The Term Structure of Expected Inflation Rates

by

HANS-JÜRG BÜTTLER
Research Group, Swiss National Bank [#], Switzerland
and
Department of Economics, University of Zurich, Switzerland

ABSTRACT

The goal of this paper is to determine empirically the term structure of expected inflation rates of investors of riskless non-indexed bonds. We proceed in three steps. In the first step, we use a non-linear optimization to determine the instantaneous forward interest rates from observed prices of coupon-bearing government bonds. The objective of the optimization are instantaneous forward rates as smooth as possible; no explicit model is assumed. The term structure of nominal spot interest rates is deduced from the optimized instantaneous forward rates by numerical integration. In the second step, the nominal discount bond price model proposed by Cox, Ingersoll and Ross is fitted, by means of another optimization, to the term structure of nominal spot rates, which, in turn, determines the term structure of real spot interest rates. In the third step, the expected instantaneous forward inflation rates are calculated from the model parameters estimated in the previous step. As for interest rates, the expected instantaneous forward inflation rates determine the whole term structure of expected inflation rates. Observe that the difference between the nominal and real spot interest rate is not equal to the expected spot inflation rate, rather it is equal to the difference between the expected spot inflation rate and the so-called interest premium. The expected inflation rates coincide surprisingly well with the observed inflation rates. The paper investigates the Swiss term structure.

Appeared in *Economic and Financial Modelling*, Vol. 7, No. 1, Spring 2000, 3 – 45.

This paper has been presented at the 8th Annual Meeting of the European Financial Management Association which was held in Paris (France) in June 1999, at the International Symposium on Economic Modelling which was held in Maribor (Slowenia) in June/July 1999, and at the Econometrics Workshop at the University of York (UK) in June 1999.

[#] Mailing address: Swiss National Bank, P. O. Box, 8022 Zurich, Switzerland.
Phone (direct dialling): +41–1–631 34 17, Fax: +41–1–631 39 01.

0 Introduction

The goal of this paper is to determine empirically the term structure of expected inflation rates of investors of riskless non-indexed bonds. Earlier attempts as, for instance, the one in Frankel [1982], which relies on a macroeconomic framework, assumes that the expected inflation rates are equal to the difference between nominal and real spot interest rates. This is Irving Fisher's hypothesis that the nominal interest rate moves one for one with the expected inflation rate (Fisher, 1930). Probably the first author, who has shown that the Fisher hypothesis does not hold true in an uncertain world, was his namesake Stanley Fischer [1975] in his pathbreaking paper on indexed bonds.[1] He shows that the difference between the nominal and real interest rate is equal to the expected inflation rate minus a term which I call the "interest premium", which may have either sign.[2] Many other authors including, for instance, Bakshi and Chen [1996], Benninga and Protopapadakis [1985], Breeden [1986], Cox, Ingersoll and Ross[3] [1981, 1985a & b], Evans and Wachtel [1992], Fama and Farber [1979], or Lucas [1982] have confirmed this result within quite different frameworks. To my knowledge, the empirical studies, however, have neglected the interest premium so far.[4] There are two exceptions to this observation. One exception is the recent paper written by Evans [1998], who is able to estimate the time-varying interest premium in his investigation of index-linked bonds. However, he fails to estimate both the term structure of expected inflation rates and the interest premia endogeneously within his framework. Instead, he uses an exogenous variable for the expected inflation rate, namely the Barclay's survey measure of expected inflation.[5] The other exception is the recent paper written by Remolona, Wickens and Gong [1998]. Using time series data of both nominal and real discount bond prices, they are able to estimate simultaneously the expected inflation rates and the interest premia in the course of time. They find that the expected inflation rate obtained from their bond price model is an unbiased estimator of future inflation for the period 1982 - 1997. Our approach is different in that we estimate the term structures of both expected inflation rates and interest premia entirely from *nominal* bonds by means of the extended CIR model at a given moment in time; *no* time series data are used.

We proceed in three steps. In the first step, we use a non-linear optimization, as proposed by Delbaen and Lorimier [1992, 1995], to determine the instantaneous forward interest

---

1 For a criticism of Fischer's equilibrium condition, see Fama and Farber [1979, p. 643].

2 Fischer calls it just "premium". Other authors call it the "inflation risk premium" or the "purchasing power risk of the nominal bond". We prefer the neutral term "interest premium" over the term "risk premium", because the latter associates in general positive values only.

3 Henceforth, CIR.

4 For a comprehensive list of empirical studies on index-linked bonds, see Evans [1998]. In particular, Brown and Schaefer [1994], although applying the CIR model, do not investigate the term structure of expected inflation rates and interest premia.

5 Evans also uses the ex-post realized inflation rate as a proxy for the expected inflation rate.

rates from observed prices of coupon-bearing government bonds. The objective of the optimization are instantaneous forward rates as smooth as possible. The term structure of nominal spot interest rates is deduced from the optimized instantaneous forward rates by numerical integration. This approach has two advantages. First, it is able to explain *any* term structure of interest rates, because no functional form of the instantaneous forward interest rates is assumed. Second, the numerical integration is more exact than the numerical differentiation. To my knowledge, the methods proposed in the literature are inferior to the one put forward by Delbaen and Lorimier. For instance, we will show by an example that the Bootstrap method is not reliable if the yield curve is sufficiently bent, or if there are no discount bonds available in the sample of bonds under consideration. Other methods such as the regression of prices of coupon-bearing bonds on discount factors as proposed in Carleton and Cooper [1976] or various spline methods as proposed in McCulloch [1971, 1975] or in Vasicek and Fong [1982] have many drawbacks as mentioned in Shea [1984, 1985]. The recent models proposed by Nelson and Siegel [1987] as well as Svensson [1995] assume an exponential function for the instantaneous forward rates. This approach has two disadvantages: it does not obey the fundamental partial differential equation to value a discount bond (Björk and Christensen, 1997) and it assumes rather than extracts the term structure of interest rates from observed data.

In the second step, we fit the model to value a nominal discount bond proposed by Cox, Ingersoll and Ross [1985a & b] to the term structure of nominal spot rates obtained in the first step. We use a non-linear regression subject to the constraint that the CIR model is fixed to as many observed nominal spot interest rates as possible. From the estimated model parameters, the term structure of real spot interest rates is calculated.

In the third step, the expected instantaneous forward inflation rates are calculated from the model parameters estimated in the previous step. As for interest rates, the expected instantaneous forward inflation rates determine the whole term structure of expected inflation rates. In the framework of the CIR model, we estimate the term structure of interest premia. As mentioned before, the difference between the nominal and real spot interest rate is equal to the difference between the expected spot inflation rate and the interest premium. Finally, we observe that the expected inflation rates coincide surprisingly well with the observed inflation rates. The paper investigates the Swiss term structure as well as the European term structure.

1 Preliminaries

In the following, we will use continuously compounded rates, however, the results presented in all charts are annually compounded rates. To clarify the use of various yields, we start with the definition of the term structure. A list of variables is given in the appendix to the paper. In order to distinguish between nominal and real variables or variables associated with the inflation rate, we use two subscripts, if necessary. The first subscript of a variable, $\nu = \{n, r, y\}$, denotes nominal values for $\nu = n$, real values for $\nu = r$, and values associated with the inflation rate for $\nu = y$. The second subscript, $k = \{m, c\}$, denotes the compounding frequency with the understanding that m denotes a compounding m times a year and c denotes the continuous compounding ($m \to \infty$). The spot interest rate, denoted as $R_\nu(t, T)$, is defined as the yield of a pure discount bond with spot price $P_\nu(t, T)$:

$$P_\nu(t, T) = \exp(-R_{\nu,c}(t, T)\,[T - t]), \quad \nu = n, r. \tag{1-1}$$

We assume that the first derivative of the pure discount bond price with respect to time exists and is bounded for any life time of the bond. Solving for the spot interest rate yields the following expression.

$$R_{\nu,c}(t, T) = -\frac{\ln(P_\nu(t, T))}{T - t}, \quad \nu = n, r. \tag{1-2}$$

The term structure of spot interest rates or the yield curve, respectively, is defined by equation (1-2). The instantaneous spot interest rate, denoted as $r_\nu(t)$, is equal to the spot interest rate with a vanishing life time:

$$r_{\nu,c}(t) \equiv R_{\nu,c}(t, t) = \lim_{t \leftarrow T}\left\{-\frac{\ln(P_\nu(t, T))}{T - t}\right\}, \quad \nu = n, r. \tag{1-3}$$

The $(\tau - T)$-year forward interest rate, denoted as $F_\nu(t, T, \tau)$, correspondes with a forward contract on a pure discount bond with the agreement that the forward price, denoted as $\mathcal{P}_\nu(t, T, \tau)$, is fixed at date t and paid at a later date T when the discount bond is delivered. The discount bond matures at a later date τ ($\tau \geqq T \geqq t$).

$$\mathcal{P}_\nu(t, T, \tau) = \exp(-F_{\nu,c}(t, T, \tau)\,[\tau - T]), \quad (t \leqq T \leqq \tau), \quad \nu = n, r. \tag{1-4}$$

In this case, the forward price is equal to the futures price (see Hull, 1997, p. 95). Again, we assume that the first derivative of the forward pure discount bond price with respect to time exists and is bounded for any life time of the bond. Solving for the forward interest rate yields the following expression.

$$F_{\nu,c}(t, T, \tau) = -\frac{\ln(\mathcal{P}_\nu(t, T, \tau))}{\tau - T}, \quad \nu = n, r. \tag{1-5}$$

The price of a pure discount bond fixed at date t with maturity date τ should be equal to the price of a portfolio at date t, which consists of a pure discount bond maturing at date T plus a $(\tau - T)$-year forward pure discount bond (see e. g. Hull, 1997). This leads to the following well-known relationship.

$$\begin{aligned} P_\nu(t,\, \tau) &= P_\nu(t,\, T)\, \mathcal{P}_\nu(t,\, T,\, \tau)\,, \quad (t \leqq T \leqq \tau) \quad \Rightarrow \\ \exp(-R_{\nu,c}(t,\, \tau)\,[\tau - t]) &= \exp(-R_{\nu,c}(t,\, T)\,[T - t])\, \exp(-F_{\nu,c}(t,\, T,\, \tau)\,[\tau - T]) \quad \Rightarrow \\ F_{\nu,c}(t,\, T,\, \tau) &= \frac{R_{\nu,c}(t,\, \tau)\,[\tau - t] - R_{\nu,c}(t,\, T)\,[T - t]}{\tau - T} \\ &= R_{\nu,c}(t,\, \tau) + \frac{R_{\nu,c}(t,\, \tau) - R_{\nu,c}(t,\, T)}{\tau - T}\,[T - t]\,, \quad \nu = n,\, r. \end{aligned} \tag{1-6}$$

It holds true that $F_\nu(t, t, T) = R_\nu(t, T)$. The instantaneous forward interest rate is obtained for a forward contract that expires in the same instant it has been initiated. Using the above equation, we obtain the following relationship.

$$\begin{aligned} f_{\nu,c}(t,\, T) &= F_{\nu,c}(t,\, T,\, T) \\ &= \lim_{\tau \downarrow T} F_{\nu,c}(t,\, T,\, \tau)\,, \quad (\tau \geqq T) \\ &= \lim_{\tau \downarrow T} \left\{ R_{\nu,c}(t,\, \tau) + \frac{R_{\nu,c}(t,\, \tau) - R_{\nu,c}(t,\, T)}{\tau - T}\,[T - t] \right\} \\ &= R_{\nu,c}(t,\, T) + \frac{\partial R_{\nu,c}(t,\, T)}{\partial T}\,[T - t]\,, \quad \nu = n,\, r. \end{aligned} \tag{1-7}$$

It holds true that $f_\nu(t, t) = R_\nu(t, t) = r_\nu(t)$, because we have assumed that the first derivative of the pure discount bond price with respect to time exists and is bounded for any life time of the bond. Integration by parts of the above equation leads to the well-known relationship that the spot interest rate is equal to the integral of the instantaneous forward interest rate divided by the corresponding period of time.

$$\int_{\tau = t}^{\tau = T} f_{\nu,c}(t,\, \tau)\, d\tau = R_{\nu,c}(t,\, T)\,[T - t]\,, \quad \nu = n,\, r. \tag{1-8}$$

Substituting the above equation into equation (1-1), it follows that the spot price of a pure discount bond can be written in terms of the instantaneous forward interest rate.

$$P_\nu(t,\, T) = \exp(-R_{\nu,c}(t,\, T)\,[T - t]) = \exp\left(-\int_{\tau = t}^{\tau = T} f_{\nu,c}(t,\, \tau)\, d\tau \right), \quad \nu = n,\, r. \tag{1-9}$$

Differentiation of the logarithm of the above equation with respect to the maturity date leads to following relationship for the instantaneous forward interest rate.

$$f_{\nu,c}(t,\, T) = -\frac{\partial \ln(P_\nu(t,\, T))}{\partial T} = -\frac{\dfrac{\partial P_\nu(t,\, T)}{\partial T}}{P_\nu(t,\, T)}\,, \quad \nu = n,\, r. \tag{1-10}$$

Later, we will modify some of the equations of this section to hold for the expected inflation rate as well.

2 The Term Structure of Nominal Interest Rates

Given a sample of quoted prices of coupon-bearing bonds at date t, the first task is to extract the term structure of nominal spot interest rates from these observed prices. We apply the methodology proposed by Delbaen and Lorimier [1992]. The goal of this methodology is to minimize the sum of squared differences in instantaneous forward interest rates in order to obtain a forward rate curve as smooth as possible.[6] The optimization is subject to the constraints that the relative errors of the deviations of the theoretical prices of coupon-bearing bonds from the observed prices lie within a given tolerance. If desired, the tolerance may be set equal to zero. The tolerance is justified by the fact that bonds prices are subject to measurement errors, which may be due to rounding errors, illiquid markets, or bid-ask spreads. By equation (1-8), the spot interest rates can be obtained from the instantaneous forward interest rates by integration, which is, from a numerical point of view, much more accurate than the numerical differentiation. The Delbaen-Lorimier methodology has two advantages over the existing procedures. First, it is numerically reliable due to the integration rather than differentiation, and second, it does not assume any functional form of the instantaneous forward rate curve. Hence, the Delbaen-Lorimier methodology is able to explain *any* term structure.

For the purpose of this paper, we modify the Delbaen-Lorimier methodology three-fold. First, we do not apply a multiobjective goal attainment optimization as proposed in Delbaen and Lorimier [1992] or Lorimier [1995], because we feel that both the weight vector and the goal vector to be used in such problems may be chosen in a somewhat arbitrary way. Instead, we apply a constrained non-linear optimization procedure to be explained in the next paragraph. Second, we apply an improved numerical integration in order to reduce significantly the number of forward rates to be optimized. By this procedure, we are able to use a relatively large time step of 90 days, as opposed to a time step of one day in the study of Lorimier [1995]. The extra computing time necessary for the improved numerical integration is moderate and by far less than the saving in computing time due to the reduced number of optimized forward rates by a factor of 90. Third, we optimize the instantaneous forward interest rates rather than their differences as proposed in Delbaen and Lorimier [1992] or Lorimier [1995]. By this, the optimization is simplified.

Suppose that we observe the cash prices of L coupon-bearing bonds, denoted as $B_{\text{obs}}(t, T_\ell \mid \cdot)$, $\ell = 1, 2, \ldots, L$, at date t. We wish to compute the continuously compounded nominal spot interest rates $R_{n,\,c}(t, \cdot)$ for time periods between zero and the maximum time period as given by the sample of observed coupon-bearing bonds. Arrange the maturity dates of the coupon-bearing bonds in ascending order $T_1 < T_2 < \ldots < T_L$. Let time be divided into equal time steps of length Δt. Determine the number of time steps of length Δt, denoted as H, according to

---

[6] In the continuous case, the goal is to minimize the integral of the squared derivatives of the instantaneous forward rate curve (Lorimier, 1995).

$$H \equiv \mathscr{B}\left(\frac{T_L - t}{\Delta t}\right) \tag{2-1}$$

where $\mathscr{B}$ denotes the floor of a real number, that is, the real number rounded towards minus infinity. We wish to find the nominal instantaneous forward interest rates at the end points of these time steps. Since the nominal instantaneous forward interest rate with a remaining time period of one instant is equal to the observed nominal instantaneous spot interest rate, there are H unknown nominal instantaneous forward interest rates to be determined by the optimization procedure. With a term structure covering a time span of 30 years into the future, and given a time step of 90 days, there are about 120 forward rates to be optimized.

The theoretical cash price of the ℓth coupon-bearing bond can be written in terms of the spot prices of the underlying pure discount bonds as follows.

$$\begin{aligned} B_n\left(t, T_\ell \mid N_\ell, c_\ell, q_\ell\right) &= \sum_{k=0}^{k=\kappa_\ell} c_{\ell,k}\, P_n\left(t, t + \rho_\ell + \frac{k}{q_\ell}\right) + N_\ell\, P_n\left(t, T_\ell\right) \\ &= \sum_{k=0}^{k=\kappa_\ell} c_{\ell,k} \exp\left(-R_{n,c}\left(t, t + \rho_\ell + \frac{k}{q_\ell}\right)\left[\rho_\ell + \frac{k}{q_\ell}\right]\right) \\ &\quad + N_\ell \exp\left(-R_{n,c}\left(t, T_\ell\right)\left[T_\ell - t\right]\right) \end{aligned} \tag{2-2}$$

where

$$\kappa_\ell = \mathscr{B}\left(q_\ell\left[T_\ell - t\right]\right), \quad \rho_\ell = T_\ell - t - \frac{\kappa_\ell}{q_\ell}, \quad \ell = 1, 2, \ldots, L.$$

The number of coupon dates of the ℓth coupon-bearing bond is denoted as κ_ℓ, the fraction of the first coupon period as ρ_ℓ, the periodicity of the coupon payments as q_ℓ, the principal or redemption value of the bond, respectively, as N_ℓ, and the coupon stream as c_ℓ. There are $\sum_{\ell=1}^{\ell=L}\left[1 + \kappa_\ell\right]$ spot interest rates to be determined at most for the L coupon-bearing bonds. Each of the nominal spot interest rate in the above equation can be replaced by the corresponding integral of nominal instantaneous forward interest rates by equation (1-8) as follows.

$$R_{n,c}\left(t, t + \rho_\ell + \frac{k}{q_\ell}\right)\left[\rho_\ell + \frac{k}{q_\ell}\right] = \int_{\tau=t}^{\tau = t + \rho_\ell + \frac{k}{q_\ell}} f_{n,c}(t, \tau)\, d\tau, \quad k = 0, \ldots, \kappa_\ell; \quad \ell = 1, \ldots, L. \tag{2-3}$$

The integral in the above equation is approximated by the mean value of the upper and lower step function, including the fraction of a time step due to the fact that rather large time steps of mostly 90 days will be applied.

$$\int_t^{t+\rho_\ell+\frac{k}{q_\ell}} f_{n,c}(t,\tau)\,d\tau = \frac{1}{2}\,\Delta t \sum_{j=0}^{j=\kappa_{\ell,k}-1} \left[\varphi_j + \varphi_{j+1}\right] + \left[\varphi_{\kappa_{\ell,k}} + \frac{\Delta\varphi_{\kappa_{\ell,k}+1}\,\rho_{\ell,k}}{2\,\Delta t}\right]\rho_{\ell,k}$$

$$\text{where} \quad \ell = 1, 2, \ldots, L; \quad k = 0, 1, \ldots, \kappa_\ell; \tag{2-4}$$

$$\kappa_{\ell,k} = \mathcal{B}\left(\frac{\rho_\ell + \frac{k}{q_\ell}}{\Delta t}\right), \quad 0 \leqq \kappa_{\ell,k} \leqq H; \quad \rho_{\ell,k} = \rho_\ell + \frac{k}{q_\ell} - \kappa_{\ell,k}\,\Delta t$$

$$\varphi_j = f_{n,c}(t, t + j\,\Delta t)\,, \quad (j = 0, 1, \ldots, \kappa_{\ell,k}, \ldots, H); \quad \varphi_{H+1} \equiv \varphi_H.$$

The number of instantaneous forward rates to be considered for the kth coupon date of the ℓth coupon-bearing bond is denoted as $\kappa_{\ell,k}$, and the fraction of the last time step for the kth coupon date of the ℓth coupon-bearing bond as $\rho_{\ell,k}$. To simplify the notation, the nominal instantaneous forward interest rate evaluated at the jth time step is abbreviated as φ_j. Inserting the above two equations into the one for the price of a coupon-bearing bond, a non-linear equation system consisting of L equations in H unknown forward rates is obtained.

In general, the number of unknown forward rates is much larger than the sample size of coupon-bearing bonds, in particular, for a small time step of, say, one day. We now look for a smooth forward rate curve such that the squared differences in nominal instantaneous forward interest rates is as small as possible subject to the condition that the relative pricing errors fall into a given tolerance range.

$$\mathcal{F}(\varphi_1, \ldots, \varphi_H) = \min_{\{\varphi_1,\,\varphi_2,\,\ldots,\,\varphi_H\}} \left\{\sum_{j=1}^{j=H} \left(\Delta\varphi_j\right)^2\right\}, \quad \text{where } \Delta\varphi_j \equiv \varphi_j - \varphi_{j-1}\,, \quad \varphi_0 \text{ given,}$$

$$\text{subject to} \tag{2-5}$$

$$-\,\epsilon_\ell \leqq \left(\frac{B_n\left(t, T_\ell \,\middle|\, N_\ell, c_\ell, q_\ell\right)}{B_{\text{obs}}\left(t, T_\ell \,\middle|\, N_\ell, c_\ell, q_\ell\right)} - 1\right) 100 \leqq \epsilon_\ell\,, \quad \ell = 1, 2, \ldots, L.$$

There are $(2 \cdot L)$ constraints and L non-negative pricing error tolerances ϵ_ℓ ($\ell = 1, 2, \ldots, L$). If measurement errors are absent, then we set the corresponding tolerance equal to zero, in particular, when short-term discount bonds or other money-market instruments are included in the sample of coupon-bearing bonds under consideration. An inspection of the above optimization programme reveals that the number of floating point operations increases with the square of H, that is, the number of nominal instantaneous forward interest rates, and linearly with L, that is, the sample size of the bonds considered. Any non-linear optimization programme, however, solves a set of sub-problems, in particular, a quadratic optimization subject to the constraints considered in the above equation.[7] The number of floating point operations for the latter sub-problem increases with the square of L.

In order to test the performance of the Delbaen-Lorimier methodology, we have conducted many experiments applied to the theoretical term structure proposed by Vasicek

7 We apply the optimization procedure “constr” of the optimization toolbox of Matlab.

[1977]. For a term structure with a maximum of 15 years remaining to maturity, and with a sample of as few as six coupon-bearing bonds equally spaced in time, we find a maximum numerical error for the term structure of nominal *spot* interest rates of about one basis point, although the maximum error for the optimized instantanteous *forward* rates may be in the order of magnitude of 5 to 8 basis points. In contrast, the maximum numerical error of the bootstrap method, when applied to the same sample of coupon-bearing bonds, is in the order of magnitude of more than hundred basis points.[8]

Another striking example is reported here. In order to test whether the Delbaen-Lorimier methodology is able to extract *any* term structure from a sample of bonds, we consider an arbitrary wave-like forward rate curve of the following shape.

$$f_{n,c}(t, T) = a + b\,(T - t) + \frac{1}{100}\sin\big(c + d\,(T - t)\big)$$
$$\text{where} \qquad (2\text{-}6)$$
$$a = 0.01, \quad b = 0.002667, \quad c = \frac{2\,\pi}{1000} = 0.006283, \quad d = \frac{4\,\pi}{15} = 0.83776$$

with the corresponding spot interest rate curve:

$$R_{n,c}(t, T) = a + \frac{1}{2}\,b\,(T - t) - \frac{2}{d\,(T - t)\,100}\sin\left(c + \frac{d\,(T - t)}{2}\right)\sin\left(-\frac{d\,(T - t)}{2}\right) \qquad (2\text{-}7)$$

The result is shown in figure 1 for a sample of 15 equally spaced coupon-bearing bonds, and for a time step of 90 days. In all of the following charts, the interest or inflation rates, respectively, are annually compounded rates according to the following conversion.

$$x_{\nu,m} = m\left[\exp\left(\frac{x_{\nu,c}}{m}\right) - 1\right], \quad x = R, F, f; \quad m = 1; \quad \nu = n, r, y. \qquad (2\text{-}8)$$

In figure 1, the magenta curve is the theoretical instantaneous forward interest rate curve according to equation (2-6) and the cyan curve is the theoretical spot interest rate curve according to equation (2-7). The red curve shows the optimized instantaneous forward interest rates and the blue curve, which coincides with the cyan curve, shows the optimized spot interest rates as a result of the Delbaen-Lorimier (DL) methodology. In contrast, both the bootstrap method, as shown by the green curve, and the extended Nelson-Siegel (NS) method, as shown by the red dotted line for the forward rates and by the blue dotted line for the spot rates, yield poor results which are not sufficient for practical purposes. The black curve shows the yields to maturity of the underlying coupon-bearing bonds considered.[9]

The empirical results for three recent samples are shown in figure 2 for the case of Switzerland. As expected, the instantaneous forward rates are more volatile than the spot rates. Any change of the slope of the latter forces the former to move rapidly. The recent term

8 We apply the procedure “zbtprice” of the financial toolbox of Matlab.

9 We apply the procedure “yldbond” of the financial toolbox of Matlab.

structure of nominal spot interest rates is unusually steep up to 19 years. We feel that there are three possible explanations for this phenomenon. First, the investors of government bonds might expect a high future *domestic* inflation. In view of the present non-existent inflation, and in view of the present slackness of the Swiss economy, this seems rather an unplausible explanation. Second, the investors might expect a high foreign, that is, imported inflation. Third, the investors might expect the Swiss nominal spot rates to adhere to the higher interest rate level of the European Union (EU), in particular, because they might expect that Switzer-landwill join the EU in the near future. We will discuss this issue in more length in the fourth section. Note that the term structure for a date at the end of 1991 is also shown in figure 2. At the end of 1991, the Swiss National Bank (SNB) conducted a tight monetary policy. Hence, the term structure of nominal spot rates turns out to be inverse.

3 The Term Structure of Real Interest Rates

Since there are no indexed bonds issued in Switzerland, we must rely on a economic model which is able to explain simultaneously nominal and real interest rates. To my knowledge, there are two candidate models to be used, the one by Cox, Ingersoll and Ross [1985a & b] on the one hand, and the one by Bakshi and Chen [1996] on the other hand. We choose the CIR model for the reason of tractability.

To be specific, CIR propose two competitive models to explain nominal and real interest rates. Again, we choose the simpler one, which is their model 2. In equilibrium, the real instantaneous spot interest rate is given by the following square-root process.

$$\mathrm{d}r_{r,c}(t) = \varkappa\left[\theta - r_{r,c}(t)\right]\mathrm{d}t + \sigma\sqrt{r_{r,c}(t)}\,\mathrm{d}z_1(t)\,, \quad 0 \leqq \varkappa,\ \theta,\ \sigma < \infty \qquad (3\text{-}1)$$

The above process corresponds to a continuous time first-order autoregressive process where the randomly moving interest rate is elastically pulled toward a long-term equilibrium value, θ. The parameter $\varkappa$ determines the speed of adjustment, and σ denotes the constant volatility parameter, and z_1 a Gauss-Wiener process. With the square-root process, the real instantaneous spot interest rate remains non-negative. By means of their fundamental partial differential equation, CIR derive the price of a real pure discount bond in real terms as follows.

$$P_r(t, T) = A(t, T)^{\psi} \exp\big(-B(t, T)\, r_{r,c}(t)\big)\text{, where}$$

$$A(t, T) = \frac{2\,\gamma \exp\left(\frac{[\varkappa + \lambda + \gamma]\,[T - t]}{2}\right)}{[\varkappa + \lambda + \gamma]\,[\exp(\gamma\,[T - t]) - 1] + 2\,\gamma}$$

$$B(t, T) = \frac{2\,[\exp(\gamma\,[T - t]) - 1]}{[\varkappa + \lambda + \gamma]\,[\exp(\gamma\,[T - t]) - 1] + 2\,\gamma} \tag{3-2}$$

$$\gamma = \sqrt{[\varkappa + \lambda]^2 + 2\,\sigma^2}\,, \quad \psi = \frac{2\,\varkappa\,\theta}{\sigma^2}$$

The parameter λ denotes the factor risk premium.

Let $p(t)$ denote the price level of consumer goods or the cost of living index, respectively, at date t and let $r_{y,c}(t)$ denote the continuously compounded instantaneous spot inflation rate at date t. CIR propose the following two random paths for the instantaneous spot inflation rate and the consumer price level to be tested empirically.

$$\begin{aligned} \mathrm{d}r_{y,c}(t) &= \varkappa_2\,[\theta_2 - r_{y,c}(t)]\,\mathrm{d}t + \sigma_2 \sqrt{r_{y,c}(t)}\,\mathrm{d}z_3(t)\,, \quad 0 \leqq \varkappa_2, \theta_2, \sigma_2 < \infty \\ \mathrm{d}p(t) &= r_{y,c}(t)\,p(t)\,\mathrm{d}t + \sigma_p\,p(t) \sqrt{r_{y,c}(t)}\,\mathrm{d}z_2(t)\,, \quad \mathscr{C}\{\mathrm{d}r_{y,c}, \mathrm{d}p\} = \zeta\,\sigma_2\,\sigma_p\,r_{y,c}\,p, \quad 0 \leqq \sigma_p < 1 \end{aligned} \tag{3-3}$$

where $\mathscr{C}$ denotes the covariance operator, ζ the correlation coefficient between the Wiener processes z_2 and z_3, and all other variables have the same meaning as above.[10] CIR derive the following price of a nominal pure discount bond in nominal terms from their fundamental partial differential equation.

$$P_n(t, T) = P_r(t, T)\, C(t, T)^{\psi_2} \exp\big(-D(t, T, r_{y,c}(t))\big)\text{, where}$$

$$C(t, T) = \frac{2\,\xi \exp\left(\frac{[\varkappa_2 + \zeta\,\sigma_2\,\sigma_p + \xi]\,[T - t]}{2}\right)}{[\varkappa_2 + \zeta\,\sigma_2\,\sigma_p + \xi]\,[\exp(\xi\,[T - t]) - 1] + 2\,\xi}$$

$$D(t, T, r_{y,c}(t)) = \frac{2\,[\exp(\xi\,[T - t]) - 1]\,[1 - \sigma_p^2]\,r_{y,c}(t)}{[\varkappa_2 + \zeta\,\sigma_2\,\sigma_p + \xi]\,[\exp(\xi\,[T - t]) - 1] + 2\,\xi} \tag{3-4}$$

$$\xi = \sqrt{[\varkappa_2 + \zeta\,\sigma_2\,\sigma_p]^2 + 2\,\sigma_2^2\,[1 - \sigma_p^2]}\,, \quad \psi_2 = \frac{2\,\varkappa_2\,\theta_2}{\sigma_2^2}$$

where $P_r(t, T)$ denotes the price of a real discount bond given above. Denote the 11-by-one parameter vector of the CIR model to be estimated at date t when the term structure of interest

---

10 We use the unusual symbol ζ to denote the correlation coefficient in order to avoid any confusion with the fraction of a coupon period denoted as ρ in the text.

rates is observed as $\boldsymbol{\beta} = [\varkappa, \theta, \sigma, \lambda, \varkappa_2, \theta_2, \sigma_2, \sigma_p, \zeta, r_{r,c}(t), r_{y,c}(t)]$, then the nominal and real spot interest rates according to the CIR model 2 can be written by equation (1-2) as follows.

$$
\begin{aligned}
R_{r,c}(t, T \mid \boldsymbol{\beta}) &= \frac{B(t, T)\, r_{r,c}(t) - \psi \ln\bigl(A(t, T)\bigr)}{T - t} \\
R_{n,c}(t, T \mid \boldsymbol{\beta}) &= \frac{B(t, T)\, r_{r,c}(t) - \psi \ln\bigl(A(t, T)\bigr) - \psi_2 \ln\bigl(C(t, T)\bigr) + D\bigl(t, T, r_{y,c}(t)\bigr)}{T - t}
\end{aligned}
\tag{3-5}
$$

Taking the limit as $T \to t$, we find for the nominal and real instantaneous spot interest rates by means of L'Hopital's rule:

$$
\begin{aligned}
R_{r,c}(t, t \mid \boldsymbol{\beta}) &= r_{r,c}(t) \\
R_{n,c}(t, t \mid \boldsymbol{\beta}) &= r_{n,c}(t) = r_{r,c}(t) + \bigl[1 - \sigma_p^2\bigr]\, r_{y,c}(t)
\end{aligned}
\tag{3-6}
$$

By Ito's lemma, the nominal instantaneous spot interest rate follows the process given by the equation below.

$$
\mathrm{d}r_{n,c}(t) = \bigl\{\varkappa\,[\theta - r_{r,c}(t)] + \varkappa_2\,[\theta_2 - r_{y,c}(t)]\bigr\}\,\mathrm{d}t + \sigma\sqrt{r_{r,c}(t)}\,\mathrm{d}z_1(t) + \bigl[1 - \sigma_p^2\bigr]\,\sigma_2\sqrt{r_{y,c}(t)}\,\mathrm{d}z_3(t) \tag{3-7}
$$

To estimate the model parameters $\boldsymbol{\beta}$, we apply a non-linear regression such that the nominal spot interest rates as given by the CIR model fit the H observed or optimized nominal spot interest rates, respectively, of the previous section for as many data points as possible. Let the m-by-1 index vector $\boldsymbol{\iota}$ denote those elements of the theoretical spot interest rates which should be equal to the observed spot interest rates, where $m \leqq 11$, then we can write the non-linear least squares optimization as follows.

$$
\begin{aligned}
&\min_{\{\boldsymbol{\beta}\}} \left\{ \sum_{j=1}^{H} \bigl[R_{n,c}(t, t + j\,\Delta t \mid \boldsymbol{\beta}) - R_{n,c}(t, t + j\,\Delta t)\bigr]^2 \right\} \\
&s.\,t.\; R_{n,c}(t, t + \boldsymbol{\iota}(k)\,\Delta t \mid \boldsymbol{\beta}) = R_{n,c}(t, t + \boldsymbol{\iota}(k)\,\Delta t), \quad k = 1, \ldots, m; \quad m \leqq 11.
\end{aligned}
\tag{3-8}
$$

Given the estimated model parameters $\boldsymbol{\beta}$, we calculate the real spot interest rates according to equation (3-6). The result for three recent samples is shwon in figure 3. The dotted lines denote the estimated term structures of nominal spot interest rates according to equation (3-6), whereas the solid lines denote the observed nominal spot interest rates obtained in the previous section. The coincidence is almost perfect. The dash-dotted lines correspond to the real spot interest rates according to equation (3-6) which are implied by the CIR model. Again, we find a steep yield curve up to 19 years. The exercise is repeated for a date at the end of 1991, exhibiting an inverse term structure of nominal spot interest rates but a normal term structure of real spot interest rates.

4 The Term Structure of Expected Inflation Rates

In the final step, we calculate the expected inflation rates from the model parameters estimated in the previous section. To start with, we define the spot inflation rate and the instantaneous forward inflation rate, respectively, in an analoguous way as for the interest rates. Let $P_y(t, T)$ denote the purchasing power of money at the future date T in nominal terms at current prices as seen from date t.

$$P_y(t, T) \equiv \frac{p(t)}{p(T)} = \exp\left(-R_{y,c}(t, T)\,[T-t]\right) = \exp\left(-\int_{\tau=t}^{\tau=T} f_{y,c}(t, \tau)\,d\tau\right), \quad t \leqq T. \tag{4-1}$$

Since the future consumer price level $p(T)$ is a random variable, both the spot inflation rate, $R_{y,c}(t, T)$, and the instantaneous forward inflation rate, $f_{y,c}(t, T)$, are random variables, too. Hence, we are interested in their expected or mean values, respectively. Let $\mathscr{E}$ denote the expectation operator given the information at date t. Taking logarithms of the above equation, the expected inflation rates become:

$$\mathscr{E}\left\{R_{y,c}(t, T)\right\}[T-t] = \int_{\tau=t}^{\tau=T} \mathscr{E} f_{y,c}(t, \tau)\,d\tau = \mathscr{E}\ln\left(\frac{p(T)}{p(t)}\right) = -\,\mathscr{E}\ln\left(P_y(t, T)\right) \tag{4-2}$$

Since $\mathscr{E}\ln(p(T) / p(t)) \leqq \ln(\mathscr{E}\{p(T) / p(t)\})$ by Jensen's inequality, the relationship (4-1) does not apply to the expected inflation rates as well. Rather, it holds true that

$$\exp\left(\mathscr{E}\left\{\ln\left(P_y(t, T)\right)\right\}\right) = \exp\left(-\,\mathscr{E}\left\{R_{y,c}(t, T)\right\}[T-t]\right) = \exp\left(-\int_{\tau=t}^{\tau=T} \mathscr{E} f_{y,c}(t, \tau)\,d\tau\right), \quad t \leqq T. \tag{4-3}$$

Given the estimated model parameters, it would be possible to compute the expected spot inflation rates as given in equation (4-2) by a Monte-Carlo simulation. However, this would be like breaking butterflies on the wheel because the expected instantaneous forward inflation rate is equal to the expected instantaneous spot inflation rate.

To prove this assertion, consider the relationship (4-1) for a future date T as seen from date t as follows.

$$\ln\left(p(t)\right) + \int_{\tau=t}^{\tau=T} f_{y,c}(t, \tau)\,d\tau = \ln\left(p(T)\right), \quad t \leqq T. \tag{4-4}$$

Since the differentiation of the integral on the left-hand side with respect to the "maturity" date T proceeds in the same way as for a deterministic variable, we obtain

$$f_{y,c}(t, T)\,dt = d\ln\left(p(T)\right) = \frac{dp(T)}{p(T)}, \quad t \leqq T. \tag{4-5}$$

Hence, the process of the instantanteous forward inflation rate is equal to that of the consumer price level as given in equation (3-3). The above relationship is the stochastic equivalence of equation (1-10). Taking expectations, it follows that the expected instantaneous forward inflation rate is equal to the expected instantaneous spot inflation rate.

$$\mathscr{E} f_{y,c}(t, T) = \mathscr{E} r_{y,c}(T), \quad t \leqq T. \tag{4-6}$$

The expected instantaneous spot inflation rate is calculated according to equation (3-3) for small time steps of, say, one day, denoted as Δt, by the following recurrence relationship.

$$\begin{aligned} \mathscr{E} r_{y,c}(t + j\,\Delta t) &= \left[1 - \varkappa_2\,\Delta t\right] \mathscr{E} r_{y,c}\left(t + [j-1]\,\Delta t\right) + \varkappa_2\,\theta_2\,\Delta t, \quad j = 2, 3, \ldots \\ \mathscr{E} r_{y,c}(t + \Delta t) &= \left[1 - \varkappa_2\,\Delta t\right] r_{y,c}(t) + \varkappa_2\,\theta_2\,\Delta t, \quad j = 1. \end{aligned} \tag{4-7}$$

Finally, the interest premium, denoted as η_k, is calculated as a residual from the following relationship.

$$R_{n,k}(t, T) = R_{r,k}(t, T) + \mathscr{E} R_{y,k}(t, T) - \eta_k(t, T), \quad k = m, c. \tag{4-8}$$

In the CIR framework, the interest premium consists of two terms, the variance of the future consumer price level and a term which we call the wealth premium. The latter depends on both the investor’s attitude towards risk, as measured by the relative risk aversion, and on the covariance between future real wealth and future inflation, which may have either sign. Hence the interest premium may have either sign, too. If investors expect to gain real wealth from future inflation, then this covariance will be positive. In this case, investors do not ask for a full compensation of the expected inflation rate. On the other hand, if investors expect to lose real wealth from future inflation, then this covariance will be negative. In this case, they may ask for a compensation in excess of the expected inflation rate.

The term structure of expected inflation rates is shown in figure 4 for a recent sample of bonds and in figure 5 for a sample of coupon-bearing bonds observed at the end of 1991. The expected one-year forward inflation rates are calculated according to equation (1-6) given the expected spot inflation rates. The interest premia are calculated as residuals of the annually compounded interest and inflation rates. As you can see from figures 4 and 5, the differences between nominal and real spot interest rates are not equal to the expected spot inflation rates.

Observed inflation rates are compared with expected inflation rates in figure 6 for the term structure considered at the end of 1991. Both the expected spot and one-year forward inflation rates coincide quite well with the observed spot and one-year forward inflation rates, respectively.

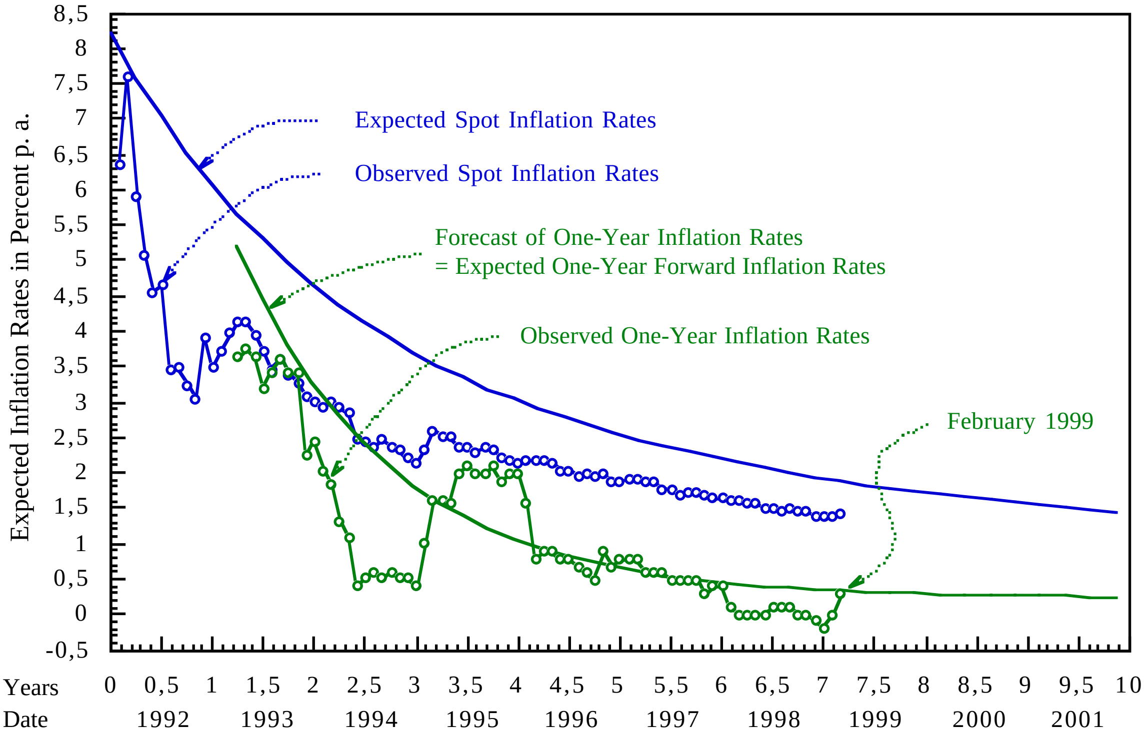

Figure 6: Observed and Expected Inflation Rates on 30 December 1991

The high expected spot inflation rates for the recent term structure as shown in figure 4 pose a question because the present inflation rate is almost zero. Why do investors expect a high future inflation? We feel that there are three possible explanations for this phenomenon. First, the investors of government bonds might expect a high future *domestic* inflation. In view of the present non-existent inflation, and in view of the present slackness of the Swiss economy, this seems rather an unplausible explanation. Second, the investors might expect a high foreign, that is, imported inflation. Again, this seems an unplausible explanation because the inflation is presently very low in those European countries which are the main trading partners of Switzerland. Third, the investors might expect the Swiss nominal spot rates to adhere to the higher interest rate level of the European Union (EU), in particular, because they might expect that Switzerland will join the EU in the near future. We feel that the third explanation is the most conceivable one for three reasons. First, the CIR model to evaluate the term structure is a model of a closed economy as well as a model that is not able to explain other effects than the expected inflation rate and the interest premium. Second, a possible Swiss memberhip in the EU was not a political issue at the end of 1991. Hence, the term structure considered for this date should not contain the possible effect that the Swiss interest rates could adhere to the higher interest rate level of the EU. The term structure considered at the end of 1991 forecasts very low future inflation rates of about 0.3% per annum for the years 1999 to 2001 as shown in figure 6. Third, an inspection of the variance of the consumer price level in the course of future time, shows that this variance calculated for a recent term structure is twice as big as the variance for the term structure considered at the end of 1991. Hence, the investor's uncertainty about future variations of the consumer price level has risen substantially, although the inflation rate fell substantially during the past decade.

5 Conclusions

In this paper, we infer the term structure of expected inflation rates from a sample of observed prices of coupon-bearing bonds by means of a three-step procedure. In the first step, the nominal instantaneous forward interest rates are optimized. In the second step, the extended CIR model 2 is fitted to the optimized nominal spot interest rates. By this procedure, the real spot interested rates are determined. Finally, the estimated CIR model parameters allow the calculation of the expected instantaneous forward inflation rates by means of a simple recurrence relationship. The expected inflation rates coincide quite well with the observed inflation rate for a sample observed at the end of 1991. We conjecture that the present high expected inflation rate obtained from the CIR model is a spurious inflationary expectation. Rather, the result relates to the difference in interest rates between Switzerland and the EU countries. Therefore, we plan to investigate the European term structure of expected inflation rates in another paper.

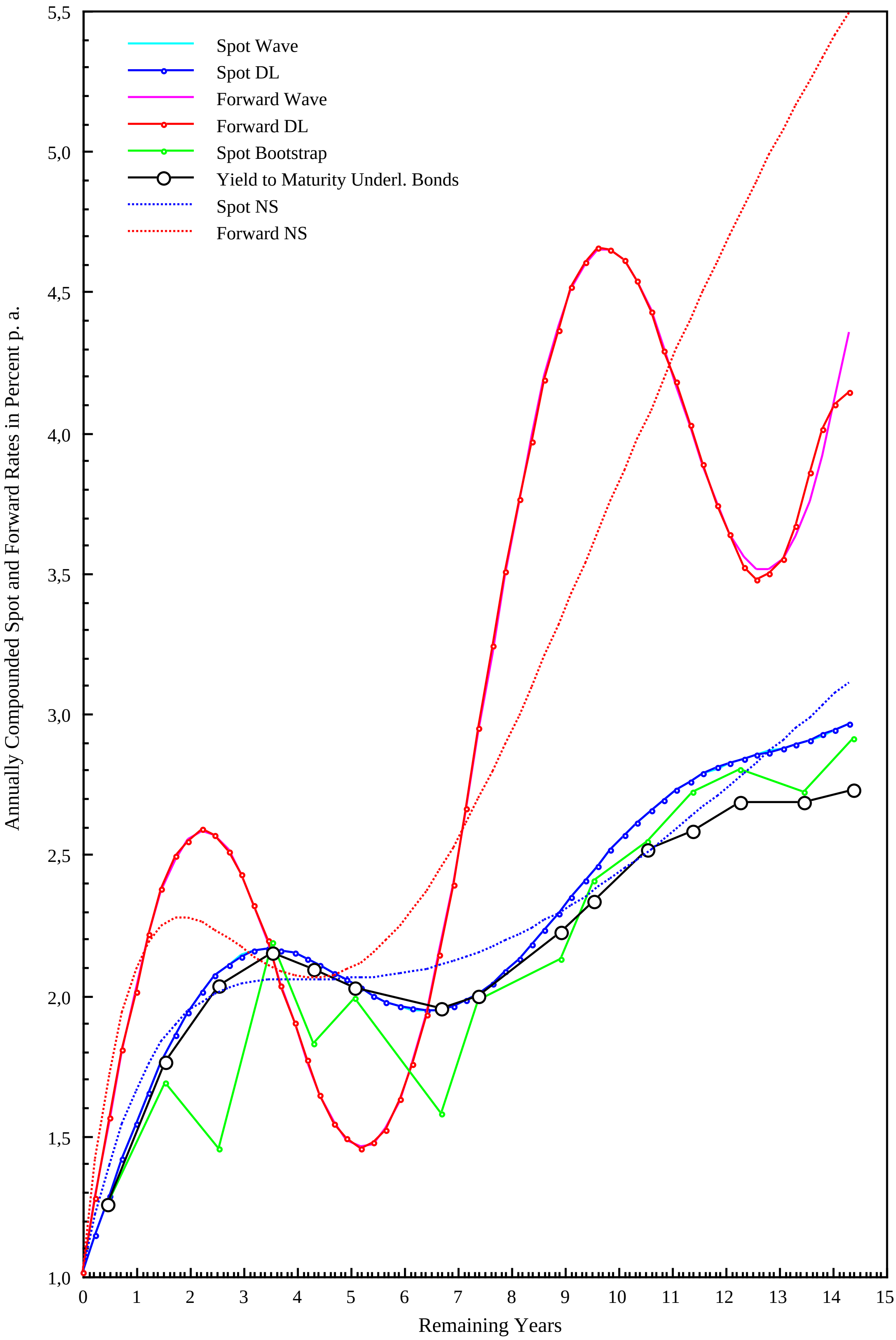

Figure 1: Optimized Forward and Spot Interest Rates for the Wave Term Structure

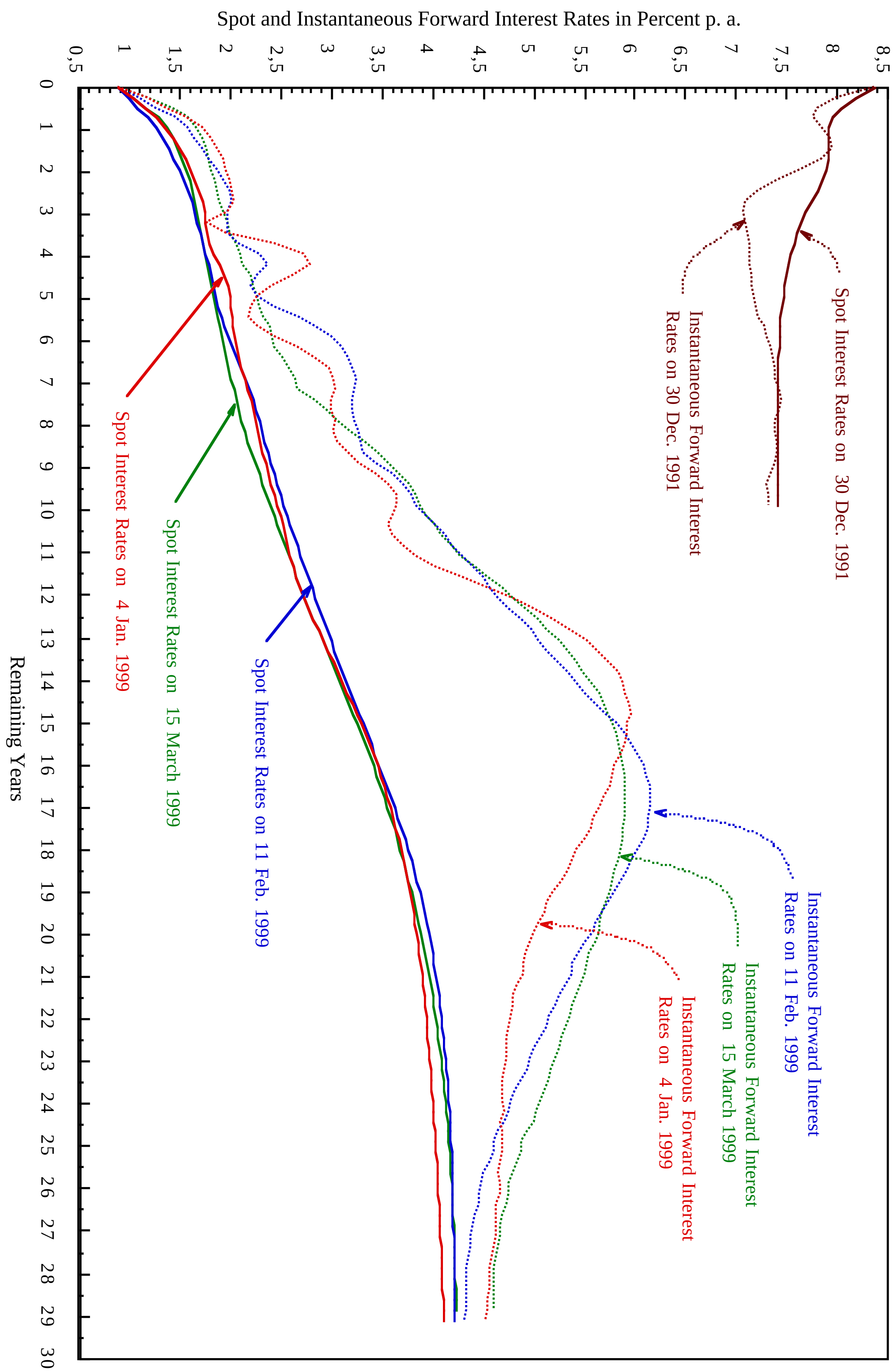

Figure 2: Nominal Spot and Instantaneous Forward Interest Rates

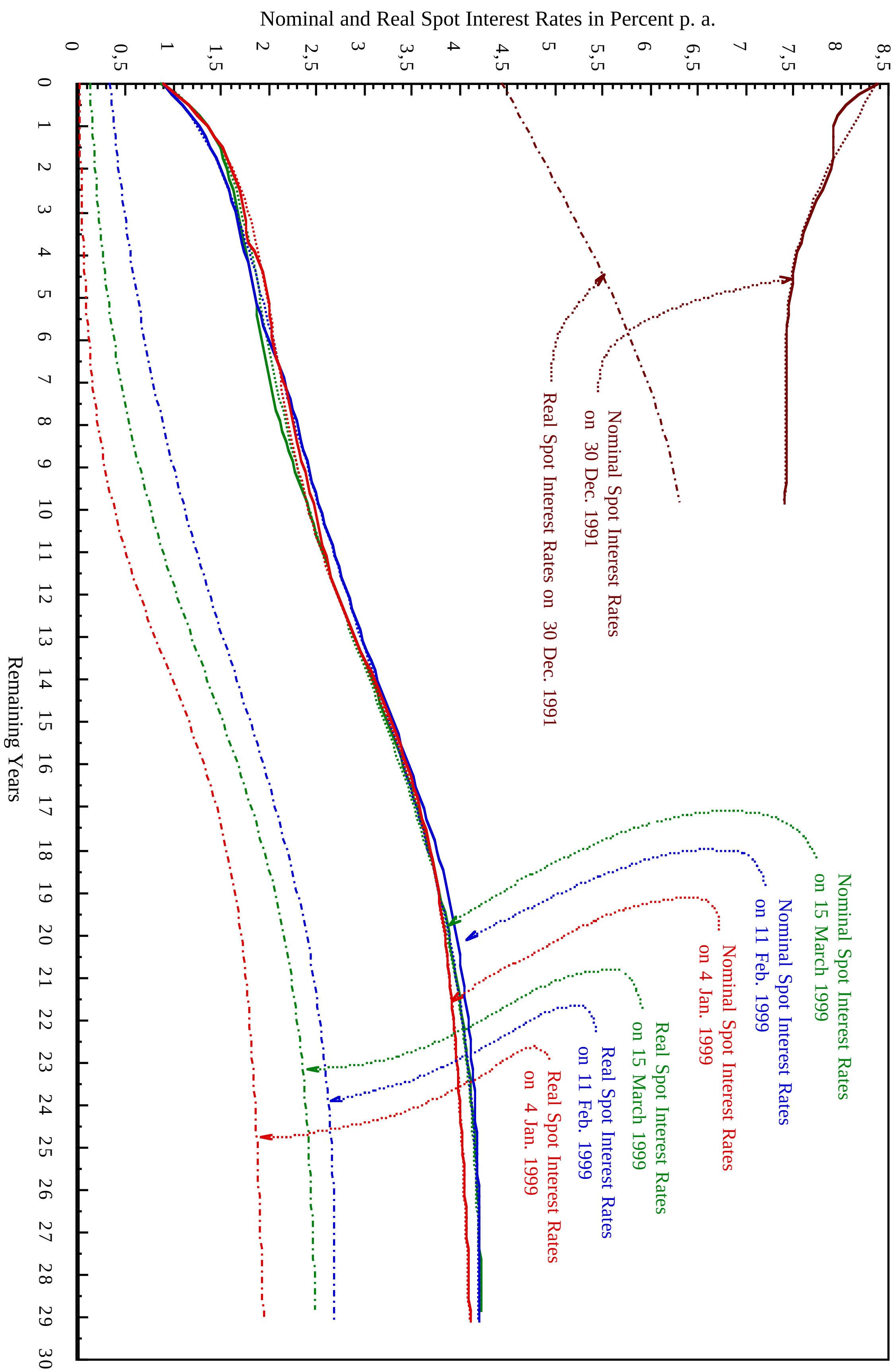

Figure 3: Real Spot Interest Rates

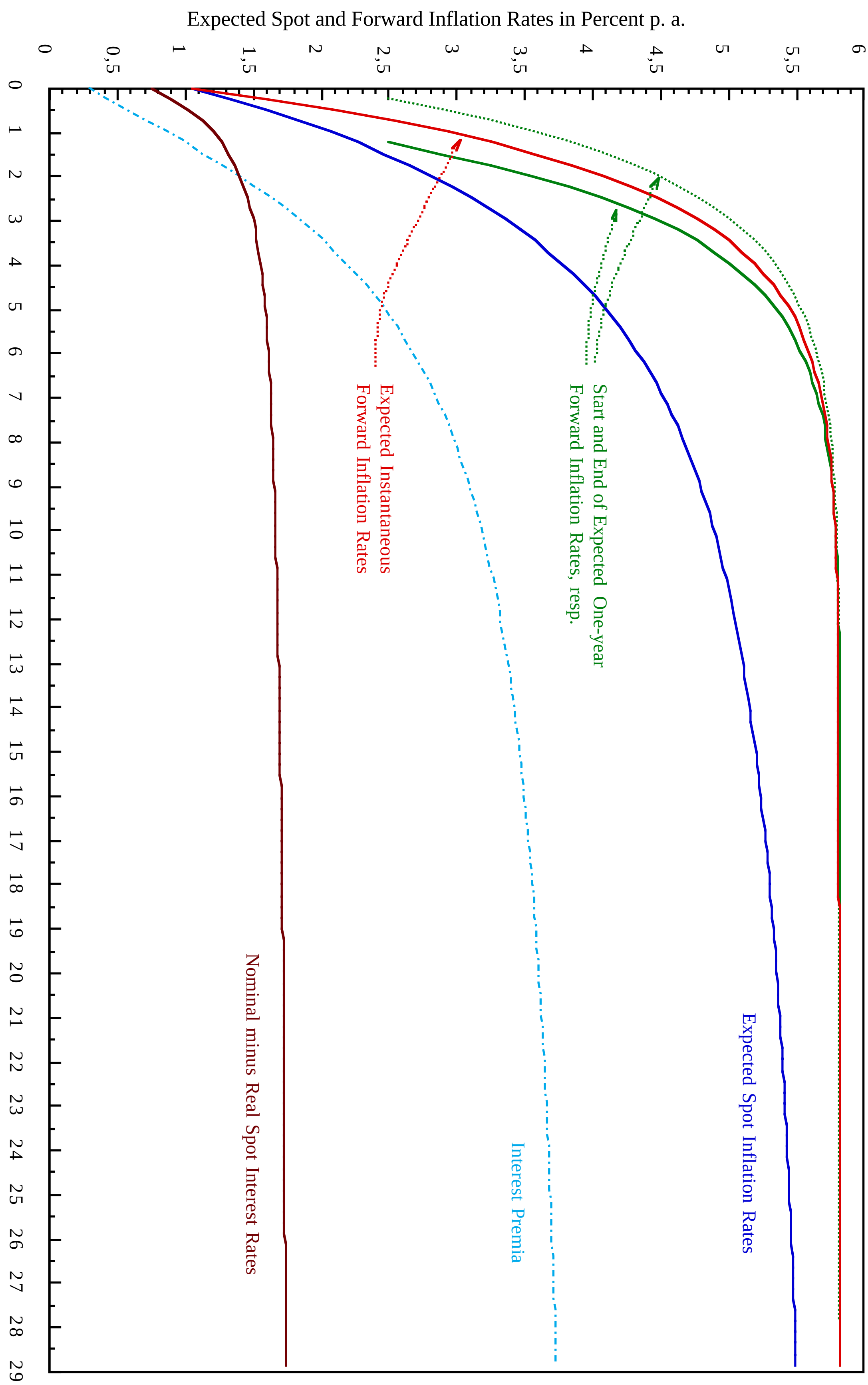

Figure 4: The Term Structure of Expected Inflation Rates on 15 March 1999

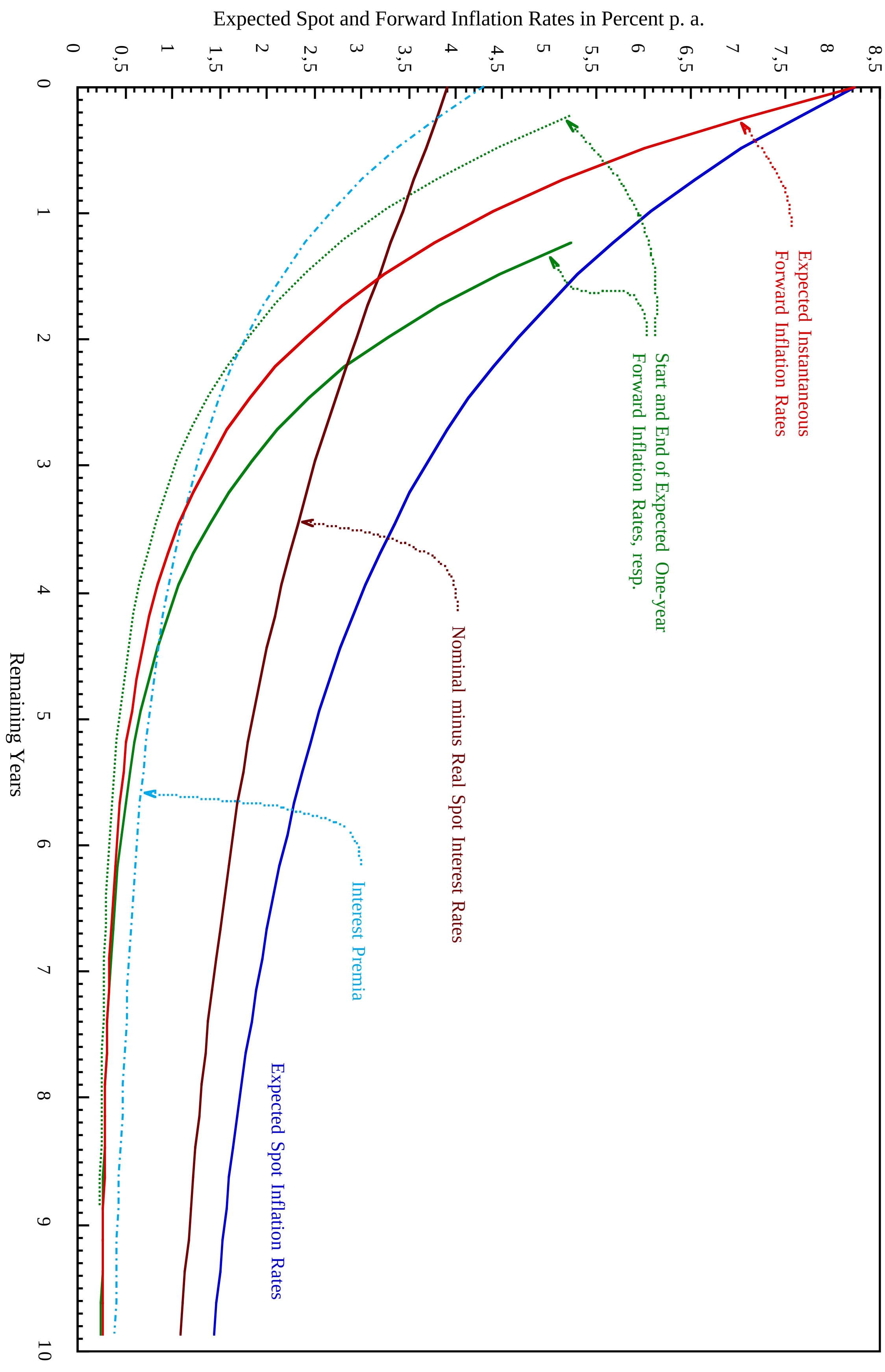

Figure 5: The Term Structure of Expected Inflation Rates on 30 December 1999

References

BAKSHI, GURDIP S. AND ZHIWU CHEN [1996]: „Inflation, Asset Prices, and the Term Structure of Interest Rates in Monetary Economies“, *The Review of Financial Studies*, 9 (1): 241 – 275.

BENNINGA, SIMON ANDARIS PROTOPAPADAKIS [1985]: „Real and Nominal Interest Rates Under Uncertainty: The Fisher Theorem and the Term Structure“, *The Journal of Political Economy*, 91: 856 – 867.

BJöRK, TOMAS AND BENT JESPER CHRISTENSEN [1997]: „Forward Rate Models and Invariant Manifolds“, *Working Paper*, Stockholm School of Economics.

BREEDEN, DOUGLAS T. [1986]: „Consumption, Production, Inflation and Interest Rates: A Synthesis“, *The Journal of Financial Economics*, 16: 3 – 39.

BROWN, ROGER H. AND STEPHEN M. SCHAEFER [1994]: „The Term Structure of Real Interest Rates and the Cox, Ingersoll, and Ross Model“, *The Journal of Financial Economics*, 35: 3 – 42.

CARLETON, WILLARD T.AND IAN A. COOPER [1976]: „Estimation and Uses of the Term Structure of Interest Rates, *The Journal of Finance*, 31 (4): 1067 – 1083.

COX, JOHN C., JONATHAN E. INGERSOLL, JR., AND STEPHEN A. ROSS [1981]: “A Re-Examination of Traditional Hypotheses about the Term Structure of Interest Rates”, *The Journal of Finance*, 36 (4): 769 – 799.

COX, JOHN C., JONATHAN E. INGERSOLL, JR., AND STEPHEN A. ROSS [1985a]: “An Intertemporal General Equilibrium Model of Asset Prices”, *Econometrica*, 53: 363 – 384.

COX, JOHN C., JONATHAN E. INGERSOLL, JR., AND STEPHEN A. ROSS [1985b]: “A Theory of the Term Structure of Interest Rates”, *Econometrica*, 53: 385 – 407.

DELBAEN FREDDY AND SABINE LORIMIER [1992]: “Estimation of the Yield Curve and the Forward Rate Curve starting from a Finite Number of Observations”, *Insurance. Mathematiccs and Economics*, 11: 259 – 269.

EVANS, MARTIN D. D. AND PAUL WACHTEL [1992]: „Interpreting the Movements in Short-term Interest Rates“, *The Journal of Business*, 65: 395 – 430.

EVANS, MARTIN D. D. [1998]: „Real Rates, Expected Inflation, and Inflation Risk Premia“, *The Journal of Finance*, 53 (1): 187 – 218.

FAMA, EUGENE F. AND ANDRé FARBER [1979]: “Money, Bonds, and Foreign Exchange”, *The American Economic Review*, 69 (4): 639 – 649.

FISCHER, STANLEY [1975]: “The Demand for Index Bonds”, *The Journal of Political Economy*, 83: 509 – 534.

FISHER, IRVING [1930]: *The Theory of Interest*, New York: Macmillan.

FRANKEL, JEFFREY A. [1982]: “A Technique for Extracting a Measure of Expected Inflation From the Interest Rate Term Structure”, *The Review of Economics and Statistics*, 64 (1): 135 – 142.

HULL, JOHN C. [1997]: *Options, Futures, and Other Derivatives*, New York: Prentice Hall International, Inc., third edition.

LORIMIER, SABINE [1995]: *Interest Rate Term Structure Estimation Based on the Optimal Degree of Smoothness of the Forward Rate Curve*, Doctoral Dissertation of the University of Antwerp (Belgium).

LUCAS, ROBERT E. JR [1982]: “Interest Rates and Currency Prices in a Two-Country World”, *The Journal of Monetary Economics*, 10: 335 – 359.

MCCULLOCH, J. HUSTON [1971]: „Measuring the Term Structure of Interest Rates“, *The Journal of Business*, 34 (1): 19 – 31.

MCCULLOCH, J. HUSTON [1975]: „The Tax-adjusted Yield Curve“, *The Journal of Finance*, 30 (3): 811 – 830.

NELSON, CHARLES R. AND ANDREW F. SIEGEL [1987]: „Parsimonious Modeling of Yield Curves“, *The Journal of Business*, 60 (4): 473 – 489.

REMOLONA, ELI M., MICHAEL R. WICKENS AND FRANK F. GONG [1998]: „What Was the Market’s View of U. K. Monetary Policy? Estimating Inflation Risk and Expected Inflation with Indexed Bonds“, *The Federal Reserve Bank of New York Staff Reports*, 57: 1 – 43.

SHEA, GARY S. [1984]: „Pitfalls in Smoothing Interest Rate Term Structure Data: Equilibrium Models and Spline Approximations“, *Journal of Financial and Quantitative Analysis*, 19 (3): 253 – 269.

SHEA, GARY S. [1985]: „Interest Rate Term Structure Estimation with Exponential Splines: A Note“, *The Journal of Finance*, 40 (1): 319 – 325.

SVENSSON, LARS E. O. [1995]: „Estimating Forward Rates with the Extended Nelson & Siegel Method“, *Sveriges Riksbank Quarterly Review*, 3: 13 – 26.

VASICEK, OLDRICH [1977]: “An Equilibrium Characterization of the Term Structure”, *Journal of Financial Economics*, 5: 177 – 188.

VASICEK, OLDRICH A. AND H. GIFFORD FONG [1982]: “Term Structure Modeling Using Exponential Splines”, *The Journal of Finance*, 37 (2): 339 – 348.

Appendix: List of Main Variables, Functions and Symbols

Subscripts:

$x_{\nu,k}(\cdot)$ — The first subscript of the variable x, $\nu = \{n, r, y\}$, denotes nominal values for $\nu = n$, real values for $\nu = r$, and values associated with the inflation rate for $\nu = y$. If necessary, we use a second subscript, $k = \{m, c\}$, which denotes the compounding frequency with the understanding that m denotes a compounding m times a year and c denotes the continuous compounding ($m \to \infty$).

Variables:

$B_\nu(t, T \mid N, c, q)$ — $\nu = \{n, r\}$. The cash price of a coupon-bearing bond which is fixed and paid at the settlement date t. The debtor of the bond pays out coupons q times a year and redeems the principal value N when the bond matures at date T ($T \geqq t$.). The symbol c denotes the vector of coupon payments c_k, $k = 0, 1, \ldots, \kappa$. The cash price is sometimes denoted as “dirty price”.

$F_\nu(t, T, \tau)$ — $\nu = \{n, r\}$. The ($\tau - T$)-year forward interest rate corresponding with a forward contract on a pure discount bond with the agreement that the forward price is fixed at date t and paid at a later date T when the discount bond will be delivered. The discount bond matures at a later date τ ($\tau \geqq T \geqq t$). It holds true that $F_\nu(t, t, T) = R_\nu(t, T)$.

$F_y(t, T, \tau)$ — The ($\tau - T$)-year forward inflation rate corresponding with future consumer price levels at future dates τ and T as seen from date t ($\tau \geqq T \geqq t$). It holds true that $F_y(t, t, T) = R_y(t, T)$.

$f_\nu(t, T)$ — $\equiv F_\nu(t, T, T)$; $\nu = \{n, r\}$. The instantaneous forward interest rate corresponding with a forward contract on a pure discount bond with the agreement that the forward price is fixed at date t and paid at a later date T when the discount bond will be delivered. The discount bond matures at the same instant when it is delivered. It holds true that $f_\nu(t, t) = r_\nu(t)$.

$f_y(t, T)$ — $\equiv F_y(t, T, T)$. The instantaneous forward inflation rate corresponding with a consumer price level at the future date T as seen from date t ($T \geqq t$). It holds true that $f_y(t, t) = r_y(t)$.

$p(t)$ — The price level of consumer goods or the cost of living index, respectively, at date t.

$P_\nu(t, T)$ — $\nu = \{n, r\}$. The spot price of a pure discount bond which is fixed and paid at the settlement date t. The debtor of the pure discount bond redeems one monetary unit when the bond matures at date T, but does not pay out any coupons during the bond’s life.

$P_y(t, T)$	$\equiv p(t) / p(T)$, $(T \geqq t)$. The purchasing power of money at the future date T in nominal terms at current prices as seen from date t.
$\mathcal{P}_\nu(t, T, \tau)$	$\nu = \{n, r\}$. The $(\tau - T)$-year forward price of a forward contract on a pure discount bond with the agreement that the forward price is fixed at date t and paid at a later date T when the pure discount bond will be delivered. The pure discount bond matures at a later date τ $(\tau \geqq T \geqq t.)$. In this case, the forward price is equal to the futures price of a discount bond (see Hull, 1997, p. 95). It holds true that $\mathcal{P}_\nu(t, t, T) = P_\nu(t, T)$.
$R_\nu(t, T)$	$\nu = \{n, r\}$. The spot interest rate of a pure disount bond with its price fixed at date t and which matures at date T $(T \geqq t)$. The spot interest rate is also denoted as the yield of the discount bond.
$R_y(t, T)$	The spot inflation rate corresponding with a consumer price level at the future date T as seen from date t $(T \geqq t)$.
$r_\nu(t)$	$\equiv R_\nu(t, t)$; $\nu = \{n, r\}$. The instantaneous spot interest rate of a pure dis-ount bond with its price fixed at date t and which matures at the same in-stant.
$r_y(t)$	$\equiv R_y(t, t)$. The instantaneous spot inflation rate at date t.
η_k	$k = \{m, c\}$. The interest premium.
κ	$\equiv \mathcal{B}(q\,[T - t])$. The number of remaining coupon periods during the re-maining life of the coupon-bearing bond. The number of outstanding coupons is equal to $(\kappa + 1)$.
ρ	$\equiv T - t - \kappa / q$. The time period between date t (today) and the next coupon date as measured in years.
φ_j	$\equiv f(t, t + j\,\Delta t)$, $j = 0, 1, \ldots, H$. Abbreviation for the instantaneous forward interest rate evaluated at equidistant time points.
Functions:	
$\mathcal{B}(x)$	Floor of the real number x, that is, the real number rounded towards mi-nus infinity.
exp	$\equiv$ e. The exponential function.
ln	The natural logarithm.
Symbols:	
$\mathcal{C}$	The covariance operator.
$\mathcal{E}$	The expectation operator.
$\mathcal{V}$	The variance operator.