An Orthogonal Polynomial Approach to Estimate the Term Structure of Interest Rates

by

HANS-JÜRG BÜTTLER *
Research, Swiss National Bank #
and University of Zurich
Switzerland

ABSTRACT

In this paper, we introduce a new algorithm to estimate the term structure of interest rates. It is obtained from a constrained optimization, where the objective is to minimize the integral of squared first derivatives of the instantaneous forward interest rate subject to the condition that the estimated bond prices lie within the range of observed bid and ask prices. We use a finite series of ordinary Laguerre polynomials to approximate the unknown function of the instantaneous forward interest rate. The objective function can be written explicitly as a quadratic form of the Laguerre constants and the nonlinear constraints can be obtained from a recurrence relationship. The estimation error is less than one basis point, given a sufficient number of bonds.

First version: February 2007
Revised version: March 2007

Appeared in *Economic and Financial Modelling*,
Swiss National Bank Working Papers, 2007-8

KEYWORDS: Term structure of interest rates, orthogonal polynomial

* I am indebted to David Jamieson Bolder (Bank of Canada) and an anonymous referee of the SNB working paper series for their valuable comments. The Matlab programs can be downloaded from www.forschung.snb.ch/lehrpersonen/Buettler/Publikationen/Unveröffentlichte_Beiträge.html.

Mailing address: Swiss National Bank, P. O. Box, 8022 Zurich, Switzerland.
Phone (direct dialling): +41–1–631 34 17, Fax: +41–1–631 39 01, E-Mail: hans-juerg.buettler@snb.ch.

0 Introduction

The term structure of interest rates or the yield curve, respectively, is a key variable of economics and finance. By definition, it relates the zero-coupon rates or spot interest rates, respectively, to the terms to maturity of zero-coupon bonds or discount bonds, respectively. Unfortunately, the spot interest rates can rarely be observed except for short terms to maturity. The task, therefore, is to estimate the yield curve from a given set of quoted prices of coupon-bearing bonds *independently* from any term structure model.

Several approaches have been proposed in the literature, including bootstrapping (Hull, 2003; Choudhry, 2005), a regression of the bond prices (Carleton and Cooper, 1976), various spline functions to approximate the unknown spot rate function (McCulloch, 1971; Vasicek and Fong, 1982; Shea 1984, 1985), a small number of exponential functions to approximate the unknown forward rate function (Nelson and Siegel, 1987; Svensson, 1995), Fourier series and the "forward-rate" method (Delbaen and Lorimier, 1992; Lorimier, 1995; Büttler, 2000). For a comprehensive review of the literature see Bolder and Gusba [2002] as well as Ioannides [2003].

In this paper, we follow the "forward-rate" method. It requires that the estimated instantaneous forward interest rate is a continuous function as smooth as possible. It does not suppose a particular function or model, respectively. Moreover, the bond prices implied by the estimated yield curve should lie within the bid-ask spread of quoted bond prices. For the problem at hand, the Laguerre polynomial is the appropriate choice to approximate the unknown function of forward rates.

In the next section, the objective function as well as the constraints are derived in terms of Laguerre polynomials. In the second section, we present two applications, followed by a comparison with the methods mentioned above. Appendix A collects some properties of the Laguerre polynomial used in the text, shows the advantage of our approximating function over the standard one and demonstrates the numerical accuracy of a recurrence relationship used in the text. Finally, appendix B collects the proofs of the equations of the first section.

1 The Orthogonal Polynomial Approach

Note that all the interest rates considered are meant to be continuously compounded. Let $P(\tau) \equiv P(t, t + \tau)$ denote the price of a pure discount bond whose price is fixed at time t. The discount bond is delivered at time t and matures at time $(t + \tau)$. Its term to maturity is denoted as τ. The discount bond pays off one unit of money on the maturity day, but no coupons during its life. The corresponding spot interest rate, yield to maturity or zero-coupon rate, respectively, is denoted as $R(\tau) \equiv R(t, t + \tau)$, that is, $P(\tau) = \exp(-R(\tau) \cdot \tau)$. The function $R(\tau)$ is usually denoted as the *yield curve* or the *term structure of interest rates*, respectively. The instantaneous spot interest rate, denoted as r, is the yield of a discount bond with a vanishing term to maturity, hence $r \equiv \lim_{\tau \to 0} R(\tau) = R(0)$. Let $B(\tau) \equiv B(t, t + \tau)$ denote the cash price of a coupon-bearing bond with the same properties as a discount bond except for the coupon payments during its life time. The coupon payment on the maturity day includes the redemption value of the bond.

The ω-year forward interest rate, denoted as $F(\tau, \omega) \equiv F(t, t + \tau, t + \tau + \omega)$, corresponds with a forward contract on a pure discount bond with the agreement that the forward price is fixed at the date t and paid at the later date $(t + \tau)$ when the discount bond will be delivered. The discount bond delivered matures at the later date $(t + \tau + \omega)$. The instantaneous forward interest rate, denoted as $f(\tau) \equiv f(t, t + \tau)$, corresponds with a forward contract on a pure discount bond with a vanishing term to maturity. Hence, $f(\tau) = \lim_{\omega \to 0} F(\tau, \omega)$. It holds true that $f(0) = r = R(0)$ and $R(\omega) = F(0, \omega)$.

We approximate the instantaneous forward interest rate by a finite sum of ordinary Laguerre polynomials of degree n, denoted as $L_n(\tau)$, multiplied by the constants c_n ($n = 0, 1, \ldots, N$) as well as by the square root of the corresponding weight function. Since the corresponding weight function is the exponential function, all the terms of the sum die out as the term to maturity increases. Hence, we consider the deviation of the instantaneous forward interest rate from its long-run value $f(\infty)$, that is,

$$f(\tau) = f(\infty) + \mathrm{e}^{-\tau/2} \sum_{n=0}^{N} c_n \, L_n(\tau) \tag{1}$$

which has the advantage to remove undesired oscillations in particular near the boundaries when compared with the standard approach; see appendix A. The $(N + 2)$ constant parameters to be determined are $\{f(\infty), c_0, c_1, \ldots, c_N\}$. We require that $f(\infty) \geqq 0$ as well as $f(\tau) \geqq 0$ for every τ.

Since the instantaneous forward interest rate, evaluated at a vanishing term to maturity, must be equal to the observed instantaneous spot interest rate, denoted as r^{obs}, this allows us to express the long-run instantaneous forward interest rate in terms of the Laguerre constants as follows

$$f(\infty) = r^{\mathrm{obs}} - \sum_{n=0}^{N} c_n \geqq 0 \tag{2}$$

by equation (1) and $L_n(0) = 1$. If the observed yield curve is either normal, flat or inverse, then the sum of Laguerre constants must be negative, zero or positive, respectively.

The objective is to find a continuous function for the instantaneous forward interest rate as smooth as possible subject to three sets of constraints. First, the estimated price of each coupon-bearing bond should not deviate from the observed price by more than a given tolerance, ϵ. The tolerance may arise from the fact that the bond prices are subject to measurement errors, in particular in illiquid markets where not all of the bonds outstanding are traded every day. If a bond is not traded on a particular trading day, the price reported in a common data base is merely the price of the last transaction before this trading day. Hence, we consider half the bid-ask spread in percent of the respective coupon-bond price as the relevant tolerance. Second, the instantaneous forward interest rate should be non-negative for every term to maturity. Third, the long-run instantaneous forward interest rate should be non-negative. Suppose that the sample consists of M coupon-bearing bonds. Let $B^{\text{obs}}(\tau_m)$ denote the observed cash price of the mth coupon-bearing bond with its corresponding term to maturity τ_m, let $B(\tau_m)$ denote the estimated cash price of the mth coupon-bearing bond with its corresponding term to maturity τ_m, let ϵ_m denote the tolerance of the mth bond constraint, then the optimization can be written as follows.

$$\begin{aligned} &\min_{\langle c_0, \ldots, c_N \rangle} G = \int_0^\infty \left(\frac{\mathrm{d}f(t)}{\mathrm{d}t} \right)^2 \mathrm{d}t \,, \quad \text{subject to} \\ &\epsilon_m \geqq \left| \frac{B(\tau_m)}{B^{\text{obs}}(\tau_m)} - 1 \right| 100 \,, \quad \text{for } m = 1, 2, \ldots, M, \\ &f(\tau) \geqq 0, \quad \text{for every } \tau, \\ &f(\infty) = r^{\text{obs}} - \sum_{n=0}^{N} c_n \geqq 0 \,. \end{aligned} \tag{3}$$

There remain $(N + 1)$ instruments, that is, the Laguerre constants $\{c_0, c_1, \ldots, c_N\}$. Disregarding the last two constraints and requiring that all the estimated bond prices match exactly the corresponding observed bond prices, then it must hold true that $N > M - 1$ due to the classical programming condition (Intriligator, 1971). For ‘well-behaved’ yield curves, the last two constraints are always satisfied.[1]

The objective function[2] can be written explicitly as the following quadratic form of the Laguerre constants c_n $(n = 0, 1, \ldots, N)$ by equations (1) and (3).

$$\begin{aligned} G &= \frac{1}{4} \left(\sum_{n=0}^{N} \sum_{m=0}^{N} [c_n + c_{n+1}]\, [c_m + c_{m+1}]\, Q_{n,m} \right), \quad \text{for } c_{N+1} \equiv 0, \\ &= \frac{1}{4}\, \mathbf{c}'\, \mathbf{E}'\, \mathbf{Q}\, \mathbf{E}\, \mathbf{c} \end{aligned} \tag{4}$$

In the first line of the equation above, $Q_{n,m}$ is defined by the following integral

$$Q_{n,m} \equiv \int_0^\infty \mathrm{e}^{-t} L_n^{(1)}(t)\, L_m^{(1)}(t)\, \mathrm{d}t \tag{5}$$

1 It is essential that the objective function is integrated from zero to infinity rather than to a finite upper limit. The solution for a finite term to maturity is shown in appendix B. For a finite upper limit, the objective function is no longer strictly convex.

2 One might argue that the second derivative rather than the first derivative is the appropriate measure of smoothness. However, this is not the case for two reasons. First, the second derivative cannot discriminate between an upward sloping straight line and a horizontal straight line. In both cases, the second derivative is zero, whereas the first derivative is a constant or zero, respectively. Second, the second derivative introduces an “overshooting” of the forward rate in general.

where $L_n^{(\alpha)}(\tau)$ denotes the generalized Laguerre polynomial of nth degree and with (in general real) parameter α ($\alpha > -1$). Note that the generalized Laguerre polynomial reduces to the ordinary Laguerre polynomial for $\alpha = 0$. In the second line of equation (4), the prime denotes transposition and $\mathbf{c} = [c_0, \ldots, c_N]'$ denotes the $(N + 1)$ column vector of Laguerre constants. The diagonal band matrix $\mathbf{E}$ with bandwidth 1 and the symmetric matrix $\mathbf{Q}$ have the following representations.

$$\underset{(N+1)\times(N+1)}{\mathbf{E}} = \begin{bmatrix} 1 & 1 & & 0 \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & \ddots & \ddots \\ 0 & & & & 1 \end{bmatrix} \qquad \underset{(N+1)\times(N+1)}{\mathbf{Q}} = \begin{bmatrix} Q_{0,0} & Q_{0,1} & Q_{0,2} & \cdots & Q_{0,N} \\ & Q_{1,1} & Q_{1,2} & \cdots & Q_{1,N} \\ & & Q_{2,2} & \cdots & Q_{2,N} \\ & & & \ddots & \vdots \\ \text{symmetric} & & & & Q_{N,N} \end{bmatrix} \tag{6}$$

Further, the integrals $Q_{n,m}$ can be reduced to a cumulated sum of integrals in terms of the ordinary Laguerre polynomials, that is,

$$Q_{n,m} \equiv \sum_{k=0}^{n} \sum_{l=0}^{m} I_{k,l} \tag{7}$$

where $I_{k,l}$ is defined by the following integral.

$$I_{k,l} \equiv \int_0^\infty e^{-t} L_k(t)\, L_l(t)\, dt = \begin{cases} 0 & \text{for } k \neq l, \\ 1 & \text{for } k = l. \end{cases} \tag{8}$$

Due to the orthogonality relationship of the Laguerre polynomial, the integrals $I_{k,l}$ are either zero or one. By equations (7) and (8), the matrix $\mathbf{Q}$ becomes the following simple expression.

$$\underset{(N+1)\times(N+1)}{\mathbf{Q}} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ & 2 & 2 & 2 & \cdots & 2 \\ & & 3 & 3 & \cdots & 3 \\ & & & \ddots & \cdots & \vdots \\ & & & & N & N \\ \text{symmetric} & & & & & N+1 \end{bmatrix} \tag{9}$$

The matrix $\mathbf{Q}$ is positive definite and has the Cholesky decomposition $\mathbf{Q} = \mathbf{A}'\,\mathbf{A}$ with

$$\underset{(N+1)\times(N+1)}{\mathbf{A}} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ & 1 & 1 & 1 & \cdots & 1 \\ & & 1 & 1 & \cdots & 1 \\ & & & \ddots & \ddots & \vdots \\ & & & & 1 & 1 \\ 0 & & & & & 1 \end{bmatrix} \tag{10}$$

where $\mathbf{A}$ is an upper triangle matrix. As a consequence, the matrix product $(\mathbf{E}'\,\mathbf{Q}\,\mathbf{E})$ is also positive definite and the objective function G is strictly convex. Since the opportunity set is compact and convex (see appendix B), the solution is both unique and a global minimum (Intriligator, 1971). Since $G = 0$ for $\mathbf{c} = \mathbf{0}$, possible start values for the optimization (3) are $\mathbf{c} = \mathbf{0}$, which, in turn, implies a flat yield curve $f(\tau) = r^{\text{obs}} = f(\infty)$.

The M inequality constraints of the optimization (3) for the coupon-bearing bonds can be expressed in the following way. Since the price of a discount bond is related to its yield to maturity by

$$P(\tau) = \exp(-\,R(\tau)\,\tau) \tag{11}$$

we can write the estimated cash price of a coupon-bearing bond as a portfolio of the discounted coupon payment stream as follows

$$\begin{aligned} B(\tau_m) &= \sum_{j=1}^{D_m} d_{m,j}\, P(\tau_{m,j}) \\ &= \sum_{j=1}^{D_m} d_{m,j} \exp(-\,R(\tau_{m,j})\,\tau_{m,j}), \quad \text{for } m = 1, 2, \ldots, M. \end{aligned} \tag{12}$$

where $d_{m,j}$ ($j = 1, 2, \ldots, D_m$) denotes the jth coupon payment of the mth bond with its corresponding term to payment $\tau_{m,j}$ and D_m the number of coupon cash flows of the mth bond during its life. Remind that the coupon payment on the maturity day includes the redemption value of the bond. The yield to maturity of a discount bond can be expressed in terms of the instantaneous forward interest rate as follows.

$$\begin{aligned} R(\tau) &= \frac{1}{\tau}\int_0^{\tau} f(t)\,\mathrm{d}t \\ &= f(\infty) + \frac{1}{\tau}\sum_{n=0}^{N} c_n\, J_n(\tau) \end{aligned} \tag{13}$$

where $J_n(\tau)$ is given by the following integral of the Laguerre polynomial of the nth degree

$$\begin{aligned} J_n(\tau) &\equiv \int_0^{\tau} \mathrm{e}^{-t/2} L_n(t)\,\mathrm{d}t = (-1)^n\, 2\left[1 - S_n(\tau)\right], \quad \text{where} \\ S_n(\tau) &\equiv \sum_{m=0}^{n} (-1)^{n-m}\, 2^m\, \mathrm{e}^{-\tau/2} L_{n-m}^{(m)}(\tau), \quad \text{with } S_n(0) = 1, \quad S_n(\infty) = 0 \end{aligned} \tag{14}$$

where again $L_n^{(\alpha)}(\tau)$ denotes the generalized Laguerre polynomial. For $N > 30$ and $\tau = 0$, the evaluation of the sum in equation (14) leads to a disastrous loss of digits; see appendix A. Moreover, since we need to calculate the integral $J_n(\tau)$ many times during the optimization, we calculate the sum in the equation above recursively by the following relationship which is numerically reliable (see appendix A).

$$S_n(\tau) = S_{n-1}(\tau) + (-1)^n \, e^{-\tau/2} L_n(\tau) + (-1)^{n-1} \, e^{-\tau/2} L_{n-1}(\tau), \quad \text{for } n = 1, 2, \ldots, \quad \text{where}$$
$$S_0(\tau) = e^{-\tau/2}, \quad \text{and} \tag{15}$$
$$L_n(\tau) = \frac{1}{n}\left\{[2\,n-1-\tau]\,L_{n-1}(\tau) - [n-1]\,L_{n-2}(\tau)\right\}, \quad \text{for } n = 2, 3, \ldots, \quad \text{where}$$
$$L_0(\tau) = 1, \quad L_1(\tau) = 1 - \tau.$$

Note that as $\tau \to 0$, $J_n(0) = 0$, and as $\tau \to \infty$, $J_n(\infty) = (-1)^n\, 2$. In the limit, therefore, we get from equations (2) and (14) $R(0) = r^{\text{obs}}$ and $R(\infty) = f(\infty)$.

The second set of constraints of the optimization (3) can be treated in the following way. The instantaneous forward interest rate is non-negative for every time period if the minima of this function remain non-negative. The first derivative of the instantaneous forward interest rate can be written as follows.

$$\frac{\mathrm{d}f(\tau)}{\mathrm{d}\tau} = -\frac{1}{2}\, e^{-\tau/2}\, Z_N(\tau, \mathbf{c}), \quad \text{with } Z_N(\tau, \mathbf{c}) \equiv \sum_{n=0}^{N} \left[c_n + c_{n+1}\right] L_n^{(1)}(\tau), \quad (c_{N+1} \equiv 0) \tag{16}$$

where $Z_N(\tau, \mathbf{c})$ denotes a polynomial of degree N and, again, $L_n^{(\alpha)}(\tau)$ the generalized Laguerre polynomial. An extremum point is obtained if, and only if, $Z_N(\tau, \mathbf{c}) = 0$. Let ζ_j denote the roots of $Z_N(\tau, \mathbf{c})$ (which are all real, positive and distinct; see Szegö, 1939) for which the second derivative of the instantaneous forward interest rate

$$\frac{\mathrm{d}^2 f(\zeta_j)}{\mathrm{d}\zeta_j^2} = \frac{1}{4}\, e^{-\zeta_j/2} \left\{ \sum_{n=0}^{N} \left[c_n + 2\,c_{n+1} + c_{n+2}\right] L_n^{(2)}(\zeta_j) \right\}, \quad (j \leqq \frac{N}{2},\, c_{N+1} \equiv c_{N+2} \equiv 0) \tag{17}$$

is positive, then the second set of constraints, $f(\tau) \geqq 0$ for every τ, can be replaced by

$$f(\zeta_j) = f(\infty) + e^{-\zeta_j/2} \sum_{n=0}^{N} c_n\, L_n(\zeta_j) \geqq 0, \quad (j \leqq \frac{N}{2}) \tag{18}$$

These are $(N / 2)$ equations at most.

Before presenting the applications in the next section, three comments are noteworthy. First, our approach does not need any discount bond as with the bootstrap method. The whole sample may consist of coupon-bearing bonds, only. Second, our approach can treat bonds with the same maturity dates as often observed in liquid markets (e. g., the market for U. S. treasury bonds, notes and bills). Of course, all the bonds considered are meant to belong to the same credit class. Third, bonds are traded infrequently in illiquid markets. Although all the outstanding bonds are reported in today's data base, some bonds may have been traded yesterday or earlier for the last time. Hence, not all of the reported bond prices may be in line with today's yield curve. For zero tolerances, the estimated yield curve is more volatile in illiquid markets than in liquid markets, because the estimated bond prices must match the observed bond prices. Since the bid-ask spread is larger in illiquid markets than in liquid markets, this helps to smooth the estimated yield curve at the expense of less accuracy. This will be made clear in the first example of the next section.

2 Applications

We present two examples. The first example considers an exponential forward rate, the second example an oscillating forward rate which, in turn, implies a humped spot rate. The maximum term to maturity is six years in the first example and thirty years in the second example. Both samples consist of coupon-bearing bonds only. In both examples, the tolerances, ϵ_m for $m = 1, 2, \ldots, M$, have been set initially equal to zero to investigate the accuracy of our approach. To show the effect of an illiquid market, this assumption will be relaxed later.

In the first example, we consider an exponential instantaneous forward interest rate which, in turn, implies an exponential spot interest rate as follows.

$$\begin{aligned} f^*(\tau) &= f_\infty^* + \left[r^{\text{obs}} - f_\infty^*\right] \exp(-b\,\tau)\,, \quad \text{where } b = 0.1, \quad r^{\text{obs}} = 0.09, \quad f_\infty^* = 0.04, \\ R^*(\tau) &= f_\infty^* + \frac{\left[r^{\text{obs}} - f_\infty^*\right]}{\tau\, b} \left[1 - \exp(-b\,\tau)\right]. \end{aligned} \tag{19}$$

This is the function considered in equation (A-11) of appendix A. The forward rate and the spot rate of equation (19) are depicted in Figure 1a. We consider three supporting bonds with a maximum term to maturity of six years. The characteristics of these bonds are summarized in Table 1. The error of the estimated instantaneous forward interest rates is depicted in Figure 1b. The maximum error is 11 basis points for $N = 20$. The error of the estimated spot interest rates is depicted in Figure 1c. The maximum error is one-half basis point for $N = 20$.

Table 1: Bonds of example 1

Number	Term to maturity	Coupon rate p.a.	Cash price [1]	Perturbation [2]
1	6	10%	118.3893	0
2	4.5	5%	90.1447	– 0.5
3	2.8	7%	97.1891	0

[1] Cash price implied by the theoretical spot interest rate of equation (19).

[2] Perturbation of cash prices for an illiquid market.

Next, we extend the first example to show the effect of an illiquid market. Suppose that the second bond price is not in line with today’s yield curve as given by equation (19) because, say, the bond has been traded some days ago for the last time. For this reason, its theoretical cash price is perturbed by a large value; see last column of Table 1. We distinguish two cases. In the first case, the tolerances are still zero. In order to get an equality of estimated and observed bond prices, the estimated forward rate must oscillate around the theoretical forward rate. In the second case, the tolerances are set to the extremely large value of one percent, sometimes encountered in illiquid markets. Since the estimated bond prices can now deviate from the observed prices, the estimated forward rate becomes much smoother. The result is depicted in Figures 1d and 1e.

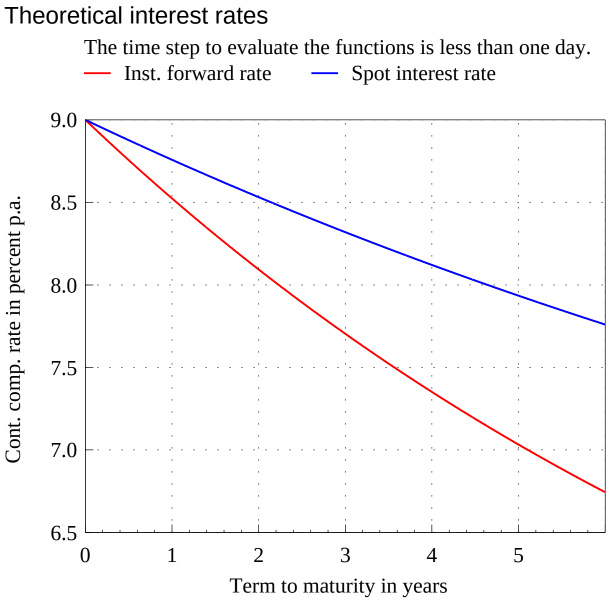

Figure 1a

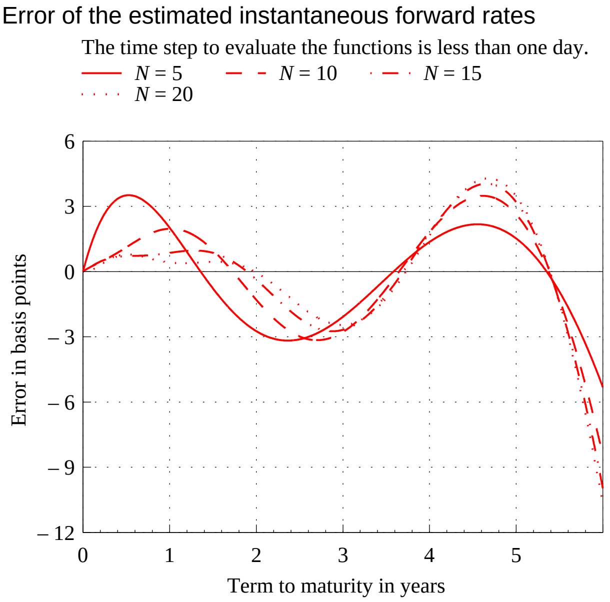

Figure 1b

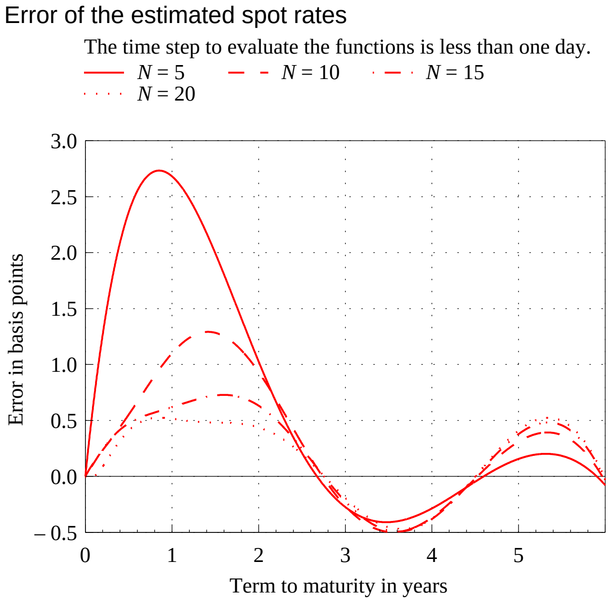

Figure 1c

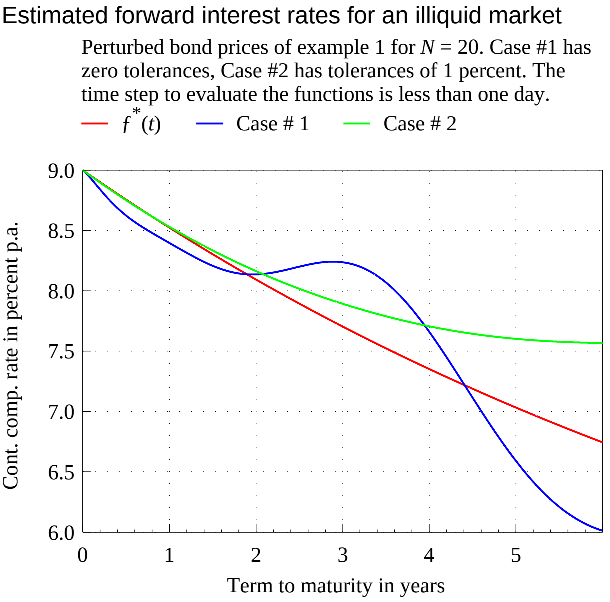

Figure 1d

In the second example, we consider an oscillating instantaneous forward interest rate as follows.

$$f^*(\tau) = \exp(-a\,\tau)\left[\sin(n_1\,\kappa\,\tau) + \cos(n_2\,\kappa\,\tau) + \sin^2(n_3\,\kappa\,\tau) - \cos^2(n_4\,\kappa\,\tau)\right] + k, \quad \text{where}$$
$$\kappa = \frac{2\,\pi}{T}, \quad a = 0.1, \quad n_1 = 5, \quad n_2 = 1, \quad n_3 = 2, \quad n_4 = 3, \quad k = 6, \quad T = 30. \tag{20}$$

Note that equation (20) measures the forward rate in percent p. a. The forward rate above implies a humped spot interest rate, measured in percent p. a., as follows.

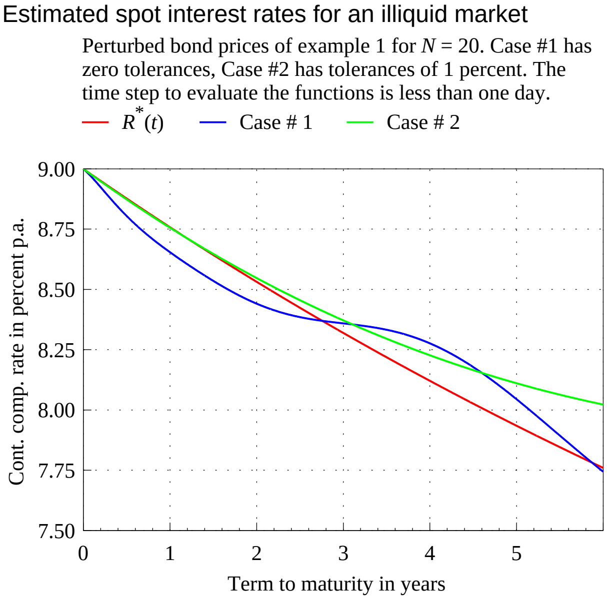

Figure 1e

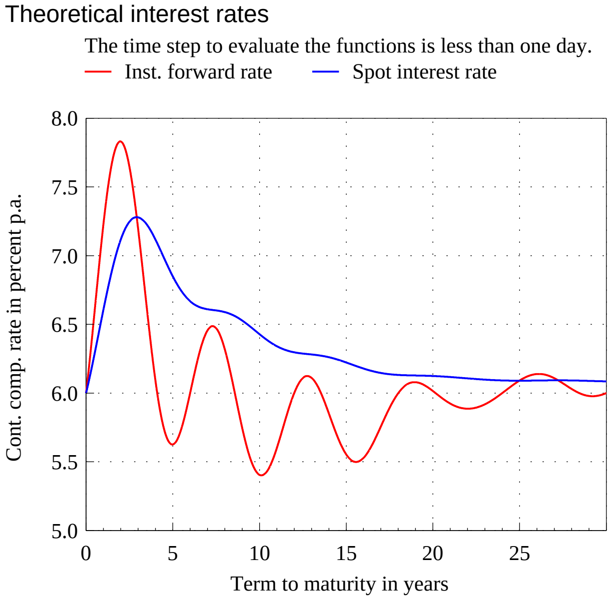

Figure 2a

$$
\begin{aligned}
R^*(\tau) &= \frac{1}{\tau}\left[\exp(-a\,t)\left(I_1(t)+I_2(t)+I_3(t)-I_4(t)\right)\right]_{t=0}^{t=\tau}+k, \quad \text{where}\\
I_1(t) &= \frac{-a\sin(n_1\,\kappa\,t)-n_1\,\kappa\cos(n_1\,\kappa\,t)}{a^2+(n_1\,\kappa)^2}\\
I_2(t) &= \frac{-a\cos(n_2\,\kappa\,t)+n_2\,\kappa\sin(n_2\,\kappa\,t)}{a^2+(n_2\,\kappa)^2}\\
I_3(t) &= \frac{\left[-a\sin(n_3\,\kappa\,t)-2\,n_3\,\kappa\cos(n_3\,\kappa\,t)\right]\sin(n_3\,\kappa\,t)-\frac{2\,(n_3\,\kappa)^2}{a}}{a^2+4\,(n_3\,\kappa)^2}\\
I_4(t) &= \frac{\left[-a\cos(n_4\,\kappa\,t)+2\,n_4\,\kappa\sin(n_4\,\kappa\,t)\right]\cos(n_4\,\kappa\,t)-\frac{2\,(n_4\,\kappa)^2}{a}}{a^2+4\,(n_4\,\kappa)^2}
\end{aligned}
\tag{21}
$$

The forward rate and the spot rate of equations (20) and (21) are depicted in Figure 2a. The spot interest rate first rises monotonically and then declines monotonically. Note that a humped spot interest rate is one of the three possible shapes which are explained by all the well-known one-factor models of the term structure of interest rates including Vasicek [1977] and Cox, Ingersoll and Ross [1985]. These one-factor models rely on the liquidity preference theory which excludes oscillating forward rates. Indeed, a humped spot rate is derived from a non-oscillating forward rate in these one-factor models. The purpose of our second example is to show that a humped spot interest rate could be the outcome of an oscillating forward interest rate as well, often observed in illiquid markets.

We consider 31 evenly distributed supporting bonds with a maximum term to maturity of thirty years. The characteristics of these bonds are summarized in Table 2. The error of the estimated instantaneous forward interest rates is depicted in Figure 2b. The maximum error is

1.5 basis points for $N = 80$. The error of the estimated spot interest rates is depicted in Figure 2c. The maximum error is 0.27 basis point for $N = 80$.

Table 2: Bonds of example 2

Number	Term to maturity	Coupon rate p.a.	Cash price [1]
1	0.5000	2.0000	98.8416
2	1.0000	2.2500	97.9637
3	2.0000	2.5000	93.7408
4	3.0000	2.7500	90.3022
5	4.0000	3.0000	88.3028
6	5.0000	3.2500	87.4767
7	6.0000	3.5000	87.1113
8	7.0000	3.7500	86.8404
9	8.0000	4.0000	86.8651
10	9.0000	4.2500	87.5051
11	10.0000	4.5000	88.7579
12	11.0000	4.7500	90.3369
13	12.0000	5.0000	92.0119
14	13.0000	5.2500	93.8084
15	14.0000	5.5000	95.8854
16	15.0000	5.7500	98.3112
17	16.0000	6.0000	100.9994
18	17.0000	6.2500	103.8181
19	18.0000	6.5000	106.7081
20	19.0000	6.7500	109.6954
21	20.0000	7.0000	112.8249
22	21.0000	7.2500	116.1097
23	22.0000	7.5000	119.5304
24	23.0000	7.7500	123.0572
25	24.0000	8.0000	126.6650
26	25.0000	8.2500	130.3389
27	26.0000	8.5000	134.0764
28	27.0000	8.7500	137.8849
29	28.0000	9.0000	141.7710
30	29.0000	9.2500	145.7291
31	30.0000	9.5000	149.7425

[1] Cash price implied by the theoretical spot interest rate of equation (21).

3 Comparison with Other Methods

The most popular method among practitioners is the bootstrap method (Hull, 2003; Choudhry, 2005). It starts from a set of discount bonds and calculates the spot interest rate for coupon-bearing bonds by linear interpolation and extrapolation of the spot rates obtained from the discount bonds. It works quite well, first, if a set of discount bonds is avalaible and, second, if a few interpolations and extrapolations have to be done. A sufficient accuracy is obtained, if the terms to maturity of the coupon-bearing bonds are evenly distributed and, if there is a sufficient number of bonds available. A disadvantage of the bootstrap method is the fact that the spot interest rates can be obtained only at the cash flow dates of the coupon-bearing bonds. Moreover, it does not work without discount bonds.

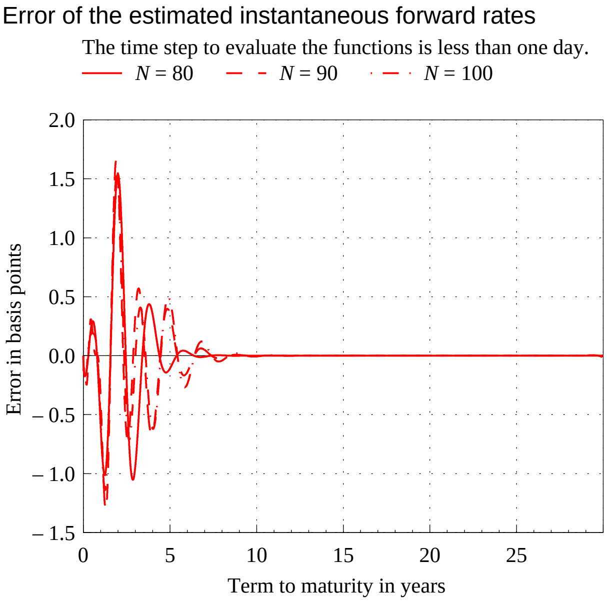

Figure 2b

Figure 2c

Carleton and Cooper [1976] derive the prices of discount bonds from a regression of coupon-bearing bonds. Shea [1984] has shown that this method has two disadvantages. First, it generates cash flow matrices which are mostly singular. To avoid this problem, Carleton and Cooper selected bonds with the same cash flow dates only which is unduely restrictive. Second, discount bond prices can be obtained only at the cash flow dates.

The use of a quadratic or cubic spline function has been proposed first by McCulloch [1971]. Vasicek and Fong [1982] applied first an exponential cubic spline. Shea [1984, 1985] has shown that this method has three disadvantages. First, the system of the spline function is not closed. Two parameters can be chosen by the user. Second, the forward interest rate may be instable for long terms to maturity. Third, the numerical accuracy of the estimated spot interest rate depends crucially both on the location of the spline knots and the number of the spline knots. Moreover, many spline methods introduce a tradeoff between the smoothness of the estimated spot rate and the accuracy by considering a penalty function multiplied by a penalty weight. The latter can be chosen by the user.

Nelson and Siegel [1987] proposed to approximate the forward interest rate by a combination of three exponential functions. Svensson [1995] extended this approach by considering one more exponential function. Clearly, the Nelson-Siegel-Svensson method cannot fit our second example because it can treat two local extremal points at most. Moreover, first, it is model-dependent because it cannot converge to any observed forward rate and, second, it often violates the condition that the estimated bond prices should lie within the bid-ask spread.

The “forward-rate” method has been proposed first by Delbaen and Lorimier [1992] and Lorimier [1995]. The focus of this method lies on the estimation of the instantaneous forward interest rate rather than the spot interest rate. Since the latter is obtained from the average of the integrated forward rates, it clearly improves the numerical accuracy. The “forward-rate” method makes the forward rate as smooth as possible while attaining the required accuracy of the estimated bond prices by means of a multi-objective goal attainment algorithm. The advantage of this method is fivefold. First, it can estimate the spot rate for any term to maturity. Second, it converges to any observed forward rate, i. e., it is model-independent. Third, the estimated bond prices lie within the bid-ask spread. Fourth, it can estimate the forward rate from coupon-bearing bonds only. Discount bonds are not required. Five, it does not require a large set of bonds. One bond is sufficient. For instance, the bootstrap method fails with one coupon-bearing bond only. The disadvantage of this method is that the user can choose the weights of the multi-objective goal algorithm.

In this paper, we use a convex nonlinear programming version of the “forward-rate” method in continuous time which removes the subjective choice as regards the weights of the objectives (Büttler, 2000). It also removes the subjective tradeoff between smoothness and accuracy as encountered with many spline methods. In our approach, the tradeoff between smoothness and accuracy is obtained in the following way. Setting all the Laguerre constants equal to zero at the beginning of the optimization, the global unconstrained minimum is attained which is zero, i. e, $G = 0$, which in turn implies a flat yield curve (horizontal line), i. e., $f(\tau) = r^{\text{obs}} = f(\infty)$. If the observed prices of the coupon-bearing bonds imply a flat yield curve, then we have found already the global constrained minimum. No further optimization step is needed. If the observed prices of the coupon-bearing bonds do *not* imply a flat yield curve, then we have to climb up the “hill” of the strictly convex objective function until the estimated prices of the coupon-bearing bonds do not deviate from the observed prices by more than the given tolerances. Since the solution is both unique and a global constrained minimum, we are sure that we get the smoothest possible forward rate for the required accuracy.

Finally, it should be noted that there exists also a discrete-time version of the optimization of equation (3) as desribed in Büttler (2000), which has been used to estimate more than 23,000 yield curves of the Swiss bond markets since the beginning of 2003.

References

Abramowitz, Milton and Irene A. Stegun [1972]: *Handbook of Mathematical Functions*, New York: Dover Publications, Inc., tenth edtion.

Bolder, David Jamieson and Scott Gusba [2002]: “Exponentials, Polynomials, and Fourier Series: More Yield Curve Modelling at the Bank of Canada”, *Bank of Canada Working Paper*, 2002-29.

Büttler, Hans-Jürg [2000]: “The Term Structure of Expected Inflation Rates”, *Economic and Financial Modelling*, 7 (1): 3 – 45.

Carleton, Willard T.and Ian A. Cooper [1976]: “Estimation and Uses of the Term Structure of Interest Rates”, *The Journal of Finance*, 31 (4): 1067 – 1083.

Choudhry, Moorad [2005]: “The Market Yield Curve and Fitting the Term Structure of Interest Rates”, in Frank J. Fabozzi (ed.): *The Handbook of Fixed Income Securities*, chapter 41: 939 – 965, seventh edition.

Cox, John C., Jonathan E. Ingersoll, Jr., and Stephen A. Ross [1985]: “A Theory of the Term Structure of Interest Rates”, *Econometrica*, 53: 385 – 407.

Delbaen Freddy and Sabine Lorimier [1992]: “Estimation of the Yield Curve and the Forward Rate Curve starting from a Finite Number of Observations”, *Insurance, Mathematiccs and Economics*, 11: 259 – 269.

Erdélyi, Arthur, Wilhelm Magnus, Fritz Oberhettinger, and Franceso G. Tricomi [1954]: *Tables of Integral Transforms*, New York: McGraw-Hill Book Company, Inc., volume 1.

Hull, John C. [2003]: *Options, Futures, and Other Derivatives*, New York: Prentice Hall International, Inc., fifth edition.

Intriligator, Michael D. [1971]: *Mathematical Optimization and Economic Theory*, Englewood Cliffs (N.J.): Prentice-Hall, Inc.

Ioannides, Michalis [2003]: “A comparison of yield curve estimation techniques using UK data”, *Journal of Banking and Finance*, 27: 1 – 26.

Lorimier, Sabine [1995]: *Interest Rate Term Structure Estimation Based on the Optimal Degree of Smoothness of the Forward Rate Curve*, Doctoral Dissertation of the University of Antwerp (Belgium).

McCulloch, J. Huston [1971]: “Measuring the Term Structure of Interest Rates”, *The Journal of Business*, 34 (1): 19 – 31.

Nelson, Charles R. and Andrew F. Siegel [1987]: “Parsimonious Modeling of Yield Curves”, *The Journal of Business*, 60 (4): 473 – 489.

Shea, Gary S. [1984]: “Pitfalls in Smoothing Interest Rate Term Structure Data: Equilibrium Models and Spline Approximations”, *Journal of Financial and Quantitative Analysis*, 19 (3): 253 – 269.

Shea, Gary S. [1985]: “Interest Rate Term Structure Estimation with Exponential Splines: A Note”, *The Journal of Finance*, 40 (1): 319 – 325.

Svensson, Lars E. O. [1995]: “Estimating Forward Rates with the Extended Nelson & Siegel Method”, *Sveriges Riksbank Quarterly Review*, 3: 13 – 26.

Szegö, Gabor [1939]: *Orthogonal Polynomials*, Providence (R. I.): American Mathematical Society, Colloquium Publications, volume XXIII.

Vasicek, Oldrich [1977]: “An Equilibrium Characterization of the Term Structure”, *Journal of Financial Economics*, 5: 177 – 188.

Vasicek, Oldrich A. and H. Gifford Fong [1982]: “Term Structure Modeling Using Exponential Splines”, *The Journal of Finance*, 37 (2): 339 – 348.

Appendix A: Some Properties of the Laguerre Polynomial

In this appendix, we collect some properties of the Laguerre polynomial which are relevant for the text, show the advantage of our approximating function of equation (1) and demonstrate the numerical accuracy of equations (14) and (15).

The generalized Laguerre polynomial, denoted as $L_n^{(\alpha)}(x)$, of degree n and parameter α can be expressed explicitly as follows.

$$L_n^{(\alpha)}(x) = \sum_{m=0}^{n} (-1)^m \binom{n+\alpha}{n-m} \frac{1}{m!} x^m, \quad 0 \leqq x \leqq \infty, \quad \alpha > -1. \tag{A-1}$$

The ordinary Laguerre polynomial, denoted as $L_n(x) \equiv L_n^{(0)}(x)$, is obtained from setting $\alpha = 0$. It satisfies the following orthogonality relationship, where $\omega(x)$ denotes the weight function corresponding with the Laguerre polynomial.

$$\int_0^\infty \omega(x)\, L_n(x)\, L_m(x)\, dx = \begin{cases} 0, & \text{for } n \neq m;\ n = m = 0, 1, 2, \ldots \\ 1, & \text{for } n = m;\ n = 0, 1, 2, \ldots \end{cases} \tag{A-2}$$

In the case of the ordinary Laguerre polynomial, the weight function is defined as $\omega(x) \equiv \exp(-x)$. In the derivation of some equations of the text, we use also Rodrigues' formula for the generalized Laguerre polynomial as follows.

$$L_n^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{n!} \frac{d^n\{e^{-x} x^{\alpha+n}\}}{dx^n} \tag{A-3}$$

The Laguerre polynomials of degree 0 through n can efficiently be computed from the following recurrence relationship.

$$\begin{aligned} &L_0(x) = 1, \quad L_1(x) = 1 - x, \\ &L_n(x) = \frac{1}{n}\{[2n - 1 - x]\, L_{n-1}(x) - [n-1]\, L_{n-2}(x)\}, \quad \text{for } n = 2, 3, \ldots \end{aligned} \tag{A-4}$$

The following special values are used in the text.

$$L_0^{(\alpha)}(x) = 1, \quad L_1^{(\alpha)}(x) = 1 - x + \alpha, \quad L_n^{(\alpha)}(0) = \binom{n+\alpha}{n}, \quad L_n(0) = 1. \tag{A-5}$$

Next, we show that the square root of the Laguerre weight function is appropriate for a least-squares determination of the constants, c_n. Let $g_n(t)$ denote the Laguerre polynomial of degree n multiplied by the square root of its corresponding weight function, for n= 0, 1, ..., N. Let g(t) denote the finite series of the functions $g_n(t)$ multiplied by the constants c_n as follows.

$$g(t) \equiv \sum_{n=0}^{N} c_n\, g_n(t), \quad g_n(t) \equiv \sqrt{\omega(t)}\, L_n(t), \quad \sqrt{\omega(t)} \equiv e^{-t/2}. \tag{A-6}$$

Let $h(t)$ denote a given real-valued function defined on $(0, \infty)$, possibly after a suitable transformation. Suppose that we wish to minimize the squared deviation of the approximating function $g(t)$ from the given function $h(t)$ with respect to the constants c_n as follows.

$$\min_{\langle c_0,\, ...,\, c_N\rangle} \left\{ F \equiv \int_0^\infty \left[h(t) - g(t)\right]^2 dt = \int_0^\infty \left[h(t) - \sum_{n=0}^{N} c_n\, g_n(t)\right]^2 dt \right\} \tag{A-7}$$

Setting the first derivatives with respect to the constant c_m equal to zero, i. e. $\partial F / \partial c_m = 0$, for $m = 0, 1, ..., N$, then we get the following equations.

$$\int_0^\infty h(t)\sqrt{\omega(t)}\; L_m(t)\, dt = \sum_{n=0}^{N} \int_0^\infty c_n\, \omega(t)\, L_n(t)\, L_m\, dt\,, \quad m = 0, 1, 2, ..., N. \tag{A-8}$$

Due to the fact that we used the square root of the weight function rather than the weight function itself, the integrals on the right-hand side reduce by the orthogonality relationship (A-2) to the following expression.

$$\sum_{n=0}^{N} \int_0^\infty c_n\, \omega(t)\, L_n(t)\, L_m\, dt = c_m \int_0^\infty \omega(t)\, L_m^2(t)\, dt = c_m\,, \quad m = 0, 1, 2, ..., N. \tag{A-9}$$

Combining equation (A-9) with equation (A-8) results in the following expression for the constants c_m.

$$c_m = \int_0^\infty h(t)\, g_m(t)\, dt = \int_0^\infty h(t)\sqrt{\omega(t)}\; L_m(t)\, dt, \quad m = 0, 1, 2, ..., N. \tag{A-10}$$

Note that this result holds true for an infinite series of $g(t)$ as well.

In the following example, we show that our approximating function of equation (1) rather than the standard approach removes undesired oscillations. Suppose that the instantaneous forward interest rate, $h(t)$, is given by the following exponential function

$$h(t) = h_\infty + \left[r^{\text{obs}} - h_\infty\right] \exp(-\, b\, t)\,, \quad \text{where } b = 0.1, \quad r^{\text{obs}} = 0.09, \quad h_\infty = 0.04, \tag{A-11}$$

which we first approximate by the standard function $g(t)$ as given in equation (A-6). After inserting equation (A-11) into eq. (A-10), we get by equation (25) in Erdélyi et al. [1954, p. 174] the following expression for the constants c_m.

$$c_m = (-\,1)^m\, 2\, h_\infty + \left[r^{\text{obs}} - h_\infty\right] \left(b - \tfrac{1}{2}\right)^m \left(b + \tfrac{1}{2}\right)^{-m-1}, \quad m = 0, 1, 2, ..., N. \tag{A-12}$$

Figures A-1a and A-1b show that the standard approximating function, $g(t)$, oscillates around the given function, $h(t)$, in particular near the left-hand boundary, a phenomenon, which can be observed with Fourier series as well. However, if we use our approximating function $f(t)$

$$f(t) = h_\infty + g(t), \quad g(t) \equiv \sum_{n=0}^{N} c_n\, g_n(t)\,, \quad g_n(t) \equiv \sqrt{\omega(t)}\; L_n(t), \quad \sqrt{\omega(t)} \equiv \mathrm{e}^{-t/2}. \tag{A-13}$$

as suggested by equation (1), then the constants c_m are given by the following expression (Erdélyi et al. [1954, p. 174], equation 25).

$$c_m = \left[r^{\text{obs}} - h_\infty \right] \left(b - \tfrac{1}{2} \right)^m \left(b + \tfrac{1}{2} \right)^{-m-1}, \quad m = 0, 1, 2, \ldots, N. \tag{A-14}$$

Figures A-1a and A-1b show that our approximating function, $f(t)$, has removed completely the oscillations around the given function, $h(t)$, near the left-hand boundary. We need $N = 10$ for our approximating function $f(t)$, to obtain a root mean squared error (RMSE) of 0.98 basis points (bps) and a maximum error in absolute value (ME) of 5.8 basis points, given a time step of 0.01 years. For $N = 20$, RMSE = 0.013 bps and ME = 0.1 bps.

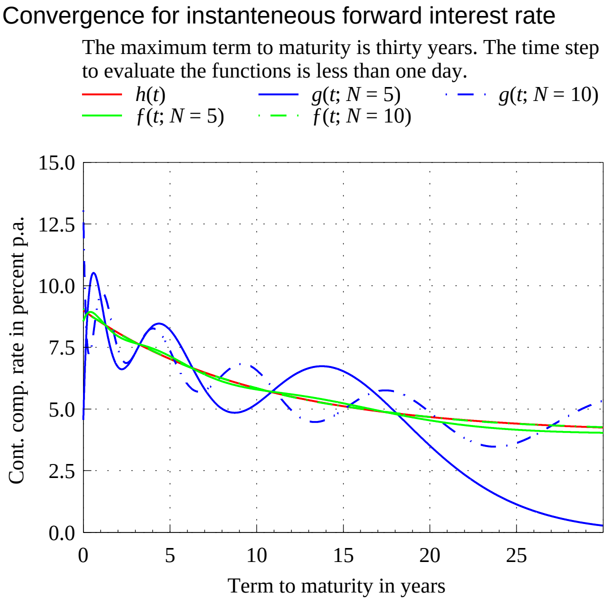

Figure A-1a

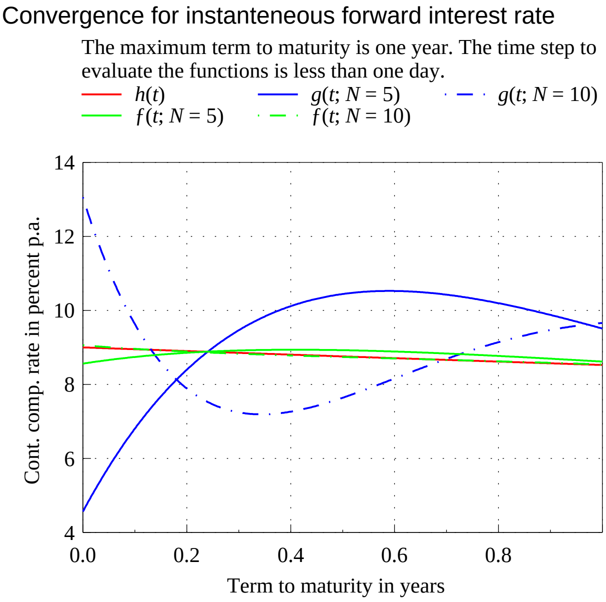

Figure A-1b

Moreover, the long-run value $f(\infty)$ of our approximating function (1) converges quite fast towards the long-run value of the given function h_∞ as follows.

$$\begin{aligned}
\lim_{N \to \infty} f(\infty) &= \lim_{N \to \infty} \left(r^{\text{obs}} - \sum_{n=0}^{N} c_n \right), && \text{by (2)} \\
&= \lim_{N \to \infty} \left(r^{\text{obs}} - \left[r^{\text{obs}} - h_\infty \right] \left[1 - \left(\frac{b - \frac{1}{2}}{b + \frac{1}{2}} \right)^{N+1} \right] \right), && \text{by (A-14)} \\
&= h_\infty
\end{aligned} \tag{A-15}$$

Figures A-2a and A-2b show the convergence of the approximated spot rate for both functions, $f(t)$ and $g(t)$. Again, our approximating function clearly outperforms the standard one.

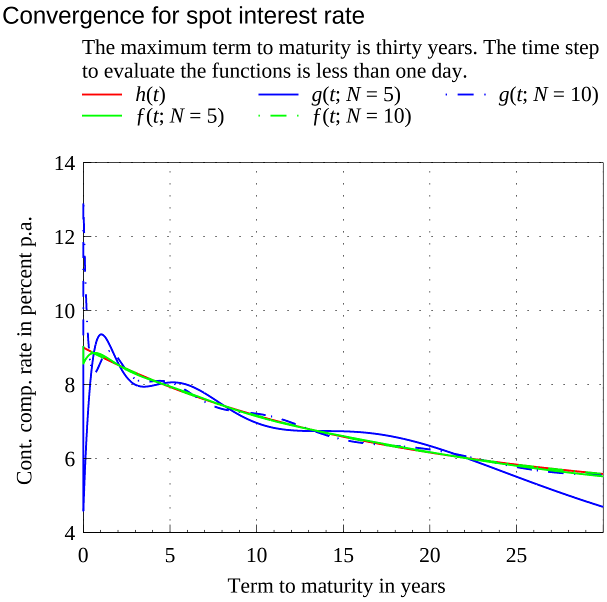

Figure A-2a

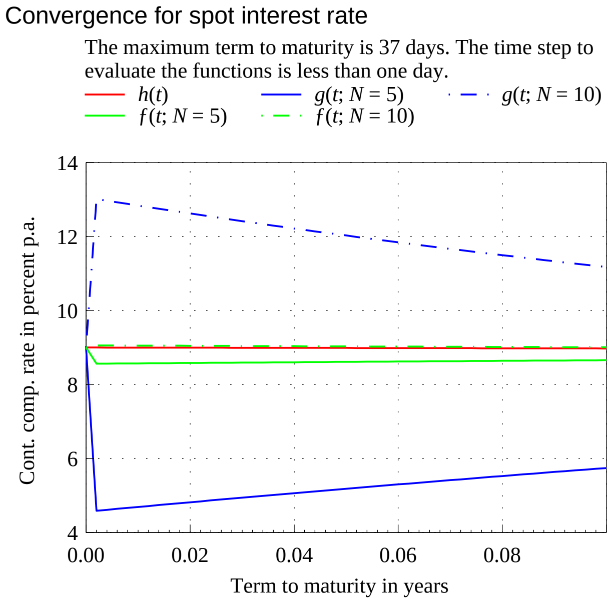

Figure A-2b

Another way to exploit the orthogonality relationship is to use a finite series of Laguerre polynomials excluding the weight function from the approximating function denoted as $z(t)$, that is,

$$z(t) \equiv \sum_{n=0}^{N} c_n L_n(t) . \tag{A-16}$$

Now, suppose that we wish to minimize the *weighted* squared deviation of the approximating function $z(t)$ from the given function $h(t)$ with respect to the constants c_n as follows.

$$\min_{\{c_0, \ldots, c_N\}} \left\{ F \equiv \int_0^\infty \omega(t) \left[h(t) - z(t)\right]^2 dt = \int_0^\infty \omega(t) \left[h(t) - \sum_{n=0}^{N} c_n L_n(t)\right]^2 dt \right\} \tag{A-17}$$

which imposes a weight on the squared errors according to the weight function of the Laguerre polynomial. Repeating the same procedure as described above, we get for the constants c_m the following expression.

$$c_m = \int_0^\infty \omega(t)\, h(t)\, L_m(t)\, dt\,, \quad m = 0, 1, 2, \ldots, N. \tag{A-18}$$

Equations (A-18) differs from equation (A-10) by the weight function. Since the approximating function $z(t)$ grows infinitely large as time approaches infinity, it is not suited for the problem at hand.

Finally, we give an example for the loss of digits when evaluating equations (14) and (15), given $\tau = 0$. In this case, the sum in equation (14) becomes

$$S_n(0) = \sum_{m=0}^{n} (-1)^{n-m} \, 2^m \binom{n}{n-m} = 1 \tag{A-19}$$

which is equal to one due to the binomial theorem.

Table A-1: Numerical error of equations (14) and (15)

	Equation (14)		Equation (15)	
n	Sum	Error	Recurrence	Error
0	1.0000e+00	0.0000e+00	1.0000e+00	0.0000e+00
10	1.0000e+00	0.0000e+00	1.0000e+00	0.0000e+00
20	1.0000e+00	0.0000e+00	1.0000e+00	0.0000e+00
30	1.0000e+00	0.0000e+00	1.0000e+00	0.0000e+00
40	0.0000e+00	1.0000e+00	1.0000e+00	0.0000e+00
50	0.0000e+00	1.0000e+00	1.0000e+00	0.0000e+00
60	– 3.0237e+12	3.0237e+12	1.0000e+00	0.0000e+00
70	– 1.6213e+17	1.6213e+17	1.0000e+00	0.0000e+00
80	– 5.9767e+21	5.9767e+21	1.0000e+00	0.0000e+00
90	– 4.7843e+26	4.7843e+26	1.0000e+00	0.0000e+00
100	– 1.8535e+32	1.8535e+32	1.0000e+00	0.0000e+00
150	– 4.7979e+55	4.7979e+55	1.0000e+00	0.0000e+00
200	2.1313e+78	2.1313e+78	1.0000e+00	0.0000e+00
250	– 5.4095e+103	5.4095e+103	1.0000e+00	0.0000e+00
300	– 1.3448e+126	1.3448e+126	1.0000e+00	0.0000e+00

Appendix B: Proofs of the Equations of the Main Text

In this appendix, we proof various equations of the text. The first derivative of the instantaneous forward interest rate is given by

$$
\begin{aligned}
\frac{\mathrm{d}f(\tau)}{\mathrm{d}\tau} &= -\,\mathrm{e}^{-\tau/2}\left\{\frac{1}{2}\sum_{n=0}^{N} c_n\, L_n(\tau) - \sum_{n=0}^{N} c_n\,\frac{\mathrm{d}L_n(\tau)}{\mathrm{d}\tau}\right\}\\
&= -\,\mathrm{e}^{-\tau/2}\left\{\frac{1}{2}\sum_{n=0}^{N} c_n\, L_n(\tau) + \sum_{n=1}^{N} c_n\, L_{n-1}^{(1)}(\tau)\right\}, && \text{by (22.5.17)}\\
&= -\,\frac{1}{2}\,\mathrm{e}^{-\tau/2}\left\{c_0 + \sum_{n=1}^{N} c_n\, L_n^{(1)}(\tau) + \sum_{n=1}^{N} c_n\, L_{n-1}^{(1)}(\tau)\right\}, && \text{by (22.7.30)}\\
&= -\,\frac{1}{2}\,\mathrm{e}^{-\tau/2}\left\{\sum_{n=0}^{N}\left[c_n + c_{n+1}\right] L_n^{(1)}(\tau)\right\}, && \text{for } c_{N+1}\equiv 0.
\end{aligned}
\tag{B-1}
$$

where the numbers in brackets in the equations above refer to the equation numbers in Abramowitz and Stegun [1972]. This is equation (16) of the text. Substituting equation (B-1) into the objective function (3) yields equations (4) – (5) of the text.

Repeated substitution of equation (22.7.30) in Abramowitz and Stegun [1972] establishes the fact that

$$
L_n^{(\alpha)}(t) = \sum_{m=0}^{n} L_m^{(\alpha-1)}(t) \tag{B-2}
$$

Substitution of this equation into equation (5) results in equations (7) - (9) of the text.

The integral $J_n(\tau)$ can be obtained in the following way.

$$
\begin{aligned}
J_n(\tau) &\overset{\text{def}}{=} \int_0^{\tau} \mathrm{e}^{-t/2}\, L_n(t)\,\mathrm{d}t\\
&= -\,2\sum_{m=0}^{n}\left[2^m\,\mathrm{e}^{-t/2}\,\frac{\mathrm{d}^m L_n(t)}{\mathrm{d}t^m}\right]_0^{\tau}, && \text{by repeated integration by parts}\\
&= (-1)^{n+1}\,2\sum_{m=0}^{n}\left[(-1)^{n-m}\,2^m\,\mathrm{e}^{-t/2}\,L_{n-m}^{(m)}(t)\right]_0^{\tau}, && \text{by (22.5.17)}\\
&= (-1)^n\,2\left[1 - \sum_{m=0}^{n}(-1)^{n-m}\,2^m\,\mathrm{e}^{-\tau/2}\,L_{n-m}^{(m)}(\tau)\right], && \text{by (22.4.7) and (3.1.1)}
\end{aligned}
\tag{B-3}
$$

This is eqution (14) of the text, where the sum in square brackets is denoted as $S_n(\tau)$. The recurrence relationship of equation (15) of the text can be derived from equation (22.7.30) in Abramowitz and Stegun [1972] as well as from changing the summation index in an appropriate manner.

Combining equations (22.5.54) and (13.4.9) in Abramowitz and Stegun [1972] establishes the fact that

$$
\frac{\mathrm{d}^m L_n^{(\alpha)}(x)}{\mathrm{d}x^m} = (-1)^m\, L_{n-m}^{(\alpha+m)}(x)\,, \quad \text{for } m \leqq n \tag{B-4}
$$

which is a generalization of equation (22.5.17) in Abramowitz and Stegun [1972]. The second derivative of the instantaneous forward interest rate can then be written as follows.

$$\begin{aligned}
\frac{\mathrm{d}^2 f(\tau)}{\mathrm{d}\tau^2} &= \frac{1}{2}\,\mathrm{e}^{-\tau/2}\left\{\frac{1}{2}\sum_{n=0}^{N}\left[c_n + c_{n+1}\right] L_n^{(1)}(\tau) - \sum_{n=0}^{N}\left[c_n + c_{n+1}\right]\frac{\mathrm{d}L_n^{(1)}(\tau)}{\mathrm{d}\tau}\right\}, && \text{by (B-1)}\\
&= \frac{1}{2}\,\mathrm{e}^{-\tau/2}\left\{\frac{1}{2}\sum_{n=0}^{N}\left[c_n + c_{n+1}\right] L_n^{(1)}(\tau) + \sum_{n=1}^{N}\left[c_n + c_{n+1}\right] L_{n-1}^{(2)}(\tau)\right\}, && \text{by (B-4)}\\
&= \frac{1}{4}\,\mathrm{e}^{-\tau/2}\left\{\sum_{n=0}^{N}\left[c_n + c_{n+1}\right] L_n^{(2)}(\tau) + \sum_{n=0}^{N-1}\left[c_{n+1} + c_{n+2}\right] L_n^{(2)}(\tau)\right\}, && \text{by (22.7.30)}\\
&= \frac{1}{4}\,\mathrm{e}^{-\tau/2}\left\{\sum_{n=0}^{N}\left[c_n + 2\,c_{n+1} + c_{n+2}\right] L_n^{(2)}(\tau)\right\}, && \text{for } c_{N+1} \equiv c_{N+2} \equiv 0.
\end{aligned} \tag{B-5}$$

This is equation (17) of the text.

Next, we show that the first set of constraints of equation (3) is convex. The second derivative of the price of a discount bond with respect to the Laguerre constants can be written as follows.

$$\frac{\partial^2 P(\tau;\mathbf{c})}{\partial \mathbf{c}^2} \overset{\text{def}}{=} \underset{(N+1)\times(N+1)}{\mathbf{H}(\tau;\mathbf{c})} = \begin{bmatrix} H_{0,0} & H_{0,1} & H_{0,2} & \cdots & H_{0,N} \\ & H_{1,1} & H_{1,2} & \cdots & H_{1,N} \\ & & H_{2,2} & \cdots & H_{2,N} \\ & & & \ddots & \vdots \\ \text{symmetric} & & & & H_{N,N} \end{bmatrix}, \text{ where}$$

$$H_{n,m} = P(\tau;\mathbf{c})\left(\tau - J_n(\tau)\right)\left(\tau - J_m(\tau)\right), \quad n, m = 0, 1, 2, \ldots, N. \tag{B-6}$$

From the equation above, it follows that for $n, m = 0, 1, 2, \ldots, N$

$$\begin{aligned}
H_{n,n} &= P(\tau;\mathbf{c})\left(\tau - J_n(\tau)\right)^2 \begin{cases} > 0 & , \tau > 0, \\ = 0 & , \tau = 0, \end{cases}\\
M_{n,m} &\overset{\text{def}}{=} H_{n,n}\, H_{m,m} - H_{n,m}^2\\
&= P(\tau;\mathbf{c})^2\left(\tau - J_n(\tau)\right)^2\left(\tau - J_m(\tau)\right)^2 - \left(P(\tau;\mathbf{c})\left(\tau - J_n(\tau)\right)\left(\tau - J_m(\tau)\right)\right)^2\\
&= 0.
\end{aligned} \tag{B-7}$$

If **H** were negative semi-definite (nonpositive definite, respectively), then it must hold true that $H_{n,n} \leqq 0$ and $M_{n,m} \geqq 0$, which is not the case by equation (B-7). If **H** were positive semi-definite (nonnegative definite, respectively), then it must hold true that $H_{n,n} \geqq 0$ and $M_{n,m} \geqq 0$, which is the case by equation (B-7). Hence, **H** must be positive semi-definite. The second derivative of the price of a coupon-bearing bond with respect to the Laguerre constants can be written by equation (12) as follows.

$$\frac{\partial^2 B(\tau_m;\mathbf{c})}{\partial \mathbf{c}^2} = \sum_{j=1}^{D_m} d_{m,j}\,\frac{\partial^2 P(\tau_{m,j};\mathbf{c})}{\partial \mathbf{c}^2} = \sum_{j=1}^{D_m} d_{m,j}\,\mathbf{H}(\tau_{m,j};\mathbf{c}), \quad m = 1, 2, \ldots, M. \tag{B-8}$$

Since the sum of positive semi-definite matrices is positive semi-definite, the first set of constraints of equation (3) is convex, but not strictly convex.

Finally, we show the solution for the objective function, if it is integrated from zero to a finite value, T, rather than to infinity. Instead of equation (8), we get equation (B-9) as follows.

$$I_{k,l}(T) \equiv \int_0^T e^{-t} L_k(t)\, L_l(t)\, dt = \begin{cases} \dfrac{1}{\mu!} \displaystyle\sum_{s=1}^{\nu+1} (\mu - s)!\, e^{-T}\, T^s\, L_{\mu-s}^{(s)}(T)\, L_{\nu-s+1}^{(s-1)}(T)\,, & \\ \mu \equiv \max(k, l), \quad \nu \equiv \min(k, l), & \text{for } k \neq l, \\ 1 - e^{-T} \left(\displaystyle\sum_{s=0}^{k} \dfrac{T^{k-s}}{(k-s)!} \right) & \\ + \dfrac{1}{k!} \displaystyle\sum_{s=1}^{k} (k-s)!\, e^{-T}\, T^s\, L_{k-s}^{(s)}(T)\, L_{k-s+1}^{(s-1)}(T)\,, & \text{for } k = l. \end{cases} \tag{B-9}$$

For $T \to \infty$, equation (8) is obtained. Although $\mathbf{Q}(T)$ does not differ significantly from $\mathbf{Q}(\infty)$ in equation (6) for $T \geqq 30$, the matrix $\mathbf{Q}(T)$ is no longer positive definite. Moreover, for $N > 200$ and $1 \leqq T \leqq 30$, the evaluation of the sums in equation (B-9) leads to a disastrous loss of digits. A recurrence relationship for equation (B-9) does not exist. The upper part of equation (B-9) is obtained for $k = 0, 1, \ldots, N$; $l > k$ as follows.

$$\begin{aligned} I_{k,l}(T) &\overset{\text{def}}{=} \int_0^T e^{-t} L_k(t)\, L_l(t)\, dt \\ &= \frac{1}{l!} \int_0^T L_k(t)\, \frac{d^l\left[e^{-t}\, t^l\right]}{dt^l}\, dt\,, && \text{by (22.11.6)} \\ &= \frac{1}{l!} \sum_{s=1}^{k+1} \left[(-1)^{s-1} \frac{d^{s-1} L_k(t)}{dt^{s-1}} \frac{d^{l-s}\left[e^{-t}\, t^l\right]}{dt^{l-s}} \right]_0^T, && \text{by repeated integration by parts} \\ &= \frac{1}{l!} \sum_{s=1}^{k+1} \left[(l-s)!\, e^{-t}\, t^s\, L_{l-s}^{(s)}(t)\, L_{k-s+1}^{(s-1)}(t) \right]_0^T, && \text{by (22.11.6) and (22.5.17)} \\ &= \frac{1}{l!} \sum_{s=1}^{k+1} (l-s)!\, e^{-T}\, T^s\, L_{l-s}^{(s)}(T)\, L_{k-s+1}^{(s-1)}(T) \end{aligned} \tag{B-10}$$

For $k = l$, the repeated integration by parts ceases as soon as we obtain the *zero*th derivative of Rodrigues' formula, that is, the index s runs from 1 to k. The remaining integral is the following.

$$\int_0^T e^{-t}\, t^k\, dt = k! \left\{ 1 - e^{-T} \left(\sum_{s=0}^{k} \frac{T^{k-s}}{(k-s)!} \right) \right\} \tag{B-11}$$

Both this equation and equation (B-10) for $s = 1, 2, \ldots, k$, establish the lower part of equation (B-9).