_1)}. \ \bigcirc\end{aligned} \tag{3.6}$$

Proof: Substituting the equations (2.5) and (2.6) into the equation (3.6) yields the first equality. Let $a \to 0$ and $b \to +\infty$, then $\mathfrak{H}$ is equal to one due to the non-central chi-square probability density function. Since $\int_{-\infty}^{+\infty} G(r, \tau_2, \rho)\, P(\rho, \tau_1)\, d\rho = P(r, \tau_1 + \tau_2)$, the so-called semi-group property discussed in FELLER [1952] which can be verified by straightforward algebraic manipulations, the second equality of the lemma follows. □

The price of the European callable bond today with the remaining time until the last possible redemption date of length $[\hat{\tau} + \tau_n + \tau]$ follows from substituting the equations (3.2) and (2.5) into the equation (3.4) and from applying lemma 1:

$$\begin{aligned}K(r, \hat{\tau} + \tau_n + \tau) &= X\, P(r, \hat{\tau} + \tau_n)\, \mathcal{H}\left(d_1 \mid 2\alpha+2,\ \Lambda(r, \hat{\tau}, \tau_n)\right) + [1+\eta]\, P(r, \hat{\tau} + \tau_n + \tau) \\ &\quad \cdot\left[1 - \mathcal{H}\left(d_2 \mid 2\alpha+2,\ \Lambda(r, \hat{\tau}, \tau_n + \tau)\right)\right] + \eta \sum_{j=1}^{m} P(r, t_r - j\tau - t_0), \text{ where} \\ \mathcal{H}(d \mid \nu, \lambda) &:= \int_0^d H(x \mid \nu, \lambda)\, dx, \quad \Lambda(r, \tau_1, \tau_2) := \frac{p(\tau_1)}{p(\tau_1) - g(\tau_2)}\, \lambda(r, \tau_1), \\ d_1 &:= 2\left[p(\hat{\tau}) - g(\tau_n)\right]\rho^*, \qquad d_2 := 2\left[p(\hat{\tau}) - g(\tau_n + \tau)\right]\rho^*.\end{aligned} \tag{3.7}$$

The first term represents the present value of the call price weighted by the forward-risk-adjusted probability that the bond will be called early, the second term represents the discounted face value of the callable bond including the last coupon payment on the maturity day weighted by the forward-risk-adjusted probability that the bond will not be called early, and the third term represents the present value of the coupon stream during the time the bond cannot be called. The last but one coupon will be received in any case whether or not the bond will be called early.

4 The Semi-American Callable Bond

Suppose there are n call dates identical to the coupon dates $[t_r - j\,\tau]$ for $j = 1, 2, \ldots, n$ (see Figure 2). Let the call (or strike) prices be denoted as $X_j := X(t_r - j\,\tau)$ for $j = 1, 2, \ldots, n$, the instantaneous interest rates prevailing at the *notice* dates as $r_j := r(t_r - j\,\tau - \tau_n)$ for $j = 1, 2, \ldots, n$, the instantaneous interest rate today as $r := r(t_0)$, the time periods between call dates as τ (one year), the period of time between today and the last notice date (when proceeding backwards in time) as $\hat{\tau} = t_r - \tau_n - n\,\tau - t_0$, and the number of whole years of the remaining time until the last possible redemption date of length $[t_r - t_0]$ as m.

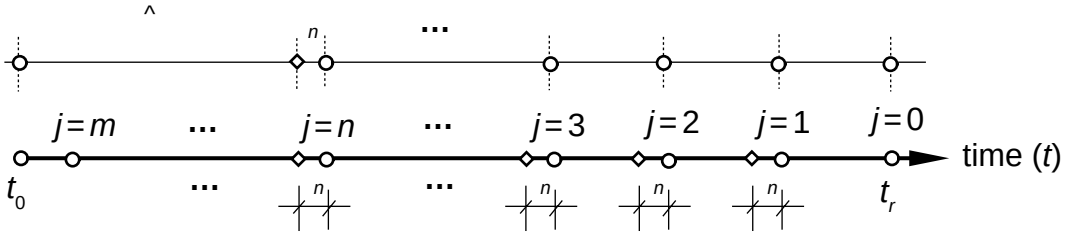


Figure 2: Semi-American Callable Bond. t_0 is the present date and t_r the last possible redemption date. The call dates are the n coupon dates $t_r - j\,\tau$, for $j = 1, 2, \ldots, n$. τ_n is the notice period. The number of whole years of the remaining time until the last possible redemption date of length $[t_r - t_0]$ is m.

As for the European callable bond, we proceed backwards in time. At each notice date, the price of the callable bond an instant before the notice date, $K(r_j, j\,\tau + \tau_n)^+$, is determined from the previous notice date by applying GREEN's function. The "break-even" interest rate, r_j^*, equates the price of the callable bond, $K(r_j, j\,\tau + \tau_n)^+$, and the time value of the call price including the coupon payment, $[X_j + \eta]\,P(r_j, \tau_n)$. Again, the optimal call policy requires the price of the callable bond an instant after the call date to be the minimum of these two prices. In summary, the procedure goes as follows:

$$
\begin{aligned}
K(r_j, j\tau+\tau_n)^+ &= \int_{-\infty}^{+\infty} G(r_j,\,\tau,\,r_{j-1})\;K\big(r_{j-1}, [j-1]\,\tau+\tau_n\big)^-\,\mathrm{d}r_{j-1} + \eta\;P(r_j,\,\tau_n),\\
\mathcal{F}\big(r_j\big) &:= \big[X_j + \eta\big]\,P\big(r_j,\,\tau_n\big) - K\big(r_j, j\,\tau+\tau_n\big)^+ = 0 \text{ for } r_j = r_j^*,\\
K(r_j, j\,\tau+\tau_n)^- &= \begin{cases} \big[X_j + \eta\big]\,P\big(r_j,\,\tau_n\big) & \text{for } r_j \leqq r_j^*,\\ K(r_j, j\,\tau+\tau_n)^+ & \text{for } r_j \geqq r_j^*. \end{cases} \qquad (j = 1, 2, \ldots, n).
\end{aligned}
\tag{4.1}
$$

At date $[t_r - \tau_n]$, the price of the callable bond is equal to the time value of the face value plus the last coupon payment, that is, $K(r_0,\,\tau_n)^- = [1 + \eta]\,P(r_0,\,\tau_n)$ with $r_0 := r(t_r - \tau_n)$. On the last notice day $[t_r - n\,\tau - \tau_n]$, the final time step is taken with $\hat{\tau}$ rather than τ in the equation (4.1).

In the lemma 2, we first determine the price of the callable bond on each notice day, which is needed to calculate the "break-even" interest rate prevailing on each notice day, r_j^* for $j = 1, 2, \ldots, n$.

LEMMA 2: *The price of the semi-American callable bond an instant before each notice date, when proceeding backwards in time, denoted as* $K(r_k, k\,\tau + \tau_n)^+$ *for* $k = 1, 2, \ldots, n$, *is given by*

$$K(r_k, k\tau+\tau_n)^+ = \sum_{j=1}^{k-1} X_{k-j}\, P(r_k, j\tau+\tau_n)\, \mathfrak{H}^+(r^*_{k-1}, r^*_{k-j}) + P(r_k, k\tau+\tau_n)\, \mathfrak{H}(r^*_{k-1}, r^*_1) + \eta \sum_{j=0}^{k} P(r_k, j\tau+\tau_n)\, \mathfrak{H}(r^*_{k-1}, r^*_{k+1-j}) \text{ , where} \tag{4.2}$$

$$\mathfrak{H}^+(r^*_{k-1}, r^*_{k-j}) := \left[\prod_{i=j}^{1} \omega(j, i)\right] \cdot \int_{r^*_{k-1}}^{\infty} \cdots \int_{r^*_{k+1-j}}^{\infty} \int_{-\infty}^{r^*_{k-j}} \left[\prod_{i=1}^{j} H\Big(\omega(j, i)\, r_{k-i} \mid 2\alpha+2,\, \hat{\lambda}(k-1, j, i)\Big)\right] \mathrm{d}r_{k-j} \ldots \mathrm{d}r_{k-1} ,$$

$$\mathfrak{H}(r^*_a, r^*_d) := \left[\prod_{i=a-d+1}^{1} \omega(a-d+2, i)\right] \cdot \int_{r^*_a}^{\infty} \cdots \int_{r^*_d}^{\infty} \left[\prod_{i=1}^{a-d+1} H\Big(\omega(a-d+2, i)\, r_{k-i} \mid 2\alpha+2,\, \hat{\lambda}(k-1, a-d+2, i)\Big)\right] \mathrm{d}r_d \ldots \mathrm{d}r_a ,$$

$$\omega(j, i) := 2\,[p(\tau) - g([j-i]\,\tau + \tau_n)], \quad g(0) = 0, \quad \hat{\lambda}(n, j, i) := \frac{p(\tau)}{p(\tau) - g([j-i]\,\tau + \tau_n)}\, \lambda(r_{n+2-i},\, \tau),$$

$$\mathfrak{H}^+(r^*_a, r^*_d) = [\cdot] \int_{-\infty}^{r^*_d} H(\cdot)\, \mathrm{d}r_d \text{ if } d+1 > a, \quad \mathfrak{H}(r^*_a, r^*_d) = 1 \text{ if } d > a. \circ$$

Proof: by induction. If $k = 1$, then the equation above reduces to $K(r_1, \tau + \tau_n)^+ = [1 + \eta]\, P(r_1, \tau + \tau_n) + \eta\, P(r_1, \tau_n)$. If $k = 2$, then the above equation reduces to the price of the European callable bond as given in the equation (3.7) with $\hat{\tau} = \tau$, $r = r_2$, $\rho = r^*_1$, and $m = 2$. Suppose that the above equation were correct for $k = n$. Substitute the expression for $K(r_n, n\,\tau + \tau_n)^+$ into the first and last lines of the equation (4.1) in order to obtain the price of the callable bond for the date $[t_r - (n + 1)\,\tau - \tau_n]$, apply lemma 1 term by term, and observe that $\hat{\lambda}(n, j, i) = \hat{\lambda}(n + 1, j + 1, i + 1)$, then the resulting expression is, after a suitable shift of the multiplication indices, equal to that of the above equation for $k = n + 1$. □

The price of the semi-American callable bond today is obtained by means of the last line of the equation (4.1) for $K(r_n, n\,\tau + \tau_n)^+$ as given in the equation (4.2), and for the final time step $\hat{\tau}$.

$$K(r, \hat{\tau}+n\,\tau+\tau_n) = \sum_{j=1}^{n} X_{n+1-j}\, P(r, \hat{\tau}+[j-1]\,\tau+\tau_n)\, \tilde{\mathfrak{H}}^+(r^*_n, r^*_{n+1-j}) + P(r, \hat{\tau}+n\,\tau+\tau_n)\, \tilde{\mathfrak{H}}(r^*_n, r^*_1) + \eta \sum_{j=1}^{n} P(r, \hat{\tau}+j\,\tau+\tau_n)\, \tilde{\mathfrak{H}}(r^*_n, r^*_{n+1-j}) + \eta \sum_{j=n}^{m} P(r, t_r - j\,\tau - t_0) \text{ , where} \tag{4.3}$$

$$\widetilde{\mathcal{H}}^{+}(r_n^*, r_{n+1-j}^*) := \left[\prod_{i=j}^{1} \widetilde{\omega}(j, i)\right]$$
$$\cdot \int_{r_n^*}^{\infty} \dots \int_{r_{n+2-j}^*}^{\infty} \int_{-\infty}^{r_{n+1-j}^*} \left[\prod_{i=1}^{j} H\left(\widetilde{\omega}(j, i)\, r_{n+1-i} \mid 2\alpha+2, \widetilde{\lambda}(n, j, i)\right)\right] \mathrm{d}r_{n+1-j} \dots \mathrm{d}r_n ,$$

$$\widetilde{\mathcal{H}}(r_a^*, r_d^*) := \left[\prod_{i=a-d+1}^{1} \widetilde{\omega}(a-d+2, i)\right]$$
$$\cdot \int_{r_a^*}^{\infty} \dots \int_{r_d^*}^{\infty} \left[\prod_{i=1}^{a-d+1} H\left(\widetilde{\omega}(a-d+2, i)\, r_{n+1-i} \mid 2\alpha+2, \widetilde{\lambda}(n, a-d+2, i)\right)\right] \mathrm{d}r_d \dots \mathrm{d}r_a ,$$

$$\widetilde{\omega}(j, i) := \begin{cases} \omega(j, i) := 2\left[p(\tau) - g([j-i]\tau+\tau_n)\right], (i > 1), \\ \widehat{\omega}(j, i) := 2\left[p(\widehat{\tau}) - g([j-i]\tau+\tau_n)\right], (i = 1), \end{cases} \quad , g(0) = 0,$$

$$\widetilde{\lambda}(n, j, i) := \begin{cases} \widehat{\lambda}(n, j, i) := \dfrac{p(\tau)}{p(\tau) - g([j-i]\tau+\tau_n)}\, \lambda(r_{n+2-i}, \tau) , (i > 1), \\ \widehat{\widehat{\lambda}}(n, j, i) := \dfrac{p(\widehat{\tau})}{p(\widehat{\tau}) - g([j-i]\tau+\tau_n)}\, \lambda(r_{n+2-i}, \widehat{\tau}) , (i = 1), \end{cases} \quad , r_{n+1} := r ,$$

$$\widetilde{\mathcal{H}}^{+}(r_a^*, r_d^*) = [\cdot] \int_{-\infty}^{r_d^*} H(\cdot)\, \mathrm{d}r_d \text{ if } d+1 > a, \quad \widetilde{\mathcal{H}}(r_a^*, r_d^*) = 1 \text{ if } d > a,$$

The first term represents the sum of the present values of call prices weighted by the forward-risk-adjusted probabilities that the bond will be called early. The second term represents the present value of the face value weighted by the forward-risk-adjusted probabilities that the bond will not be called early. The third term represents the sum of the present values of coupon payments during the call period weighted by the forward-risk-adjusted probabilities that the bond will not be called. Finally, the last term represents the present values of the coupon stream that will be received by the investor with certainty, that is, during the period the bond cannot be called. If there is a single call date, then the above equation reduces to that of the European callable bond.

5 The Numerical Algorithm

The price of the semi-American callable bond requires to compute several multiple integrals of the forward-risk-adjusted probability density function of the underlying interest rate process. The highest dimension of these integrals is equal to the number of call dates. Since each of the limits of such a multiple integral is the left-most zero of the function $\mathcal{F}$ in the equation (4.1) which involves the computation of lower-dimensional integrals as shown in the equation (4.2), the computation of a semi-American callable bond with ten call dates, say, would require to compute a sequence of integrals with a dimension ranging from one to nine, before one could finally compute a ten-dimensional integral. Since the computation of a multiple integral is rather time-consuming for higher dimensions as well as causing a considerable loss of accuracy, we choose to split the multiple integral into a *sequence of one-dimensional integrals* involving GREEN's function. The price we have to pay for this approach is that, for each time step,

the resulting price function can only be evaluated for discrete interest rate points. Hence, the price function which has been obtained from the preceding time step must be *interpolated* before it can serve as an input in the current integration. However, the *advantage* of this approach is two-fold. First, the entire numerical accuracy can mainly be controlled by that of the interpolation. Secondly, for suitably small time steps, the proposed algorithm can also be applied to American callable bonds, or to any American-type option with GREEN's function being explicitly known.

The numerical algorithm follows the method of the equation (4.1). As before, we proceed backwards in time. The starting point is the *second* notice day for which we compute the price of the callable bond an instant before this notice day with the help of the formula (3.7) for the European callable bond for a set of interpolating points $\{r_j^{(1)}, r_j^{(2)}, \ldots, r_j^{(v)}\}$ on the notice day j ($j := 2, 3, \ldots, n$). The computation of the probability distributions for the European callable bond will be described in the appendix B. The computation of the price of the European callable bond can be accomplished with approximate machine precision. There are three numerical operations to be repeated on each of the remaining notice days.

First, the set of callable bond prices an instant before each of the remaining notice days $\{K_j(r_j^{(1)}, \cdot), K_j(r_j^{(2)}, \cdot), \ldots, K_j(r_j^{(v)}, \cdot)\}$ for $j := 2, 3, \ldots, n$, which is obtained from the preceding time step, is interpolated numerically. (The index of K emphasizes the fact that the interpolated function varies from one notice day to the other). The two interpolation methods which will be discussed in the appendix B are the polynomial interpolation and the exponential cubic spline. Since the discount bond has been derived in terms of a series of orthogonal polynomials in the equation (2.4), it seems natural to use this series again for the interpolation. The callable bond price is interpolated on the finite interval $[r_l, r_u]$, but extrapolated for instantaneous interest rates which are greater than the upper interpolation boundary r_u. There is no extrapolation needed if the instantaneous interest is less than the lower interpolation boundary because the integration in the range $(-\infty, r_l]$ will be done explicitly. The interpolated price function an instant before each notice day, denoted as $K_j(\cdot, \cdot)^+$ for $j := 2, 3, \ldots, n$, may be written as:

$$K_j(r_j, j\tau + \tau_n)^+ := \begin{cases} \mathrm{e}^{b r_j} \sum_{i=0}^{N_p} d_i \mathcal{P}_i(x), \quad r_j := \vartheta_1 + \vartheta_2 x \quad \text{(polynomial interpolation)}, \\ \exp\left(\mathfrak{S}_3(r_j)\right), \quad \mathfrak{S}_3(r_j) := \sum_{i=0}^{3} a_i r_j^i \quad \text{(exponential cubic spline)}, \\ \exp\left(y(r_j)\right), \quad y(r_j) := y(r_u) + y'(r_u)\left[r_j - r_u\right] \quad \text{(extrapolation)}. \end{cases} \tag{5.1}$$

The orthogonal polynomial $\mathcal{P}_i$ of ith degree is the HERMITE polynomial in the case of VASICEK's model or the generalized LAGUERRE polynomial in the case of the CIR model. The constant b has been defined in the equation (2.4), whereas $\{d_i, \vartheta_1, \vartheta_2\}$ will be determined in the appendix B. The exponential spline method with a cubic polynomial $\mathfrak{S}_3$ computes the first derivatives $y'(\cdot)$ of the logarithm of the callable bond price function, whereas the coefficients a_i

are not needed. The extrapolation assumes that the logarithm of the callable bond price function grows linearly. An inspection of the graph of this function reveals that $\ln(K_j(\cdot, \cdot)^+)$ grows indeed approximately linearly for large interest rates. The first derivative of the callable bond price function evaluated at the upper interpolation boundary, $y'(r_u)$, can be taken from the exponential cubic spline.

Secondly, the "break-even" interest rate on each notice day, r_j^*, equates the time value of the sum of the call price and the coupon payment, $[X_j + \eta]\cdot P(r_j, \tau_n)$, with the callable bond price an instant before the notice day, $K_j(\cdot, \cdot)^+$:

$$\mathcal{F}_j(r_j) := [X_j + \eta]\, P(r_j, \tau_n) - K_j(r_j, j\,\tau + \tau_n)^+, \quad j = 2, 3, \ldots, n. \tag{5.2}$$

In the relevant range of interest rates, the function $\mathcal{F}_j(r_j)$ is found to be monotonically increasing for the numerical simulations considered in this paper. The algorithm determines the "break-even" interest rate on each notional call date as the left-most root of $\mathcal{F}_j(r_j)$ if the root should not be unique. The root can be bracketed with the help of the same set of callable bond prices $\{K_j(r_j^{(1)}, \cdot), K_j(r_j^{(2)}, \cdot), \ldots, K_j(r_j^{(\nu)}, \cdot)\}$ which is used for the interpolation mentioned above. The combined NEWTON-bisection method to find the "break-even" interest rate will be described in the appendix B.

Thirdly, the integration involving GREEN's function over the next period of time yields the price of the callable bond an instant before the next notice day, using the price of the callable bond an instant after the current notice day as the initial data:

$$\begin{aligned} K_{j+1}(r_{j+1}, [j+1]\,\tau + \tau_n)^+ &= \int_{-\infty}^{+\infty} G(r_{j+1}, \tau, r_j)\, K_j(r_j, j\,\tau + \tau_n)^- \,\mathrm{d}r_j + \eta\, P(r_{j+1}, \tau_n) \\ &:= \mathfrak{I}_1 + \mathfrak{I}_2 + \eta\, P(r_{j+1}, \tau_n), \quad (j = 2, 3, \ldots, n-1). \end{aligned} \tag{5.3}$$

GREEN's function, $G(\cdot, \cdot, \cdot)$, is given in the equation (2.6) and the callable bond price function an instant after the current notice day, $K_j(\cdot, \cdot)^-$, in the equation (4.1). The integral involving GREEN's function can be split into two parts. The first integral can be performed explicitly (see the lemmas 1 & 1′):

$$\begin{aligned} \mathfrak{I}_1 &:= [X_j + \eta] \int_{-\infty}^{r_j^*} G(r_{j+1}, \tau, r_j)\, P(r_j, \tau_n)\,\mathrm{d}r_j \\ &= [X_j + \eta]\, P(r_{j+1}, \tau + \tau_n) \cdot \begin{cases} \mathcal{N}(d_j), & d_j := \dfrac{r_j^* - \mu - s(\tau)^2\,[\,t(r_{j+1}, \tau) + g(\tau_n)]}{s(\tau)}, \\ \mathcal{H}(d_j \mid 2\alpha + 2, \Lambda(r_{j+1}, \tau, \tau_n)), & d_j := 2\, r_j^*\,[p(\tau) - g(\tau_n)]. \end{cases} \end{aligned} \tag{5.4}$$

All the symbols and functions have been defined in the equations (3.7) and (3.7)′. The computation of the GAUSSian probability integral $\mathcal{N}(\cdot)$ in VASICEK's model and of the non-central chi-square probability integral $\mathcal{H}(\cdot \mid \nu, \lambda)$ in the CIR model will be shown in the appendix B. The key operation is to integrate numerically the second integral

$$\Im_2 := \int_{r_j^*}^{+\infty} G(r_{j+1},\, \tau,\, r_j)\, K_j(r_j,\, j\,\tau + \tau_n)^+ \, \mathrm{d}r_j \,. \tag{5.5}$$

We suggest to apply a particular transformation to the above integral such that the numerical quadrature procedure may achieve machine precision (WALDVOGEL [1988]). This numerical procedure will be described in the appendix B.

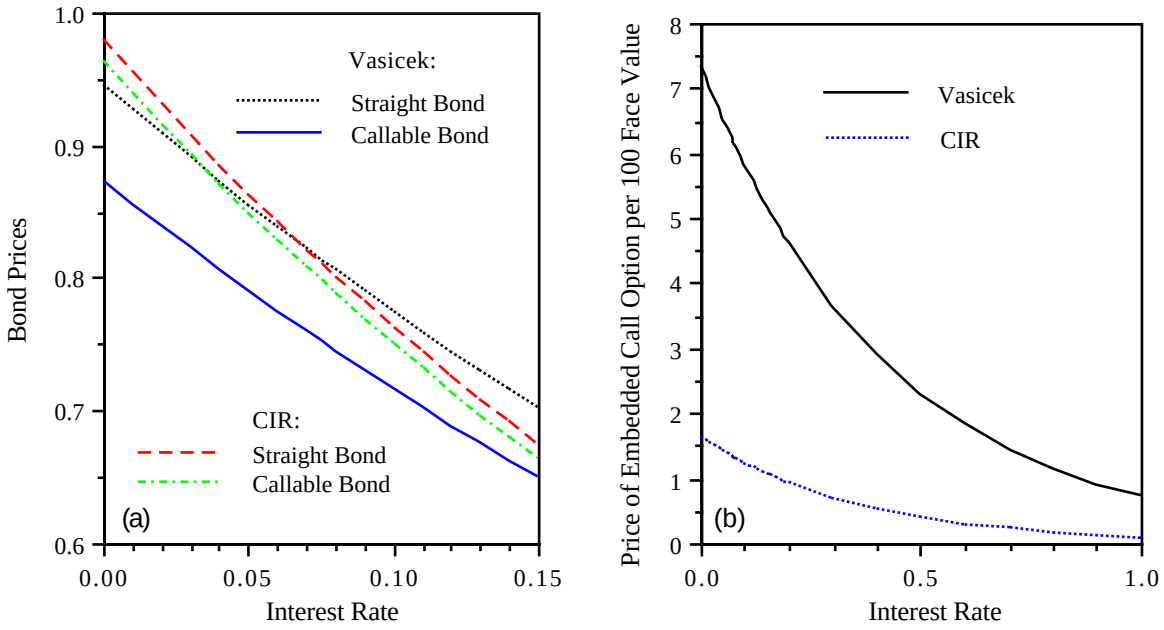
The last step of the algorithm is equal to those described so far except for the time span: replace τ by $\hat{\tau}$. While all the call dates are relevant in the case of VASICEK's model, the algorithm stops in the case of the CIR model as soon as the "break-even" interest rate becomes zero or negative. In that event, the final time step $\hat{\tau}$ has to be redefined.

Three remarks may conclude this section. First, some simulations indicate that our numerical algorithm offers both higher speed of computation and higher accuracy than finite difference methods. For the example to follow in the next section, our numerical algorithm uses only between 46% and 82% of the CPU time required by finite difference methods, while achieving a three-digit accuracy at least, in contrast to a one-digit accuracy of finite difference methods. Secondly, the overall accuracy of our algorithm is determined by the accuracy of the interpolation method considered, because the proposed numerical quadrature procedure can achieve machine precision. An efficient hybrid interpolation method is explained in the appendix B. In contrast, the accuracy of finite difference methods is difficult to control due to possible numerical instabilities or different treatments of the boundary conditions. Finally, numerical instabilities due to discontinuous boundary conditions as encountered with finite difference methods do not occur with our numerical algorithm, because the boundary conditions are explicitly built into GREEN's function.

6 An Example

In order to obtain comparable results, the bond valuation models of both VASICEK [1977] and COX, INGERSOLL AND ROSS [1985a & b] have been calibrated to the *same* term structure of interest rates which was *observed* on a particular trading day. In VASICEK's model, the partial derivatives of the theoretical yield curve with respect to two pairs of parameters are linearly dependent. These pairs are either the long-run "equilibrium" value of the instantaneous interest rate, θ, and the market price of interest rate risk, q, or the the long-run "equilibrium" value of the instantaneous interest rate, θ, and the instantaneous interest rate, r. Since we want to compute the price of the callable bond primarly for the *current* value of the instantaneous interest rate, r_0, this value is chosen as a data of the observed yield curve. Hence, the long-run "equilibrium" value of the instantaneous interest rate, θ, is estimated as the mean value of the instantaneous interest rate over the last twenty years. Although none of the partial derivatives of the theoretical yield curve with respect to the model parameters is linearly dependent in the CIR model, an inspection of these derivatives reveals that the partial derivatives with respect to the

speed of adjustment, κ, the risk premium, $\zeta - \kappa$, and the long-run "equilibrium" value of the instantaneous interest rate, θ, are almost linearly dependent. Moreover, these derivatives are linearly dependent when $\kappa = \theta$. Therefore, we estimate again the long-run "equilibrium" value of the instantaneous interest rate, θ, as the mean value of the instantaneous interest rate over the last twenty years. There remain three parameters to be estimated in both bond models. Given three *observed* discount bond yields and the current value of the instantaneous interest rate, these three parameters have been computed from the theoretical yield curves of each of the two models by means of a modified NEWTON-RAPHSON method. The estimated model parameters are reported in the Table C.1. Both the market price of interest rate risk, q, and the risk premium, $\zeta - \kappa$, have the expected sign. Since we wish to focus on the accuracy of our numerical algorithm, we feel that the simple matching of the yield curve may serve our purpose. Moreover, a reasonable econometric estimation of these parameters seems to be rather difficult due to the very small number of daily yields available in the Swiss market, which may be as small as three or four on many trading days. In real applications, one would wish to estimate these parameters with the method of forward induction (JAMSHIDIAN [1991b]), say, and make these parameters time-dependent (DUFFIE [1992]).

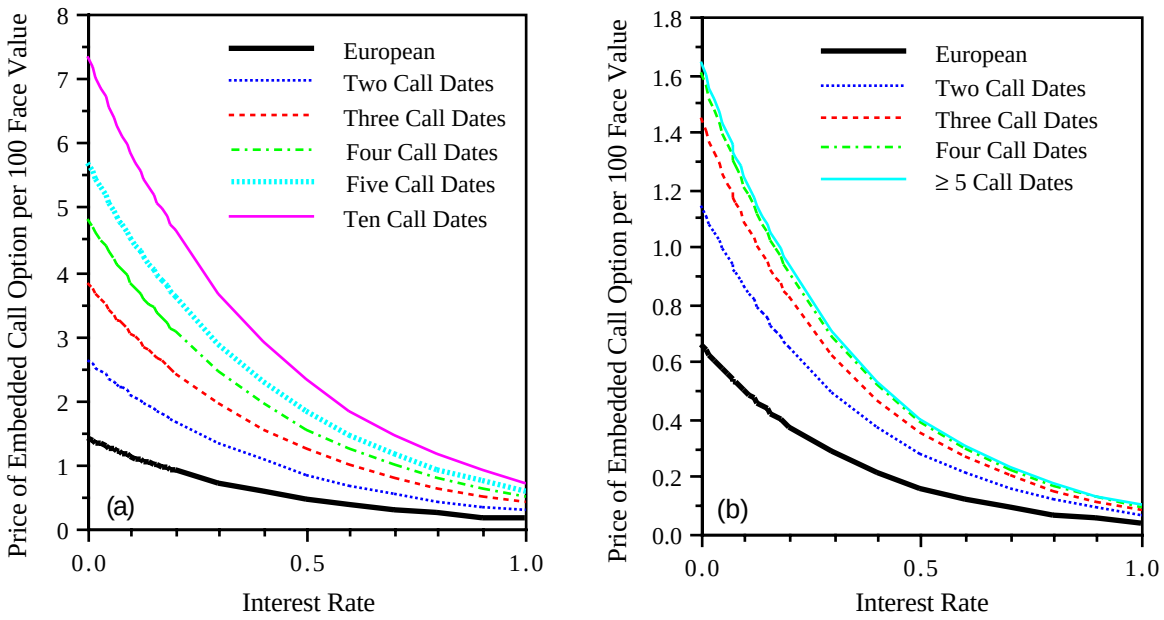


Figures 3a & b: The Price of the Callable Bond, the Underlying Straight Bond (a), and the Embedded Call Option (b). The price of the callable bond has been obtained for 650 knots. The characteristics of the callable bond are given in the Table C.2.

We report the numerical results for an exchange-traded callable bond with ten call dates and with a maximum life of more than twenty years. The characteristics of this callable bond are given in the Table C.2. An inspection of the graph of the callable bond price function reveals that $r_u = 6$ might be a reasonable choice for both interest rate models. The lower interpolation boundary has been chosen to be $r_l = -0.5$ in VASICEK's model in order to accommodate the

negative "break-even" interest rates. In the CIR model, a natural choice is zero, that is, $r_l = 0$. We report the numerical results for the exponential cubic spline interpolation with equidistant knots.

The price of the callable bond given the current value of the instantaneous interest rate, $K(r_0, \hat{\tau} + n\,\tau + \tau_n)$ where $r_0 = 0.07522$ and $\hat{\tau} + n\,\tau + \tau_n = 20.172$ years, is shown in the Table C.3 for various numbers of interpolating knots. In VASICEK's model, the price is accurate up to two decimal digits only when 325 interpolating knots are used. A computation with a larger interpolation interval shows that the price of the callable bond which is shown in the Table C.3 for 650 knots is accurate up to three significant digits. In the model of CIR, the price is accurate up to six significant digits when 325 knots are used. The difference in accuracy of these two models is due to the fact that there are twice as many integrals to be solved in VASICEK's model because there are only five positive "break-even" interest rates in the model of CIR. The "break-even" interest rates are reported in the Table C.4 for the case of 650 interpolating knots. In VASICEK's model, the smallest "break-even" interest rate is –13.91% and, surprisingly, there is only one positive "break-even" interest rate. As mentioned before, the numerical algorithm leaves the loop to compute the integral involving GREEN's function whenever the "break-even" interest becomes zero or negative in the model of CIR.



Figures 4a & b: The Price of Embedded Call Options for VASICEK's model (a) and for the CIR model (b). The call options refer to callable bonds the characteristics of which are identical to the ones given in the Table C.2 except for the number of call dates. The call dates are counted backwards in time. The price of the callable bonds have been obtained for 650 knots in panel (a) and for 325 knots in panel (b).

The price of the callable bond, the price of the underlying straight bond, and the price of the embedded call option are reported in the Table C.5 for both models as a function of the in-

stantaneous interest rate. These results are depicted in Figures 3a & b. More information on the price of various embedded call options is given in Figures 4a & b. These call options refer to callable bonds the characteristics of which are identical to the ones given in the Table C.2 except for the number of call dates.

The considerable difference between the prices of embedded call options for the VASICEK and CIR models in this example may be explained partly by the fact that we matched the theoretical yield curve to just three observed yields only. Another plausible explanation is the different number of *positive* "break-even" interest rates in the two models considered (Table C.4). Some simulations indicate that the price difference becomes moderate if the number of positive "break-even" interest rates is the same in both models. A similar result has been found by JAMSHIDIAN [1991b] within the framework of the GAUSSian interest rate model.

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Appendix A: Derivations for the Two Interest Rate Models Considered

In the first part of this appendix, we derive the equations (2.4) – (2.6) which apply to the CIR model, and then collect the corresponding equations in the case of VASICEK's model in the second part of this appendix.

A.1 The CIR model

We transform the partial differential equation (2.2) according to

$$P(r, \tau) = \mathrm{e}^{br}\, Q(r, \tau), \quad b < 0, \tag{A.1.1}$$

which reduces it to "normal" form

$$Q_\tau = \frac{1}{2}\,\sigma^2 r^\varepsilon\, Q_{rr} + [-Ar + B]\, Q_r + CQ, \quad \varepsilon \in \{0, 1\}, \tag{A.1.2}$$

where $A > 0$, $B \gtrless 0$ and $C < 0$ are given constants. In the CIR model, $\varepsilon = 1$ and the constants A, B, and C as well as the parameter b become

$$A = \gamma, \quad B = \kappa\theta\,, \quad C = \kappa\theta\,\frac{\zeta - \gamma}{\sigma^2}\,, \quad b = \frac{\zeta - \gamma}{\sigma^2} < 0, \tag{A.1.3}$$

where γ has been defined in the equation (2.4). The case $b = [\zeta + \gamma] / \sigma^2 > 0$ can be excluded in view of the discussion below.

Separating variables, we construct solutions of (A.1.2) which have the product form $Q(r, \tau) = R(r)\, T(\tau)$. It turns out that the time function is exponential, while the locational function can be reduced to KUMMER's differential equation by means of the transformation $x = 2\,A\,r\,/\,\sigma^2$ and $w(x) = R(r)$:

$$x\,\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + [\alpha + 1 - x]\,\frac{\mathrm{d}w}{\mathrm{d}x} + nw = 0, \quad n := \frac{C + \xi}{A}\,, \quad \alpha + 1 := \frac{2B}{\sigma^2} = \frac{2\kappa\theta}{\sigma^2} > 0, \tag{A.1.4}$$

where ξ denotes the parameter of separation. Any solution $w(x)$ of this equation grows polynomially or exponentially as $x \to \infty$. Since the latter would lead to an exploding discount function, the boundary conditions must be

$$\begin{aligned} &w(x) = \mathbb{O}\left(x^k\right),\ k < \infty \text{ as } x \to +\infty \text{ (right boundary)},\\ &w'(x) \text{ finite as } x \to 0 \text{ (left boundary)}. \end{aligned} \tag{A.1.5}$$

This constitutes a generalized STURM-LIOUVILLE eigenvalue problem. The general solution of KUMMER's differential equation (A.1.4) is a linear combination of the (generally) independent solutions as follows:

$$w_1(x) = M(-n,\, \alpha+1;\, x)\,, \quad w_2(x) = U(-n,\, \alpha+1;\, x)\,, \quad n \in \mathbb{R}. \tag{A.1.6}$$

where $M(\cdot, \cdot; \cdot)$ is the KUMMER function and $U(\cdot, \cdot; \cdot)$ the TRICOMI function. We exclude the case that the second argument has an integer value; this may be justified by the fact that it is determined from observed data (κ, θ, σ). Since $w_2'(x) = n\, U(-n+1, \alpha+2; x)$ is unbounded for $b = \alpha + 2 > 1$ as $x \to 0$ (ABRAMOWITZ AND STEGUN [1965, equ. (13.4.21) and (13.5.6) – (13.5.8)]), the second solution does not satisfy the left boundary condition. Since $w_1'(x)$ is finite as $x \to 0$, it satisfies the left boundary condition. From the asymptotic theory of confluent hypergeometric functions we obtain for the first solution (ABRAMOWITZ AND STEGUN [1965, equ. (13.1.4)])

$$w_1(x) = \frac{\Gamma(\alpha+1)}{\Gamma(-n)}\, e^x\, x^{-[n+\alpha+1]} \left[1 + \mathcal{O}\left(x^{-1}\right)\right] \text{ as } x \to \infty\,. \qquad \text{(A.1.7)}$$

Therefore, $w_1(x)$ grows exponentially as $x \to \infty$ except in the cases $n = 0, 1, 2, \ldots$, where $w_1(x)$ is a polynomial and thus satisfies both boundary conditions. Hence, the series solution (2.4) follows. The well-known price formula for the discount bond (2.5) can be deduced from the series solution by means of (i) the orthogonality condition for the generalized LAGUERRE polynomial, $L_n^{(\alpha)}(x)$, (ii) an application of the LAPLACE transform (28) in ERDÉLYI ET AL. [1954, vol. 1, p. 174] to the resulting orthogonality condition, and (iii) an application of the generating function (22.9.15) in ABRAMOWITZ AND STEGUN [1965] to the series solution (2.4). GREEN's function (2.6) follows in the same way except for the initial condition and the generating function (20) in ERDÉLYI ET AL. [1953, vol. 2, p. 189].

A.2 The Vasicek model

The process for the instantaneous interest rate is the following ORNSTEIN-UHLENBECK process:

$$dr = \kappa(\theta - r)\, dt + \sigma\, dz, \qquad (2.1)'$$

where all the symbols have been introduced in the second section. Note that our notation is slightly different from that in VASICEK [1977]. The spot interest rate may become negative, but "mean-reverting" will eventually pull it back to its long-run equilibrium value. VASICEK derives the following partial differential equation to determine the price of a default-free discount bond, $P(r, \tau)$, promising to pay one unit of money on the redemption date:

$$P_\tau = \frac{1}{2}\sigma^2 P_{rr} + \left[\kappa(\theta - r) + \sigma q\right] P_r - rP, \qquad (2.2)'$$

where the subscripts denote partial derivatives, τ the remaining period of time until the redemption of the discount bond, and q the market price of interest rate risk assumed to be constant. If arbitrage opportunities are ruled out, the market price of interest rate risk must be the same for all discount bonds of different maturities. Empirically, we would expect q to be positive.

In analogy to the CIR model with the goal of obtaining a STURM-LIOUVILLE problem, the boundary conditions are stated as

$$\begin{aligned} &P(r,\,\tau) \to 0 \text{ as } r \to \infty, \text{ (right boundary)},\\ &P(r,\,\tau) = \mathbb{O}\left(\mathrm{e}^{-\vartheta r}\right),\ \vartheta > 0, \text{ as } r \to -\infty, \text{ (left boundary)}. \end{aligned} \tag{2.3′}$$

Using the transformation (A.1.1) and the normal form (A.1.2) with $\varepsilon = 0$, the constants become now:

$$A = \kappa,\quad B = \kappa\left[R_\infty - \frac{1}{2}\frac{\sigma^2}{\kappa^2}\right],\quad C = -R_\infty,\quad b = -\frac{1}{\kappa} < 0,\quad R_\infty := \theta + \frac{\sigma\, q}{\kappa} - \frac{1}{2}\frac{\sigma^2}{\kappa^2}. \tag{A.2.1}$$

The abbrevation R_∞ denotes the yield of a discount bond with infinite time to maturity. In a meaningful model, the nominal yield of a consol bond should be positive.

Separating variables again, we construct solutions of (A.1.2) which have the product form $Q(r,\,\tau) = R(r)\,T(\tau)$. It turns out that the time function is exponential, while the locational function can be reduced to KUMMER's differential equation by means of the transformation $x = [A\,r - \mathrm{B}] / [\sigma\sqrt{A}\,]$, $w(x) = R(r)$ and $z = x^2$. The general solution of KUMMER's differential equation is a linear combination of the (generally) independent solutions:

$$w_1(x) = M\left(-\frac{n}{2},\,\frac{1}{2};\,x^2\right),\quad w_2(x) = U\left(-\frac{n}{2},\,\frac{1}{2};\,x^2\right),\quad n \in \mathbb{R}. \tag{A.2.2}$$

which must satisfy the transformed boundary conditions

$$w(x) = \mathbb{O}\left(x^k\right),\ k < \infty \text{ as } x \to \pm\infty. \tag{A.2.3}$$

From the asymptotic theory of confluent hypergeometric functions there follows that $w_1(x)$ grows like $\mathbb{O}(\exp(x^2))$ as $x \to \pm\infty$ except in the cases $n = 0, 2, 4, \ldots$, when $w_1(x)$ is a polynomial and thus satisfies both boundary conditions. However, more eigenfunctions may be produced by means of the second solution $w_2(x)$. It can be shown that the non-negative integers $n = 0, 1, 2, \ldots$ are the only eigenvalues, and that the corresponding eigenfunction $w_2(x)$ is a polynomial. It follows from ERDÉLYI ET AL. [1953, vol. 2, equ. (16), p. 194] that the second solution can be expressed as $w_2(x) = 2^{-n}\,H_n(x)$, where $H_n(x)$ denotes the HERMITE polynomial. The even eigenfunctions, but not the odd ones, may also be obtained from the first solution $w_1(x)$; thus for $n = 0, 2, 4, \ldots$, $w_1(x)$ and $w_2(x)$ are linearly dependent. Other linear combinations of $w_1(x)$ and $w_2(x)$ do not produce new eigenfunctions. The series solution now follows:

$$P(r,\,\tau) = \mathrm{e}^{br} \sum_{n=0}^{\infty} c_n\, \mathrm{e}^{[C - nA]\,\tau}\, H_n\!\left(\frac{A\,r - B}{\sigma\sqrt{A}}\right). \tag{2.4′}$$

The well-known price formula for the discount bond is obtained from the series solution by means of (i) the orthogonality condition for the HERMITE polynomial, (ii) a repeated integration by parts of the resulting orthogonality condition using the equation (22.13.15) in ABRAMOWITZ

AND STEGUN [1965], and (iii) an application of the generating function (22.9.17) in ABRAMOWITZ AND STEGUN [1965] to the series solution:

$$P(r,\tau) = \exp(f(\tau) + g(\tau)\, r), \text{ where}$$
$$f(\tau) := \frac{1}{\kappa} R_\infty \left[1 - \mathrm{e}^{-\kappa\tau}\right] - R_\infty \tau - \frac{\sigma^2}{4\,\kappa^3}\left[1 - \mathrm{e}^{-\kappa\tau}\right]^2, \quad g(\tau) := -\frac{1}{\kappa}\left[1 - \mathrm{e}^{-\kappa\tau}\right]. \tag{2.5′}$$

GREEN's function now follows from the generating function (22) in ERDÉLYI ET AL. [1953, vol. 2, p. 194]:

$$G(r,\tau,\rho) = P(r,\tau)\left\{\frac{1}{s(\tau)\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left[\frac{\rho - \mu - s(\tau)^2\, t(r,\tau)}{s(\tau)}\right]^2\right)\right\}, \text{ where}$$
$$t(r,\tau) := \frac{2\,[A\,r - B]}{\sigma^2}\,\frac{\mathrm{e}^{-A\,\tau}}{1 - \mathrm{e}^{-2A\,\tau}} - b = \frac{2\,[\kappa r - B]}{\sigma^2}\,\frac{\mathrm{e}^{-\kappa\tau}}{1 - \mathrm{e}^{-2\kappa\tau}} + \frac{1}{\kappa}, \tag{2.6′}$$
$$\mu := \frac{B}{A} = R_\infty - \frac{1}{2}\,\frac{\sigma^2}{\kappa^2},$$
$$s(\tau)^2 := \frac{1}{2}\,\frac{\sigma^2}{A}\left[1 - \mathrm{e}^{-2A\,\tau}\right] = \frac{1}{2}\,\frac{\sigma^2}{\kappa}\left[1 - \mathrm{e}^{-2\kappa\tau}\right].$$

$P(r,\tau)$ and R_∞ denote the price of a discount bond and the yield of a discount bond with infinite time to maturity, respectively. The forward-risk adjustment is given by the expression $\sigma\, q + \sigma^2\, g(\tau)$.

In the sequel, we collect the corresponding results for the European and semi-American callable bond.

LEMMA 1′: *Suppose the length of the remaining time until an intermediary date, which serves as "initial" date, is τ_2, where the initial condition is given by the price of a discount bond with remaining time until maturity of length τ_1, then*

$$\int_a^b G(r,\tau_2,\rho)\, P(\rho,\tau_1)\, \mathrm{d}\rho$$
$$= P(r,\tau_2) \exp\left(f(\tau_1) + \mu\, g(\tau_1) + s(\tau_2)^2\, t(r,\tau_2)\, g(\tau_1) + \frac{1}{2}\, s(\tau_2)^2\, g(\tau_1)^2\right) \mathfrak{N}$$
$$= P(r,\tau_1 + \tau_2)\, \mathfrak{N}, \text{ where} \tag{3.6′}$$
$$\mathfrak{N} := \frac{1}{s(\tau_2)\sqrt{2\pi}} \int_a^b \exp\left(-\frac{1}{2}\left[\frac{\rho - \mu - s(\tau_2)^2\,[\,t(r,\tau_2) + g(\tau_1)]}{s(\tau_2)}\right]^2\right) \mathrm{d}\rho\,. \quad \bigcirc$$

The proof of lemma 1′ is similar to that given for lemma 1. The price of the European callable bond today with the remaining time until the last possible redemption date of length $[\hat{\tau} + \tau_n + \tau]$ now follows from lemma 1′:

$$K(r, \hat{\tau} + \tau_n + \tau) = X\, P(r, \hat{\tau} + \tau_n)\, \mathcal{N}(d_1) + [1 + \eta]\, P(r, \hat{\tau} + \tau_n + \tau)\, \mathcal{N}(d_2) + \eta \sum_{j=1}^{m} P(r, t_r - j\,\tau - t_0),$$
$$\mathcal{N}(d) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} \exp\left(-\frac{1}{2}\, x^2\right) \mathrm{d}x, \text{ (standard normal probability distribution)} \tag{3.7′}$$
$$d_1 := \frac{\rho^* - \mu - s(\hat{\tau})^2\,[\,t(r,\hat{\tau}) + g(\tau_n)]}{s(\hat{\tau})}, \quad d_2 := -\,d_1 + s(\hat{\tau})\,[\,g(\tau + \tau_n) - g(\tau_n)].$$

Lemma 2 to be used for the determination of the "break-even" interest rates now becomes lemma 2′:

LEMMA 2′: *The price of the semi-American callable bond an instant before each notice date, when proceeding backwards in time, denoted as* $K(r_k, k\tau+\tau_n)^+$ *for* $k = 1, 2, \ldots, n$, *is given by*

$$K(r_k, k\tau+\tau_n)^+ = \sum_{j=1}^{k-1} X_{k-j}\, P(r_k, j\tau+\tau_n)\, \mathfrak{N}^+(r^*_{k-1}, r^*_{k-j}) + P(r_k, k\tau+\tau_n)\, \mathfrak{N}(r^*_{k-1}, r^*_1) + \eta \sum_{j=0}^{k} P(r_k, j\tau+\tau_n)\, \mathfrak{N}(r^*_{k-1}, r^*_{k+1-j}) \text{, where} \tag{4.2′}$$

$$\mathfrak{N}^+(r^*_{k-1}, r^*_{k-j}) := \frac{1}{[s(\tau)\sqrt{2\pi}]^j} \int_{r^*_{k-1}}^{\infty} \cdots \int_{r^*_{k+1-j}}^{\infty} \int_{-\infty}^{r^*_{k-j}} \exp\left(-\frac{1}{2}\sum_{i=1}^{j} \varphi(k, j, i)^2\right) \mathrm{d}r_{k-j} \ldots \mathrm{d}r_{k-1},$$

$$\mathfrak{N}(r^*_a, r^*_d) := \frac{1}{[s(\tau)\sqrt{2\pi}]^{a-d+1}} \int_{r^*_a}^{\infty} \cdots \int_{r^*_d}^{\infty} \exp\left(-\frac{1}{2}\sum_{i=1}^{a-d+1} \varphi(k, a-d+2, i)^2\right) \mathrm{d}r_d \ldots \mathrm{d}r_a,$$

$$\varphi(k, j, i) := \frac{r_{k-i} - \mu - s(\tau)^2\left[t(r_{k+1-i}, \tau) + g([j-i]\tau+\tau_n)\right]}{s(\tau)}, \quad g(0) = 0, \quad r_{n+1} := r,$$

$$\mathfrak{N}^+(r^*_a, r^*_d) = [\ldots] \int_{-\infty}^{r^*_d} \exp(\cdot)\, \mathrm{d}r_d \text{ if } d+1 > a, \quad \mathfrak{N}(r^*_a, r^*_d) = 1 \text{ if } d > a. \ \bigcirc$$

The proof is similar to that of lemma 2. The price of the semi-American callable bond today is given by

$$K(r, \hat{\tau}+n\tau+\tau_n) = \sum_{j=1}^{n} X_{n+1-j}\, P(r, \hat{\tau}+[j-1]\tau+\tau_n)\, \tilde{\mathfrak{N}}^+(r^*_n, r^*_{n+1-j}) + P(r, \hat{\tau}+n\tau+\tau_n)\, \tilde{\mathfrak{N}}(r^*_n, r^*_1) + \eta \sum_{j=1}^{n} P(r, \hat{\tau}+j\tau+\tau_n)\, \tilde{\mathfrak{N}}(r^*_n, r^*_{n+1-j}) + \eta \sum_{j=n}^{m} P(r, t_r - j\tau - t_0) \text{, where} \tag{4.3′}$$

$$\tilde{\mathfrak{N}}^+(r^*_n, r^*_{n+1-j}) := \frac{1}{s(\hat{\tau})\, s(\tau)^{j-1}[\sqrt{2\pi}]^j} \cdot \int_{r^*_n}^{\infty} \cdots \int_{r^*_{n+2-j}}^{\infty} \int_{-\infty}^{r^*_{n+1-j}} \exp\left(-\frac{1}{2}\sum_{i=1}^{j} \tilde{\varphi}(n+1, j, i)^2\right) \mathrm{d}r_{n+1-j} \ldots \mathrm{d}r_n,$$

$$\tilde{\mathfrak{N}}(r^*_a, r^*_d) := \frac{1}{s(\hat{\tau})\, s(\tau)^{a-d}[\sqrt{2\pi}]^{a-d+1}} \cdot \int_{r^*_a}^{\infty} \cdots \int_{r^*_d}^{\infty} \exp\left(-\frac{1}{2}\sum_{i=1}^{a-d+1} \tilde{\varphi}(n+1, a-d+2, i)^2\right) \mathrm{d}r_d \ldots \mathrm{d}r_a,$$

$$\tilde{\varphi}(n, j, i) := \begin{cases} \varphi(n, j, i) := \dfrac{r_{n-i} - \mu - s(\tau)^2\left[t(r_{n+1-i}, \tau) + g([j-i]\tau+\tau_n)\right]}{s(\tau)}, & (i > 1), \\ \hat{\varphi}(n, j, i) := \dfrac{r_{n-i} - \mu - s(\hat{\tau})^2\left[t(r_{n+1-i}, \hat{\tau}) + g([j-i]\tau+\tau_n)\right]}{s(\hat{\tau})}, & (i = 1), \end{cases}$$

$$\tilde{\mathfrak{N}}^+(r^*_a, r^*_d) = [\ldots] \int_{-\infty}^{r^*_d} \exp(\cdot)\, \mathrm{d}r_d \text{ if } d+1 > a,\ \tilde{\mathfrak{N}}(r^*_a, r^*_d) = 1 \text{ if } d > a,\ g(0) = 0,\ r_{n+1} := r\ .$$

The interpretation of the various terms is the same as for the equation (4.3).

Appendix B: The Building Blocks of the Numerical Algorithm

The computer program has been written in PASCAL (JENSEN AND WIRTH [1978]) and runs on the APPLE® MACINTOSH™ family, the machine precision of which is 19 – 20 decimal digits (i. e., the fractional part or the mantissa of the floating-point format). However, most procedures do *not* use any machine constant; see, e. g., the next section.

B.1 Probability Distributions and Related Functions

The algorithm starts on the second notice day for which we compute the price of the European callable bond with $[2\ \tau + \tau_n]$ years to maturity. This requires the numerical evaluation of the GAUSSian or non-central chi-square probability integral, respectively, which we write as an infinite power series for small arguments or as a continued fraction for large arguments (HENRICI [1974, 1982]). In the case of the normal probability integral, all the terms are positive, that is, no cancellation of digits occurs. In the case of equal signs, a machine-independent stopping criterion may be obtained by truncating the sum as soon as the current partial sum becomes equal to the previous partial sum.

The GAUSSian probability integral may be written in terms of the KUMMER function $M(\cdot, \cdot, \cdot)$ for small arguments (e. g.,$x < 3$) (ABRAMOWITZ AND STEGUN [1965, equ. (26.2.37)]):

$$\begin{aligned} \mathcal{N}(x) - \tfrac{1}{2} &= x\,\mathcal{N}'(x)\,M\left(1, \tfrac{3}{2}, \tfrac{x^2}{2}\right), \quad (x > 0) \\ &:= x\,\mathcal{N}'(x) \sum_{n=0}^{\infty} a_n\,, \quad \text{with } a_0 = 1, \quad a_n = \frac{x^2}{1+2n}\,a_{n-1} > 0, \ (n = 1, 2, \ldots). \end{aligned} \tag{B.1.1}$$

Here, $\mathcal{N}'(x) = \exp(-x^2/2) / \sqrt{2\pi}$ denotes the standard normal probability density function. The terms a_n may be obtained recursively as shown in the second line of the above equation. For large arguments ($x \geqq 3$), a continued fraction of the complementary normal probability integral is used (ABRAMOWITZ AND STEGUN [1965, equ. (26.2.14)]):

$$\mathcal{N}_c(x) := 1 - \mathcal{N}(x) = \mathcal{N}'(x)\left[\frac{1}{x+}\,\frac{1}{x+}\,\frac{2}{x+}\,\frac{3}{x+}\,\frac{4}{x+}\,\ldots\right], \quad (x > 0). \tag{B.1.2}$$

This continued fraction is evaluated forwards by means of the well-known recursive algorithm (see e. g. PRESS ET AL. [1989]). Evaluating the continued fraction from the tail is, of course, much faster but has the disadvantage that the number of terms, which is necessary to achieve machine precision, is a function of the argument x.

The non-central chi-square probability integral is written as an infinite series for all arguments because no useful continued fraction representation is known.

$$\begin{aligned}\mathcal{H}(2x \mid 2\nu, 2\lambda) &= \int_0^{2x} H(t \mid 2\nu, 2\lambda)\,\mathrm{d}t = \mathrm{e}^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \frac{1}{2^{\nu+n}\,\Gamma(\nu+n)} \int_0^{2x} t^{\nu+n-1}\,\mathrm{e}^{-t/2}\,\mathrm{d}t \\ &= \mathrm{e}^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\,\mathcal{H}(2x \mid 2\nu+2n) = \mathrm{e}^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \frac{\gamma(\nu+n,\, x)}{\Gamma(\nu+n)} := \mathrm{e}^{-\lambda} \sum_{n=0}^{\infty} a_n\,.\end{aligned} \tag{B.1.3}$$

Here, the non-central chi-square distribution $\mathcal{H}(\cdot \mid \nu, \lambda)$ has been written in terms of the χ^2-distribution, denoted as $\mathcal{H}(\cdot \mid \nu)$, and in terms of the incomplete gamma function, denoted as $\gamma(\cdot, \cdot)$ (ABRAMOWITZ AND STEGUN [1965, equ. (26.4.1) and (26.4.19)]). The terms a_n may be obtained recursively as follows:

$$\begin{aligned}&a_0 = \frac{\gamma(\nu,\, x)}{\Gamma(\nu)} = \mathcal{H}(2x \mid 2\nu) > 0, \quad b_1 = \frac{\lambda\, x^{\nu}\,\mathrm{e}^{-x}}{\nu\,\Gamma(\nu)} > 0, \\ &b_n = \frac{\lambda x}{n\,[\nu + n - 1]}\, b_{n-1}, \quad (n = 2, 3, \ldots), \quad a_n = \frac{\lambda}{n}\, a_{n-1} - b_n, \quad (n = 1, 2, \ldots).\end{aligned} \tag{B.1.4}$$

The above expansion relies on the recurrence relation for the incomplete gamma function (ABRAMOWITZ AND STEGUN [1965, equ. (6.5.22)]). The term b_n is merely an auxiliary variable to compute the sequence a_n. We feel that the algorithm (B.1.4) is simpler than the one given by TIKU [1981]. The chi-square distribution $\mathcal{H}(\cdot \mid \nu)$ or the incomplete gamma function, respectively, has been computed by means of the KUMMER function for small arguments ($x < \nu + 1$) as follows (ABRAMOWITZ AND STEGUN [1965, equ. (6.5.12)]):

$$\gamma(\nu,\, x) = \frac{x^{\nu}\,\mathrm{e}^{-x}}{\nu}\, M(1,\, 1+\nu,\, x) := \frac{x^{\nu}\,\mathrm{e}^{-x}}{\nu} \sum_{n=0}^{\infty} a_n\,, \quad a_0 = 1, \quad a_n = \frac{x}{n+\nu}\, a_{n-1}\,. \tag{B.1.5}$$

All the terms a_n are positive. The complementary chi-square probability distribution is computed for large arguments ($x \geqq \nu + 1$) by means of a continued fraction (ABRAMOWITZ AND STEGUN [1965, equ. (26.4.10)]):

$$\mathcal{H}_c(2x \mid 2\nu) := 1 - \mathcal{H}(2x \mid 2\nu) = \frac{x^{\nu}\,\mathrm{e}^{-x}}{\Gamma(\nu)} \left[\frac{1}{x+} \frac{1-\nu}{1+} \frac{1}{x+} \frac{2-\nu}{1+} \frac{2}{x+} \ldots \right], \quad (x > 0). \tag{B.1.6}$$

Again, the continued fraction is evaluated forwards. Both the power series and the continued fraction are mentioned in PRESS ET AL. [1989].

The non-central chi-square density function, which is needed for GREEN's function in the case of the CIR model, may also be computed recursively from an infinite power series representation:

$$\begin{aligned}H(2x \mid 2\nu, 2\lambda) &= \mathrm{e}^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \frac{[2x]^{\nu+n-1}\,\mathrm{e}^{-x}}{2^{\nu+n}\,\Gamma(\nu+n)} \\ &:= \frac{\mathrm{e}^{-[\lambda+x]}}{2} \sum_{n=0}^{\infty} a_n\,, \quad a_0 = \frac{x^{\nu-1}}{\Gamma(\nu)}\,, \quad a_n = \frac{\lambda x}{n\,[\nu + n - 1]}\, a_{n-1} > 0.\end{aligned} \tag{B.1.7}$$

All the terms are positive, that is, no cancellation of digits occurs. It is obvious from the first term that the non-central chi-square density function, $H(\cdot \mid \nu, \lambda)$, has a singularity at the origin if $\nu < 2$.

EULER's gamma function, $\Gamma(x)$, is computed from an asymptotic formula as follows (ABRAMOWITZ AND STEGUN [1965, equ. (6.1.40)]):

$$\Gamma(x) = \frac{\Gamma(z)}{(x)_n} \sim \frac{\sqrt{2\pi}}{(x)_n} \exp\left(\left[z - \frac{1}{2}\right] \ln(z) - z + \sum_{m=1}^{M} \frac{B_{2m}}{2m\,[2m-1]\,z^{2m-1}}\right), \quad z := x + n. \qquad \text{(B.1.8)}$$

The expression on the right-hand side approximates the value of EULER's gamma function for sufficiently large arguments z. The expression $(x)_n := x\,[x+1]\,\ldots\,[x+n-1]$ denotes POCHHAMMER's symbol (ABRAMOWITZ AND STEGUN [1965, equ. (6.1.22)]) and the symbol B_m the BERNOULLI number (ABRAMOWITZ AND STEGUN [1965, Table 23.2]). In order to get a value of the gamma function with 19-digit precision, the integer n is chosen such that the argument z becomes greater than fifteen and the series involving BERNOULLI numbers, the BINET function, is truncated after $M = 14$ terms.

B.2 The Interpolation and Extrapolation of the Price Function

One of the two interpolation methods considered in this paper is the *polynomial interpolation*. The constants d_m of the equation (5.1) are obtained from the orthogonality conditions:

$$\begin{aligned} &d_m = a_m \int_{-\infty}^{+\infty} \omega(x)\, F(x)\, \mathcal{P}_m(x)\, \mathrm{d}x, \quad F(x) := K_j\left(r_j, j\cdot\tau + \tau_n\right)^+ \mathrm{e}^{-b\,r_j}, \quad m = 0, 1, \ldots, N_p, \\ &a_m := \begin{cases} \left[2^m\, m!\, \sqrt{\pi}\right]^{-1} & \text{if } \mathcal{P}_m(x) = H_m(x), \\ m!/\Gamma(\alpha+m+1) & \text{if } \mathcal{P}_m(x) = L_m^{(\alpha)}(x), \end{cases} \quad \omega(x) := \begin{cases} \mathrm{e}^{-x^2} & \text{if } \mathcal{P}_m(x) = H_m(x), \\ x^\alpha\, \mathrm{e}^{-x} & \text{if } \mathcal{P}_m(x) = L_m^{(\alpha)}(x). \end{cases} \end{aligned} \qquad \text{(B.2.1)}$$

The transformation of variable $r_j := r(x)$, which was defined in the equation (5.1) and which is used in the above equation, will be determined later. The orthogonal polynomials are computed by means of their recurrence relations (ABRAMOWITZ AND STEGUN [1965, Table (22.7)]). Again, the terms a_m may be computed recursively:

$$a_m = \begin{cases} \dfrac{a_{m-1}}{2m}, & a_0 = \left[\sqrt{\pi}\right]^{-1}, & \text{if } \mathcal{P}_m(x) = H_m(x), \\ \dfrac{m\, a_{m-1}}{\alpha + m}, & a_0 = \left[\Gamma(\alpha+1)\right]^{-1}, & \text{if } \mathcal{P}_m(x) = L_m^{(\alpha)}(x), \end{cases} \qquad m = 1, \ldots, N_p. \qquad \text{(B.2.2)}$$

The equation (B.2.1) is integrated with the help of the GAUSSian quadrature rule:

$$\int_{-\infty}^{+\infty} \omega(x)\, F(x)\, \mathcal{P}_m(x)\, \mathrm{d}x \approx \sum_{n=1}^{\nu} w_n\, \omega\left(x_n^{(\nu)}\right) F\left(x_n^{(\nu)}\right) \mathcal{P}_m\left(x_n^{(\nu)}\right), \quad m = 0, 1, \ldots, N_p\,. \qquad \text{(B.2.3)}$$

Observe that $x_n^{(\nu)}$ denotes the nth root of the νth-degree orthogonal polynomial under consideration, where the roots (which are all real and single, SZEGö [1939]) are in increasing order $x_1^{(\nu)} < x_2^{(\nu)} < \ldots < x_\nu^{(\nu)}$, $\omega(\cdot)$ denotes the weight function associated with a particular orthogonal polynomial, and w_n denote the GAUSSian weights which correspond to the roots of the orthogonal polynomial in question.

The GAUSSian quadrature algorithm of PRESS ET AL. [1989, section 4.5] to compute the GAUSSian weights is *not* reliable because the accuracy of their algorithm deteriorates very rapidly with higher polynomial degrees. To bear out our assertion, here is an example for ν = 20, 30 and 40: first, the GAUSSian weight w_{20} for the HERMITE polynomial of degree $\nu = 20$ is accurate up to only 5 decimal digits. Secondly, the weight w_{29} for the HERMITE polynomial of degree $\nu = 30$ turns out to be negative (!) and the weight w_{30} "explodes". Thirdly, with degree ν = 40 the situation becomes even worse, because the weights w_{35} to w_{40} behave like an undampened oscillation with alternating sign. Also, the combined NEWTON-RAPHSON-bisection method for real roots or the LAGUERRE method for complex roots, which are the proposed algorithms of PRESS ET AL. [1989, sections 9.4 and 9.5] to compute the roots of the orthogonal polynomials, are *not* reliable for large polynomial degrees as well. In fact, the roots of the HERMITE polynomial have been computed for the above example with these two algorithms of PRESS ET AL. [1989]. The bracketing values of the roots, which are required for the former method, are provided by SZEGö [1939].

A computationally efficient and numerically reliable method to compute both the roots and the GAUSSian weights has been proposed by GOLUB AND WELSH [1969]. In fact, we do not know a better algorithm. The basic idea is to write the three-term reccurrence relationship of an orthogonal polynomial of νth degree as an equation system which consists of ν equations. The unknowns are the values of the orthogonal polynomials of degree 0 to $[\nu - 1]$, evaluated at arguments x. If the polynomials of lower degrees are evaluated at the roots of the orthogonal polynomial of degree ν, then this equation system transforms into an eigenvalue problem for which the roots of the orthogonal polynomial of degree ν are the eigenvalues of a real, symmetric, and tridiagonal matrix **J**:

$$\mathbf{J} = \begin{bmatrix} \delta_1 & \beta_1 & & & \\ \beta_1 & \delta_2 & \beta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \beta_{\nu-2} & \delta_{\nu-1} & \beta_{\nu-1} \\ & & & \beta_{\nu-1} & \delta_\nu \end{bmatrix}, \quad \text{with} \begin{cases} \left.\begin{aligned} \delta_j &= 0 \\ \beta_j &= \sqrt{j/2} \end{aligned}\right\} & \text{if } \mathcal{P}_m(x) = H_m(x), \\ \left.\begin{aligned} \delta_j &= 2j + \alpha - 1 \\ \beta_j &= \sqrt{j\,[j + \alpha]} \end{aligned}\right\} & \text{if } \mathcal{P}_m(x) = L_m^{(\alpha)}(x). \end{cases} \tag{B.2.4}$$

GOLUB AND WELSH have shown that the GAUSSian weights are equal to the square of the first element of the normalized eigenvectors, multiplied by the integral of the polynomial weight function, and divided by the polynomial weight function (because we use the transformed GAUSSian weights):

$$w_j = \frac{q_{1,j}^2\,\mu_0}{\omega\left(x_j^{(\nu)}\right)}, \quad \text{with} \quad \mu_0 := \int_{-\infty}^{+\infty} \omega(x)\,\mathrm{d}x = \begin{cases} \sqrt{\pi} & \text{if } \mathcal{P}_m(x) = H_m(x), \\ \Gamma(\alpha+1) & \text{if } \mathcal{P}_m(x) = L_m^{(\alpha)}(x), \end{cases} \tag{B.2.5}$$

where the weight function $\omega(\cdot)$ has been introduced in the equation (B.2.1) and $q_{1,j}$ denotes the first element of the eigenvector which is associated with the jth eigenvalue. The eigenvectors have unit length, that is, $\mathbf{q}_j' \mathbf{q}_j = 1$ for $j = 1, 2, \ldots, \nu$. Both the eigenvalues and eigenvectors of the real, symmetric and tridiagonal matrix **J** have been computed with the QL algorithm with implicit shifts of PRESS ET AL. [1989, section 11.3]. As a control for accuracy, we reproduced exactly those results for the GAUSS-LAGUERRE quadrature rule which are given in CONCUS ET AL. [1963] and in ABRAMOWITZ AND STEGUN [1965, Table (25.9)], as well as those results for the GAUSS-HERMITE quadrature rule which are given in ABRAMOWITZ AND STEGUN [1965, Table (25.10)]. GOLUB AND WELSH themselves found a minor difference for the last significant digits which is due to the fact that they applied the QR rather than the QL algorithm to the tridiagonal matrix **J**.

How many polynomial terms should be considered? Although N_p could be twice the polynomial degree ν in order to yield an exact GAUSSian quadrature, our own computations indicate that the accuracy of the interpolated price function, $K_j(\cdot, \cdot)^+$, turns out to be best for the choice $N_p = \nu - 1$.

Since we need the callable bond prices at the transformed roots $\{r_j^{(k)} := r(x_k^{(\nu)})$, for $k = 1, \ldots, \nu\}$ in order to perform the GAUSSian quadrature of the equation (B.2.3), it seems natural to use the same roots for the numerical quadrature of GREEN's function of the equation (5.5) on each of the call dates as well, that is, $\{r_j^{(k)} := r^{(k)}$, for $j = 2, \ldots, n$, and $k = 1, \ldots, \nu\}$. Hence, this defines the transformation of the equation (5.1): given the choice of the polynomial degree ν, we map linearly all the polynomial roots onto the region of interpolation such that the first root corresponds to the lower interpolation boundary and the last root corresponds to the upper interpolation boundary, that is,

$$\vartheta_1 = \frac{r_l\,x_\nu^{(\nu)} - r_u\,x_1^{(\nu)}}{x_\nu^{(\nu)} - x_1^{(\nu)}}, \quad \vartheta_2 = \frac{r_u - r_l}{x_\nu^{(\nu)} - x_1^{(\nu)}}. \tag{B.2.6}$$

Observe that the series solution (2.4) has also a linear transformation of variable. It may be perceivable to apply the same transformation to the cubic spline too. Our own computations, however, indicate that such a transformation does not improve considerably the interpolation accuracy of the callable bond price function. Hence, we do *not* use this transformation of variable for the exponential cubic spline.

The second interpolation method, which is considered in this paper, is the *cubic spline* for the logarithm of the callable bond prices $y(\cdot) := \ln(K_j(\cdot, \cdot)^+)$. Again, we did *not* use the cubic spline algorithm of PRESS ET AL. [1989, section 3.3] because it leaves open the question of how to find the two missing equations of the cubic spline approach in terms of first derivatives, $y'(\cdot)$. Our choice to complete the equation system is to follow DE BOOR [1978] who equates the third derivatives of two neighbouring third-degree polynomials $\mathfrak{P}_3$ both at the second interpolat-

ing point and at the last but one interpolating point. The equation system of the cubic spline may then be written as follows:

$$\begin{bmatrix} b_1 & a_2/2 & & & \\ b_1 & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & b_{\nu-2} & a_{\nu-1} & b_{\nu-1} \\ & & & a_{\nu-1}/2 & b_{\nu-1} \end{bmatrix} \begin{bmatrix} y'_1 \\ y'_2 \\ \vdots \\ y'_{\nu-1} \\ y'_\nu \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{\nu-1} \\ d_\nu \end{bmatrix}, \qquad \text{(B.2.7a)}$$

where $y'_j := y'(r^{(j)})$. Here, $\{r_k^{(j)} := r^{(j)}$, for $k = 2, \ldots, n$, and $j = 1, \ldots, \nu\}$ denote the ν interpolating points or *knots*, respectively. Again, the knots are the same for all the notional call dates $k = 2, \ldots, n$. The different matrix and vector elements are defined as follows, given the abbreviation $y_j := y(r^{(j)})$:

$$\begin{aligned} &a_j = \frac{2}{h_{j-1}} + \frac{2}{h_j}, \quad b_j = \frac{1}{h_j}, \quad d_j = 3\left[\frac{y_{j+1}-y_j}{h_j^2} + \frac{y_j - y_{j-1}}{h_{j-1}^2}\right], \quad h_j = r_k^{(j+1)} - r_k^{(j)}, \\ &d_1 = 2\frac{y_2 - y_1}{h_1^2} + \frac{h_1}{h_1 + h_2}\frac{d_2}{3}, \quad d_\nu = 2\frac{y_\nu - y_{\nu-1}}{h_{\nu-1}^2} + \frac{h_{\nu-1}}{h_{\nu-1} + h_{\nu-2}}\frac{d_{\nu-1}}{3}. \end{aligned} \qquad \text{(B.2.7b)}$$

The tridiagonal matrix of the equations (B.2.7) is solved by means of the tridiagonal matrix algorithm of PRESS ET AL. [1989, section 2.6]. Given the partial derivatives from the equation (B.2.7), the interpolated callable bond price can be written as:

$$y\big(t(r_j)\big) = a_0 + \big[b_0 + \big[c_0 + d_0 t\big]\,[t-1]\big]\,t, \quad r^{(i)} \leqq r_j < r^{(i+1)}, \quad t := \frac{r_j - r^{(i)}}{r^{(i+1)} - r^{(i)}}, \qquad \text{(B.2.8a)}$$

with the meaning that t is the local variable, that $\{r^{(i)}$ for $i = 1, \ldots, \nu\}$ are the given knots, and that the constants are determined from the following difference scheme:

$$\begin{array}{l} \vdots \\ a_0 = y_i \\ a_1 = y_{i+1} \\ \vdots \end{array} \Big\rangle \begin{array}{l} b_{-1} = h_i\, y'_i \\ b_0 = a_1 - a_0 \\ b_1 = h_i\, y'_{i+1} \end{array} \Big\rangle \begin{array}{l} c_0 = b_0 - b_{-1} \\ c_1 = b_1 - b_0 \end{array} \Big\rangle d_0 = c_1 - c_0. \qquad \text{(B.2.8b)}$$

How should the knots $\{r^{(i)}$ for $i = 1, \ldots, \nu\}$ be chosen? Following DE BOOR [1978], a good choice for the knots are the roots of the CHEBYSHEV polynomial. However, if the number of knots, ν, is large, also equidistant knots are feasible. We have tried both types of knots and could not find a significant difference in accuracy for a large number of knots. It should be noted that, in view of the extrapolation approach considered in this paper, it may be conceivable to set the second derivative evaluated at the last knot, y''_ν, equal to zero.

Some numerical experiments show that the interpolation by means of series of orthogonal polynomials is extremely accurate, that is, accurate up to 16 – 19 decimal digits, for a moderate number of polynomials, for example $N_p = \nu - 1 \approx 20$, if the interpolation interval is small, say, [– 0.5, 0.5] in VASICEK's model or [0.0, 1.0] in the CIR model. For comparison, the expo-

nential cubic spline with the same number of knots would yield a much lower accuracy of 5 – 6 decimal digits only.[7] However, if the interval of interpolation grows, the accuracy of the polynomial interpolation deteriorates very rapidly for large interest rates because the orthogonal polynomials under consideration grow very large too. Our own computations indicate that the polynomial interpolation fails if the interpolation interval has a length of more than two, as in the example given in section 6. Considering the fact that most of the variation of the callable bond price function occurs in the two intervals just mentioned, it seems reasonable to combine the polynomial interpolation with the cubic spline interpolation in real applications. This *hybrid interpolation method* should reduce the number of interpolating points considerably. In a first subinterval, $[r_l, r_m]$, where most of the price variation occurs, the interpolation could be done with relatively few orthogonal polynomials (e. g., $\nu = 20 - 30$), whereas in the remaining subinterval, $(r_m, r_u]$, the exponential cubic spline could also be applied with relatively few knots (e. g., $\nu = 70 - 80$).

The *extrapolation* of the callable bond price function assumes that the logarithm of this function is approximately linear for large interest rates. This defines the upper interpolation boundary, r_u. In the example to follow, an inspection of the graph of this function reveals that $r_u = 6$ might be a reasonable choice for both interest rate models. The first derivative of the logarithm of the callable bond price at the last knot is obtained from the cubic spline of the equation (B.2.7). If the polynomial interpolation is used exclusively, an extra cubic spline may be applied to the interpolating callable bond prices with the polynomial roots as spline knots.

B.3 The Determination of the "Break-even" Interest Rate

The "break-even" interest rate on each notional call date is the left-most root of the equation (5.2). A modified algorithm of PRESS ET AL. [1989, section 9.4] is used, which is a combination of bisection and NEWTON-RAPHSON. The hybrid algorithm takes a bisection step whenever NEWTON-RAPHSON would take the solution out of bounds, or whenever NEWTON-RAPHSON is not reducing the size of the brackets rapidly enough.

The NEWTON-bisection algorithm requires as input both brackets for the real root and the first derivative of the function under consideration. The bracketing of the root is easily done because the interpolating values of both the callable bond price function and the time value of the call price, including the current coupon payment, is altogether the necessary information.

If the polynomial interpolation applies, the partial derivative of the function $\mathcal{F}_j(r_j)$ is given by

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7 This accuracy may be improved with a monote cubic spline interpolation as proposed by FRITSCH AND CARLSON [1980]. We are indebted to Freddy Delbaen for drawing our attention to this method.

$$\frac{\partial \mathcal{F}_j(r_j)}{\partial r_j} = [X_j + \eta] \frac{\partial P(r_j, \tau_n)}{\partial r_j} - \frac{\partial K_j(r_j, j\cdot\tau+\tau_n)^+}{\partial r_j}, \quad \frac{\partial P(r_j, \tau_n)}{\partial r_j} = g(\tau_n) P(r_j, \tau_n),$$
$$\frac{\partial K_j(r_j, j\cdot\tau+\tau_n)^+}{\partial r_j} = b\, K_j(r_j, j\cdot\tau+\tau_n)^+ + \frac{e^{br_j}}{\vartheta_2} \cdot \begin{cases} 2 \sum_{i=2}^{N_p} d_i [i-1] H_{i-2}(x), \\ \frac{1}{x} \sum_{i=1}^{N_p} d_i \left[i L_i^{(\alpha)}(x) - [\alpha + i] L_{i-1}^{(\alpha)}(x) \right]. \end{cases} \tag{B.3.1}$$

The function $g(\cdot)$ is given in the equation (2.5), b is given in equation (2.4), and x is defined in the equation (5.1). The first and second half-lines correspond to the VASICEK or the CIR model, respectively. If the exponential cubic spline applies, then

$$\frac{\partial K_j(r_j, j\cdot\tau+\tau_n)^+}{\partial r_j} = \frac{dy(t(r_j))}{dr_j} \exp(y(t(r_j))), \quad \frac{dy(t(r_j))}{dr_j} = \frac{1}{h_i} \{ b_0 + c_0 [2t-1] + d_0 [3t^2 - 2t] \}, \tag{B.3.2}$$

should be substituted into the equation (B.3.1). All the symbols have been introduced in the equations (B.2.7) and (B.2.8) above.

B.4. The Numerical Quadrature Involving Green's Function

The key operation is the computation of the integral in the equation (5.5) by means of numerical quadrature (50 – 70% of the total computing time is spent for numerical quadrature in the example considered in this paper). We suggest to apply the trapezoidal rule as described by WALDVOGEL [1988]. In the case of an analytic integrand, this operation is computationally much more efficient than the ROMBERG integration algorithm which is highly recommended by PRESS ET AL. [1989, section 4.3]. The use of the trapezoidal rule for analytic integrands has been suggested earlier by several authors, see e. g. SCHWARTZ [1969] or IRI ET AL. [1970], but the method entered the textbooks only recently, see e. g. SCHWARZ [1986]. For instance, WALDVOGEL's algorithm achieves machine precision for the probability integrals over the whole range of definition for the normal density function and the non-central chi-square density function, whereas the ROMBERG algorithm fails. However, one cannot expect to accomplish machine precision for the equation (5.5) due to the interpolation errors. Similarly, the loss of accuracy might be considerable for, say, a ten-dimensional integral of an analytic function.

The numerical quadrature algorithm considers three standard types of ranges of integration, namely $\mathcal{I}_1 = (-\infty, +\infty)$, $\mathcal{I}_2 = (0, +\infty)$ and $\mathcal{I}_3 = (-1, +1)$. Of course, any integral can be transformed into an integral with one of these three ranges by means of a linear transformation. Suppose that the integral $\mathcal{I} := \int_{t \in \mathcal{I}} f(t)\, dt$ exists and that $f(t) \to 0$ as $t \to \pm\infty$ if the limits of integration are infinite. Integrable singularities can be treated at one or both limits of integration in the case of the single-infinite range $\mathcal{I} = \mathcal{I}_2$ or the finite range $\mathcal{I} = \mathcal{I}_3$. Each of the three types of integrals is transformed into an improper integral with both limits unbounded, that is

$$\Im := \int_{t_{n+1} \in \mathcal{I}} f(t_{n+1})\, \mathrm{d}t_{n+1} = \int_{t_0 = -\infty}^{t_0 = +\infty} g(t_0)\, \mathrm{d}t_0, \quad (\mathcal{I} = \mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3). \tag{B.4.1}$$

This is exactly the reverse strategy of what is usually done, see e. g. PRESS ET AL. [1989]. The transformation of variables in the above equation consists of n optional transformations with the sinh-function and one compulsory transformation with a particular function which is suited for each of the three ranges of integration:

$$t_j = \sinh(t_{j-1}), \quad (j = 1, \ldots, n), \quad t_{n+1} = \begin{cases} t_n & \text{if } \mathcal{I} = \mathcal{I}_1, \\ \exp(t_n) & \text{if } \mathcal{I} = \mathcal{I}_2, \\ \tanh(t_n) & \text{if } \mathcal{I} = \mathcal{I}_3. \end{cases} \tag{B.4.2}$$

In the case of the first type of integral with range $\mathcal{I} = \mathcal{I}_1$, there are, of course, just the n optional trasformations with the sinh-function because the original integral is already improper. Given the transformations of the equation (B.4.2), the function $g(\cdot)$ of the equation (B.4.1) can be written as follows.

$$g(t_0) = \begin{cases} f(\sinh(\ldots\sinh(t_0)))\displaystyle\prod_{j=0}^{n-1} \cosh(t_j) & \text{if } \mathcal{I} = \mathcal{I}_1, \\ f(\exp(\sinh(\ldots\sinh(t_0))))\, \mathrm{e}^{t_n} \displaystyle\prod_{j=0}^{n-1} \cosh(t_j) & \text{if } \mathcal{I} = \mathcal{I}_2, \\ f(\tanh(\sinh(\ldots\sinh(t_0))))\, \dfrac{1}{\cosh^2(t_n)} \displaystyle\prod_{j=0}^{n-1} \cosh(t_j) & \text{if } \mathcal{I} = \mathcal{I}_3. \end{cases} \tag{B.4.3}$$

The integral (B.4.1) is now evaluated by means of the simplest quadrature scheme, that is, the trapezoidal rule. Given a choice of the initial step size h, the function values, which are evaluated at equally spaced abscissas, are summed in both the positive and negative direction of integration. Given a center of integration c as the starting point, the summation is stopped as soon as the sum does not change any more:

$$T(h) = h\, S(h), \quad \text{with } S(h) := \sum_{\substack{t_0 = c, \\ t_i = t_{i-1} + h}}^{|\, g(\cdot)\,| < \varepsilon} g(t_i) + \sum_{\substack{t_0 = c - h, \\ t_i = t_{i-1} - h}}^{|\, g(\cdot)\,| < \varepsilon} g(t_i)\,, \tag{B.4.4}$$

where $\varepsilon = 1.0 \cdot 10^{-19}$ is the machine tolerance. Next, the step size is cut in half and the new center of integration is equal to the old center shifted by the new step size. Then the summation is repeated in both directions as in the equation (B.4.4). Next, the step size is cut in half again, the summation is continued, and so forth. The repeated reduction of the step size is stopped when $| T(h) - T(h/2) | < \sqrt{\varepsilon}$. Then, $T(h/2)$ has accuracy ε. The convergence of the trapezoidal value to the integral is given by $T(h) - \Im = \mathbb{O}(\mathrm{e}^{-\gamma/h})$ with γ a positive constant, given an analytic function $g(\cdot)$.

How many transformations with the sinh-function should be used? The number of sinh-transformations, n, which depends on the type of both the integral and the integrand, may be chosen such that the numerical quadrature uses as few function calls as possible. As an exam-

ple, the number of function evaluations is smallest for the normal probability integral if there is no sinh-transformation used in the case of interval $\mathcal{I}_1$, or if there are two sinh-transformations used in the case of interval $\mathcal{I}_2$, or if there is one sinh-transformation used in the case of interval $\mathcal{I}_3$. In the case of the non-central chi-square probability integral, one or two sinh-transformations are generally efficient. Numerical experiments show that the trapezoidal rule is generally quite efficient if the decay of the integrand $g(t_0)$ as $t_0 \to \pm\infty$ is doubly exponential (see TAKAHASHI AND MORI [1974]).

The second integral (5.5) is transformed into an integral with range $\mathcal{I}_2$,

$$\begin{aligned} \mathfrak{I}_2 &:= \int_{r_j = r_j^*}^{r_j = +\infty} G(r_{j+1},\, \tau,\, r_j)\, K_j\big(r_j,\, j\,\tau + \tau_n\big)^+ \,\mathrm{d}r_j \\ &= \int_{t_{n+1}=0}^{t_{n+1}=+\infty} G(r_{j+1},\, \tau,\, t_{n+1} + r_j^*)\, K_j\big(t_{n+1} + r_j^*,\, j\,\tau + \tau_n\big)^+ \,\mathrm{d}t_{n+1}\,, \end{aligned} \tag{B.4.5}$$

by means of the shift $r_j = t_{n+1} + r_j^*$.

Appendix C: The Data for the Example

Table C.1: The Estimated Model Parameters. [1]

Parameter [2]	VASICEK	CIR
κ	0.44178462	0.54958046
σ	0.13264223	0.38757496
q or $\zeta - \kappa$	0.21166329	– 0.40663675
θ	0.0348468515	
r_0	0.0752280589	

[1] Trading day December 23, 1991. The theoretical term structure which is implied by the two models under consideration was perfectly fitted to the four observed yields $\hat{R}_i = \{r_0, 0.0775395775, 0.0664769812, 0.0629805885\}$ with $\tau_i = \{0, 1, 7.175, 10.2555555\}$ years to maturity by means of a modified NEWTON-RAPHSON method.

[2] The underlying interest rate processes are the ORNSTEIN-UHLENBECK process in the case of VASICEK's model, $dr = \kappa(\theta - r)\,dt + \sigma\,dz$, and the square-root process in the case of the CIR model, $dr = \kappa(\theta - r)\,dt + \sigma\sqrt{(r)}\,dz$, where r denotes the instantaneous interest rate, t a point in time, κ the speed of adjustment, θ the long-run "equilibrium" value of the instantaneous interest rate, σ the instantaneous volatility, and z the GAUSS-WIENER process. The long-run "equilibrium" value of the instantaneous interest rate was estimated as the mean value over the last twenty years; its discrete-time equivalent is 3.546% p. a. The parameters q and $\zeta - \kappa$ denote the market price of interest rate risk in VASICEK's model and the risk premium of the single factor in the CIR model, respectively. Finally, r_0 denotes the current value of the instantaneous interest rate, approximated by the tomorrow-next rate, which was observed on the trading day in question; its discrete-time equivalent is 7.813% p. a.

Table C.2: The Characteristics of the Numerical Example.

Name of security [1]	4 1/4% Swiss Confederation 1987–2012	Year [2]	Call Price
Maximum life of callable bond	20.172 years	1. – 5.	1.000
Notice period	0.1666 years (2 months)	6.	1.005
Interpolation method	Exponential spline with equidistant knots	7.	1.010
Center of integration	0.0	8.	1.015
Initial step size of integration	1.0	9.	1.020
Number of sinh-transformations	1	10.	1.025

[1] Security number 15 718. The bond bears an annual coupon of 4 1/4%. Trading day December 23, 1991.

[2] Ordinal number of the call date when counted backwards in time. The ten call dates are equal to the ten coupon dates prior to the last possible redemption day. The computations are based on the assumption that the debtor makes his call decision on the notice days (notional call dates) which lie two months ahead of the effective call dates.

Table C.3: The Price of the Callable Bond for Various Numbers of Knots. [1]

Number of Knots	VASICEK	CIR
20	7.69009349390688 5004e-1	7.97791748204131 1071e-1
41	7.44975087722577 6772e-1	7.98072875297476 8909e-1
81	7.44965790483799 0726e-1	7.98146760278869 3622e-1
162	7.50069425664897 8132e-1	7.98155146022675 2178e-1
325	7.51150354812794 0533e-1	7.98155680567579 2171e-1
650	7.52082229856677 1664e-1	7.98155703081802 7044e-1
r_u	6.0	6.5
r_l	– 0.5	0.0

[1] Price of the callable bond for the current value of the instantaneous interest rate, r_0. r_u denotes the upper interpolation boundary and r_l the lower interpolation boundary.

Table C.4: The Break-even Interest Rates. [1]

Year [2]	VASICEK	CIR
1.	0.0270644976	0.0338871564
2.	- 0.0101266321	0.0179273733
3.	- 0.0333187039	0.0097892562
4.	- 0.0557203925	0.0048817260
5.	- 0.0743127490	0.0015784739
6.	- 0.0935658091	0
7.	- 0.1088988626	0
8.	- 0.1214749074	0
9.	- 0.1314698228	0
10.	- 0.1391392481	0

[1] Break-even interest rates obtained for 650 knots.

[2] Ordinal number of notional call date (i. e., notice date) counted backwards in time.

Table C.5: The Price of the Callable Bond, the Underlying Straight Bond and the Embedded Call Option. [1]

	VASICEK's model			CIR model		
Interest [2]	Straight Bond	Callable Bond	Call [4]	Straight Bond	Callable Bond	Call [4]
0.000	9.462e-1	8.728e-1	7.346	9.796e-1	9.631e-1	1.644
0.010	9.274e-1	8.556e-1	7.178	9.552e-1	9.392e-1	1.599
0.020	9.089e-1	8.388e-1	7.014	9.315e-1	9.159e-1	1.554
0.030	8.908e-1	8.223e-1	6.853	9.084e-1	8.933e-1	1.511
0.040	8.731e-1	8.062e-1	6.696	8.859e-1	8.712e-1	1.469
0.050	8.558e-1	7.904e-1	6.543	8.641e-1	8.498e-1	1.428
0.060	8.389e-1	7.749e-1	6.393	8.428e-1	8.289e-1	1.388
0.070	8.223e-1	7.598e-1	6.246	8.220e-1	8.085e-1	1.350
0.075 [3]	8.137e-1	7.520e-1	6.171	8.114e-1	7.981e-1	1.330
0.080	8.060e-1	7.450e-1	6.103	8.018e-1	7.887e-1	1.312
0.090	7.901e-1	7.305e-1	5.963	7.822e-1	7.694e-1	1.276
0.100	7.746e-1	7.163e-1	5.827	7.631e-1	7.507e-1	1.240
0.110	7.594e-1	7.024e-1	5.693	7.444e-1	7.324e-1	1.206
0.120	7.445e-1	6.888e-1	5.563	7.263e-1	7.146e-1	1.172
0.130	7.299e-1	6.755e-1	5.436	7.086e-1	6.973e-1	1.140
0.140	7.156e-1	6.625e-1	5.311	6.915e-1	6.804e-1	1.108
0.150	7.016e-1	6.497e-1	5.189	6.747e-1	6.640e-1	1.077
0.160	6.879e-1	6.372e-1	5.070	6.584e-1	6.480e-1	1.047
0.170	6.745e-1	6.250e-1	4.954	6.426e-1	6.324e-1	1.018
0.180	6.614e-1	6.130e-1	4.841	6.271e-1	6.172e-1	0.990
0.190	6.486e-1	6.013e-1	4.730	6.121e-1	6.025e-1	0.962
0.200	6.360e-1	5.898e-1	4.622	5.974e-1	5.881e-1	0.936
0.300	5.240e-1	4.874e-1	3.666	4.705e-1	4.635e-1	0.706
0.400	4.334e-1	4.044e-1	2.907	3.731e-1	3.678e-1	0.532
0.500	3.601e-1	3.370e-1	2.306	2.982e-1	2.942e-1	0.402
0.600	3.006e-1	2.823e-1	1.829	2.405e-1	2.375e-1	0.303
0.700	2.523e-1	2.378e-1	1.450	1.959e-1	1.937e-1	0.228
0.800	2.130e-1	2.015e-1	1.150	1.614e-1	1.597e-1	0.172
0.900	1.810e-1	1.719e-1	0.912	1.345e-1	1.332e-1	0.130
1.000	1.549e-1	1.476e-1	0.724	1.136e-1	1.126e-1	0.098
1.500	7.986e-2	7.759e-2	0.227	5.923e-2	5.899e-2	0.024
2.000	5.028e-2	4.957e-2	0.071	4.056e-2	4.050e-2	0.006
2.500	3.718e-2	3.696e-2	0.022	3.244e-2	3.243e-2	0.001
3.000	3.039e-2	3.032e-2	0.007	2.787e-2	2.786e-2	0.000
3.500	2.623e-2	2.620e-2	0.002	2.471e-2	2.471e-2	0.000
4.000	2.327e-2	2.327e-2	0.001	2.225e-2	2.224e-2	0.000
4.500	2.096e-2	2.096e-2	0.000	2.019e-2	2.019e-2	0.000
5.000	1.904e-2	1.904e-2	0.000	1.841e-2	1.841e-2	0.000
5.500	1.737e-2	1.737e-2	0.000	1.683e-2	1.683e-2	0.000
6.000	1.590e-2	1.590e-2	0.000	1.542e-2	1.542e-2	0.000
6.500	1.458e-2	1.458e-2	0.000	1.414e-2	1.414e-2	0.000
7.000	1.339e-2	1.339e-2	0.000	1.297e-2	1.297e-2	0.000
7.500	1.230e-2	1.230e-2	0.000	1.190e-2	1.190e-2	0.000
8.000	1.131e-2	1.131e-2	0.000	1.093e-2	1.093e-2	0.000

[1] The price of the callable bond is obtained for 650 knots. The figures given for the straight bond and the callable bond are *not* rounded to four significant digits but merely cut off from the 19-digit results.

[2] Instantaneous interest rate.

[3] Current value of the instantaneous interest rate on the trading day in question, denoted as r_0.

[4] Price of the embedded call option per 100 face value. These figures are rounded.