A Flexible Yield Curve Model

by

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ABSTRACT

This paper develops a flexible yield curve model from the point of view of macroeconomics. The model does not admit arbitrage between bonds of different maturities. Whereas the price of a non-indexed discount bond in *real* terms is assumed to depend on three factors, the same bond in *nominal* terms depends on two factors only. The three factors considered in this paper are the real instantaneous spot interest rate, the "expected" instantaneous spot inflation (deflation) rate and the consumer price level. The Fisher equation is derived simply from transforming the bond price in *real* terms into the bond price in *nominal* terms rather than from a portfolio optimization. The model determines consistently the various market prices of factor risks within the framework considered, second, it allows for both "expected" inflation rates and "expected" *deflation* rates, and third, it determines the domain of the factors consistently.

In contrast to the well-known Cox-Ingersoll-Ross model, the partial differential equation (PDE) to value a non-indexed bond in this paper is not separable. Hence, a closed-form solution does not exist. We solve numerically the PDE considered by means of the method of lines which transforms the PDE into a system of first-order ordinary differential equations. Finally, we show the martinagale representation of the discount bond price in terms of the nominal instantaneous spot interest rate together with the Fokker-Planck equation of the probability density function.

Incomplete version: February 2008

KEYWORDS: Term structure of interest rates, three-factor model, Fisher equation, parabolic partial differential equation, method of lines, martinagale, Fokker-Planck equation.

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0 Introduction

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1 The Structure of the Yield Curve Model

In view of Irving Fisher's equation, the price of a nominal or non-indexed pure discount bond, respectively, in nominal terms is explained only by two factors in this paper, namely by the real instantaneous spot interest rate and the drift rate of the instantaneous spot (price) inflation rate. All rates are continuously compounded. Let P_n denote the price of a nominal pure discount bond, let r_r denote the real instantaneous spot interest rate, $\bar{r}_y$ the the drift rate of the instantaneous spot inflation rate, t the the current date and T the the maturity date of the discount bond, then we consider the function $P_n = P_n(t, T, r_r, \bar{r}_y)$ in this paper. The payoff of the nominal pure discount bond in nominal terms is equal to one unit of money at the terminal date, that is, $P_n(T, T, \cdot) = 1$.

The price of a non-indexed pure discount bond in *real* terms, denoted as Π_n, is equal to the price of the nominal discount bond in nominal terms divided by the consumer price level, denoted as p. Hence, $\Pi_n(t, T, r_r, \bar{r}_y, p) = P_n(t, T, r_r, \bar{r}_y) / p(t)$ depends on the consumer price level. The payoff is equal to the purchasing power of money at the terminal date, that is, $\Pi_n(T, T, \cdot) = 1 / p(T)$.

The real or indexed bond, respectively, is not exposed to inflation risk. In this paper, it is assumed that its price depends solely on the real instantaneous spot interest rate. Let Π_r denote the price of a real bond in units of consumption goods, then we consider the function $\Pi_r = \Pi_r(t, T, r_r)$ in this paper. The payoff of the indexed pure discount bond in real terms is equal to one unit of consumption goods at the terminal date, that is, $\Pi_r(T, T, \cdot) = 1$.

The price of an indexed pure discount bond in *nominal* terms, denoted as P_r, is equal to the price of the real pure discount bond in real terms multiplied by the consumer price level. Hence, $P_r(t, T, r_r, p) = \Pi_r(t, T, r_r)\, p(t)$ depends on the consumer price level. The payoff is equal to the consumer price level at the terminal date, that is, $P_r(T, T, \cdot) = p(T)$.

1-1 The SDEs of the Three Factors

As usual, we assume that the stochastic differential equation (SDE) of each factor or state variable, respectively, follows an Ito process. In order to develop a general structure of the model, the drift functions, denoted as $m_j(\cdot)$ (j = 1, 2, 3), and the volatility functions, denoted as $s_j(\cdot)$ (j = 1, 2, 3), will not be specified until the third section. All the drifts and volatilities could, in principle, depend on all the state variables and time. However, they cannot depend on the consumer price level as long as it is assumed that the price of a nominal pure discount bond in nominal terms depends only on the real instantaneous spot interest rate and the drift rate of the instantaneous spot inflation rate. Moreover, $m_1(\cdot)$ and $s_1(\cdot)$ below cannot depend on the drift rate of the instantaneous spot inflation rate or the consumer price level as long as it is assumed that the price of an indexed bond in real terms depends only on the real instantaneous spot interest rate. Hence, the SDE for the real instantaneous spot interest rate is given by

$$\mathrm{d}r_r(t) = m_1(t, r_r)\,\mathrm{d}t + s_1(t, r_r)\,\mathrm{d}z_1(t) \tag{1-1}$$

where z_1 denotes a first Wiener process. The SDE of the drift rate of the instantaneous spot (price) inflation rate is given by

$$\mathrm{d}\bar{r}_y(t) = m_2(t, r_r, \bar{r}_y)\,\mathrm{d}t + s_2(t, r_r, \bar{r}_y)\,\mathrm{d}z_2(t) \tag{1-2}$$

where z_2 denotes a second Wiener process. Finally the SDE of the consumer price level is given by

$$\begin{aligned} r_y(t)\,\mathrm{d}t \equiv \frac{\mathrm{d}p(t)}{p(t)} &= \bar{r}_y(t)\,\mathrm{d}t + s_3(t, r_r, \bar{r}_y)\,\mathrm{d}z_3(t) \\ &= m_3(\cdot)\,\mathrm{d}t + s_3(t, r_r, \bar{r}_y)\,\mathrm{d}z_3(t)\,, \quad m_3(\cdot) \equiv \bar{r}_y(t) \end{aligned} \tag{1-3}$$

where z_3 denotes a third Wiener process. In the general equilibrium framework of Bakshi and Chen [1996], the instantaneous spot (price) inflation rate, $r_y(t)$, consists of a random drift and a volatility term, both driven by macroeconomic variables. The same structure has been assumed by Cox, Ingersoll and Ross (CIR) [1985b].

It is assumed that the three Wiener processes are contemporaneously correlated with correlation coefficients ρ_{ij} as follows.

$$\mathscr{C}_t\big(\mathrm{d}z_i(t), \mathrm{d}z_j(t)\big) = \rho_{ij}\,\mathrm{d}t\,, \quad i, j = 1, 2, 3. \tag{1-4}$$

where $\mathscr{C}_t$ denotes the covariance operator given the information at time t. If monetary policy has real effects in the short run, then the real instantaneous spot interest rate should be correlated with the instantaneous spot inflation rate. There is empirical evidence that money has real effects in the short run, but money is neutral in the long run.

1-2 The PDEs of the Nominal Pure Discount Bond

We start with the partial differential equation (PDE) of highest dimension. Using arbitrage arguments, the PDE to value a nominal pure discount bond in *real* terms can be written as follows (see, e. g., Garman, 1976, Hull, 2006).

$$\begin{aligned} 0 = {} & \frac{1}{2}\, s_1(\cdot)^2 \frac{\partial^2 \Pi_n}{\partial r_r^2} + \rho_{12}\, s_1(\cdot)\, s_2(\cdot) \frac{\partial^2 \Pi_n}{\partial r_r\, \partial \bar{r}_y} + \rho_{13}\, s_1(\cdot)\, s_3(\cdot)\, p \frac{\partial^2 \Pi_n}{\partial r_r\, \partial p} \\ & + \frac{1}{2}\, s_2(\cdot)^2 \frac{\partial^2 \Pi_n}{\partial \bar{r}_y^2} + \rho_{23}\, s_2(\cdot)\, s_3(\cdot)\, p \frac{\partial^2 \Pi_n}{\partial \bar{r}_y\, \partial p} + \frac{1}{2}\, s_3(\cdot)^2\, p^2 \frac{\partial^2 \Pi_n}{\partial p^2} \\ & + \left[m_1(\cdot) + \psi_1(\cdot)\, s_1(\cdot)\right] \frac{\partial \Pi_n}{\partial r_r} + \left[m_2(\cdot) + \psi_2(\cdot)\, s_2(\cdot)\right] \frac{\partial \Pi_n}{\partial \bar{r}_y} + \left[m_3(\cdot) + \psi_3(\cdot)\, s_3(\cdot)\right] p \frac{\partial \Pi_n}{\partial p} \\ & - r_r\, \Pi_n + \frac{\partial \Pi_n}{\partial t} \end{aligned} \tag{1-5}$$

where $\psi_j(\cdot)$ (j = 1, 2, 3) denote the three market prices of factor risks in real terms. They depend on all three factors (r_r, $\bar{r}_y$, p) and time. We denote the algebraic product $\psi_j(\cdot)\, s_j(\cdot)$ (j = 1, 2, 3) as the market premia of the factor risks. Note that an instantaneously riskless portfolio consisting of four nominal pure discount bonds in real terms with different terms to maturity earns the real instantaneous spot interest rate during a small period of time.

Using the relationship $\Pi_n(\cdot) = P_n(\cdot) \,/\, p(t)$, we transform the above PDE into a PDE to value a nominal pure discount bond in *nominal* terms as follows.

$$\begin{aligned} 0 = {} & \frac{1}{2}\, s_1(\cdot)^2 \frac{\partial^2 P_n}{\partial r_r^2} + \rho_{12}\, s_1(\cdot)\, s_2(\cdot) \frac{\partial^2 P_n}{\partial r_r\, \partial \bar{r}_y} + \frac{1}{2}\, s_2(\cdot)^2 \frac{\partial^2 P_n}{\partial \bar{r}_y^2} \\ & + \left[m_1(\cdot) + \psi_1(\cdot)\, s_1(\cdot) - \rho_{13}\, s_1(\cdot)\, s_3(\cdot)\right] \frac{\partial P_n}{\partial r_r} + \left[m_2(\cdot) + \psi_2(\cdot)\, s_2(\cdot) - \rho_{23}\, s_2(\cdot)\, s_3(\cdot)\right] \frac{\partial P_n}{\partial \bar{r}_y} \\ & - \left[r_r + m_3(\cdot) + \psi_3(\cdot)\, s_3(\cdot) - s_3(\cdot)^2\right] P_n + \frac{\partial P_n}{\partial t} \end{aligned} \tag{1-6}$$

The price of the nominal pure discount bond in nominal terms does not depend on the consumer price level. Obviously, this type of PDE is to be solved numerically because it has both a lower dimension than the PDE of the nominal bond in real terms and a regular terminal condition.

The PDE to value a nominal pure discount bond in *nominal* terms can be obtained directly when considering the fact that an instantaneously riskless portfolio consisting of three nominal pure discount bonds in nominal terms with different terms to maturity earns the nominal instantaneous spot interest rate during a small period of time. The PDE is as follows.

$$\begin{aligned} 0 = {} & \frac{1}{2}\, s_1(\cdot)^2 \frac{\partial^2 P_n}{\partial r_r^2} + \rho_{12}\, s_1(\cdot)\, s_2(\cdot) \frac{\partial^2 P_n}{\partial r_r\, \partial \bar{r}_y} + \frac{1}{2}\, s_2(\cdot)^2 \frac{\partial^2 P_n}{\partial \bar{r}_y^2} \\ & + \left[m_1(\cdot) + \Psi_1(\cdot)\, s_1(\cdot)\right] \frac{\partial P_n}{\partial r_r} + \left[m_2(\cdot) + \Psi_2(\cdot)\, s_2(\cdot)\right] \frac{\partial P_n}{\partial \bar{r}_y} \\ & - r_n\, P_n + \frac{\partial P_n}{\partial t} \end{aligned} \tag{1-7}$$

where $\Psi_j(\cdot)$ (j = 1, 2) denote the three market prices of factor risks in nominal terms which depend on the first two factors (r_r, $\bar{r}_y$) and time.

Comparing equations (1-6) and (1-7), we find the following relationships between the market premia of factor risks in nominal and real terms

$$\begin{aligned} \Psi_1(\cdot)\, s_1(\cdot) &= \psi_1(\cdot)\, s_1(\cdot) - \rho_{13}\, s_1(\cdot)\, s_3(\cdot) = \psi_1(\cdot)\, s_1(\cdot) - \mathscr{C}_t\big(\mathrm{d}r_r(t),\, r_y(t)\big) \\ \Psi_2(\cdot)\, s_2(\cdot) &= \psi_2(\cdot)\, s_2(\cdot) - \rho_{23}\, s_2(\cdot)\, s_3(\cdot) = \psi_2(\cdot)\, s_2(\cdot) - \mathscr{C}_t\big(\mathrm{d}\bar{r}_y(t),\, r_y(t)\big) \end{aligned} \tag{1-8}$$

and for the instantaneous spot interest rates

$$r_n(t) = r_r(t) + m_3(\cdot) + \psi_3(\cdot)\, s_3(\cdot) - s_3(\cdot)^2 \tag{1-9}$$

which is Irving Fisher's eqation in an uncertain world (Stanley Fisher, 1975). The three market premia of factor risks will be specified later.

1-3 The Instantaneous Interest Premium

Defining the instantaneous interest premium or inflation risk premium, respectively, $\eta(t, t, \cdot)$, as

$$\eta(t, t, \cdot) \equiv s_3(\cdot)^2 - \psi_3(\cdot)\, s_3(\cdot) = \mathscr{V}_t\big(r_y(t)\big)\, \mathrm{d}t - \psi_3(\cdot)\, s_3(\cdot) \tag{1-10}$$

where $\mathscr{V}_t$ denotes the variance operator given the information at time t, then we can write Irving Fisher's equation in the following way.

$$r_n(t) = r_r(t) + \mathscr{E}_t r_y(t) - \eta(t, t, \cdot) \tag{1-11}$$

where $\mathscr{E}_t$ denotes the expectation operator given the information at time t. In this model, the existence of an interest premium is the consequence of transforming the bond price in real terms into the bond price in nominal terms.

1-4 The SDE of the Nominal Instantaneous Spot Interest Rate

Applying Ito's theorem to equation (1-9), the SDE of the nominal instantaneous spot interest rate can be written as follows.

$$\begin{aligned} \mathrm{d}r_n(t) = {} & \left[\frac{\partial r_n}{\partial t} + m_1(\cdot) \frac{\partial r_n}{\partial r_r} + m_2(\cdot) \frac{\partial r_n}{\partial \bar{r}_y} + \frac{1}{2}\, s_1(\cdot)^2 \frac{\partial^2 r_n}{\partial r_r^2} + \rho_{12}\, s_1(\cdot)\, s_2(\cdot) \frac{\partial^2 r_n}{\partial r_r\, \partial \bar{r}_y} + \frac{1}{2}\, s_2(\cdot)^2 \frac{\partial^2 r_n}{\partial \bar{r}_y^2} \right] \mathrm{d}t \\ & + s_1(\cdot) \frac{\partial r_n}{\partial r_r}\, \mathrm{d}z_1(t) + s_2(\cdot) \frac{\partial r_n}{\partial \bar{r}_y}\, \mathrm{d}z_2(t) \end{aligned} \tag{1-12}$$

One half the instantaneous variance of the nominal instantaneous spot interest rate is given by the last three terms within the square bracket above. The evolution of the nominal instantaneous spot interest rate is driven by the two Wiener processes associated with the real instantaneous spot interest rate and the drift rate of the instantaneous spot inflation rate. The drift rate of the nominal instantaneous spot interest rate may be non-linear, depending on the specification of the various drifts, volatilities and market prices of factor risks.

1-5 The SDEs of the Nominal Pure Discount Bond

In view of the empirical estimation of the model parameters, we derive the SDE of the price of a nominal pure discount bond in either real or nominal terms. Applying Ito's theorem to the function $\Pi_n(\cdot)$ first, substituting the partial derivative with respect to time from the PDE (1-5), then the SDE of the price of a nominal pure discount bond in real terms becomes as follows.

$$\begin{aligned} \mathrm{d}\Pi_n(t) = &\left[-\psi_1(\cdot)\, s_1(\cdot) \frac{\partial \Pi_n}{\partial r_r} - \psi_2(\cdot)\, s_2(\cdot) \frac{\partial \Pi_n}{\partial \bar{r}_y} - \psi_3(\cdot)\, s_3(\cdot)\, p(t) \frac{\partial \Pi_n}{\partial p} + r_r(t)\, \Pi_n(t) \right] \mathrm{d}t \\ &+ s_1(\cdot) \frac{\partial \Pi_n}{\partial r_r} \mathrm{d}z_1(t) + s_2(\cdot) \frac{\partial \Pi_n}{\partial \bar{r}_y} \mathrm{d}z_2(t) + s_3(\cdot)\, p(t) \frac{\partial \Pi_n}{\partial p} \mathrm{d}z_3(t) \end{aligned} \tag{1-13}$$

By this equation, the drift rate of the relative change of the price of a nominal pure discount bond in real terms is equal to the risk-adjusted real instantaneous spot interest rate. Using the relationship $\Pi_n(\cdot) = P_n(\cdot)\,/\,p(t)$, the partial derivatives can be replaced by those of the nominal discount bond in nominal terms as follows.

$$\begin{aligned} \mathrm{d}\Pi_n(t) = &\left[-\frac{\psi_1(\cdot)\, s_1(\cdot)}{p(t)} \frac{\partial P_n}{\partial r_r} - \frac{\psi_2(\cdot)\, s_2(\cdot)}{p(t)} \frac{\partial P_n}{\partial \bar{r}_y} + \left\{ \psi_3(\cdot)\, s_3(\cdot) + r_r(t) \right\} \Pi_n(t) \right] \mathrm{d}t \\ &+ \frac{s_1(\cdot)}{p(t)} \frac{\partial P_n}{\partial r_r} \mathrm{d}z_1(t) + \frac{s_2(\cdot)}{p(t)} \frac{\partial P_n}{\partial \bar{r}_y} \mathrm{d}z_2(t) - s_3(\cdot)\, \Pi_n(t)\, \mathrm{d}z_3(t) \end{aligned} \tag{1-14}$$

Next, the change of the price of the nominal pure discount bond in nominal terms can be written as

$$\mathrm{d}P_n(t) = p(t)\, \mathrm{d}\Pi_n(t) + \Pi_n(t)\, \mathrm{d}p(t) + \mathrm{d}\Pi_n(t)\, \mathrm{d}p(t) \tag{1-15}$$

by Ito's theorem. Substituting from equations (1-3), (1-9) and (1-14) into the above equation, the SDE of the price of a nominal pure discount bond in nominal terms can be written as follows.

$$\begin{aligned} \mathrm{d}P_n(t) = &\left[-\left\{ \psi_1(\cdot)\, s_1(\cdot) - \rho_{13}\, s_1(\cdot)\, s_3(\cdot) \right\} \frac{\partial P_n}{\partial r_r} - \left\{ \psi_2(\cdot)\, s_2(\cdot) - \rho_{23}\, s_2(\cdot)\, s_3(\cdot) \right\} \frac{\partial P_n}{\partial \bar{r}_y} + r_n(t)\, P_n(t) \right] \mathrm{d}t \\ &+ s_1(\cdot) \frac{\partial P_n}{\partial r_r} \mathrm{d}z_1(t) + s_2(\cdot) \frac{\partial P_n}{\partial \bar{r}_y} \mathrm{d}z_2(t) \end{aligned} \tag{1-16}$$

As it should be, the third Wiener process cancels off. The same result is obtained, if you, first, apply Ito's theorem to the nominal pure discount bond in nominal terms, second, substitute the partial derivative with respect to time from the PDE (1-7) and, third, substitute the relationships (1-8).

1-6 The PDEs of the Real Pure Discount Bond

We start with the PDE of higher dimension. Using arbitrage arguments, the PDE to value an indexed pure discount bond in *nominal* terms can be written as follows.

$$\begin{aligned} 0 = & \frac{1}{2}\, s_1(\cdot)^2 \frac{\partial^2 P_r}{\partial r_r^2} + \rho_{13}\, s_1(\cdot)\, s_3(\cdot)\, p\, \frac{\partial^2 P_r}{\partial r_r\, \partial p} + \frac{1}{2}\, s_3(\cdot)^2\, p^2 \frac{\partial^2 P_r}{\partial p^2} \\ & + \left[m_1(\cdot) + \overline{\Psi}_1(\cdot)\, s_1(\cdot) \right] \frac{\partial P_r}{\partial r_r} + \left[m_3(\cdot) + \overline{\Psi}_3(\cdot)\, s_3(\cdot) \right] p\, \frac{\partial P_r}{\partial p} \\ & - r_n\, P_r + \frac{\partial P_r}{\partial t} \end{aligned} \tag{1-17}$$

where $\overline{\Psi}_j(\cdot)$ (j = 1, 3) denote the two market prices of factor risks in nominal terms. Using the relationship $P_r(\cdot) = \Pi_r(\cdot)\, p(t)$, we transform the above PDE into a PDE to value an indexed pure discount bond in *real* terms as follows.

$$\begin{aligned} 0 = & \frac{1}{2}\, s_1(\cdot)^2 \frac{\partial^2 \Pi_r}{\partial r_r^2} + \left[m_1(\cdot) + \overline{\Psi}_1(\cdot)\, s_1(\cdot) + \rho_{13}\, s_1(\cdot)\, s_3(\cdot) \right] \frac{\partial \Pi_r}{\partial r_r} \\ & - \left[r_n - m_3(\cdot) - \overline{\Psi}_3(\cdot)\, s_3(\cdot) \right] \Pi_r + \frac{\partial \Pi_r}{\partial t} \end{aligned} \tag{1-18}$$

Also, the PDE to value an indexed pure discount bond in real terms can be obtained directly, when considering the fact that an instantaneously riskless portfolio consisting of two indexed pure discount bonds in real terms with different terms to maturity earns the real instantaneous spot interest rate during a small period of time. The PDE is as follows.

$$0 = \frac{1}{2}\, s_1(\cdot)^2 \frac{\partial^2 \Pi_r}{\partial r_r^2} + \left[m_1(\cdot) + \overline{\psi}_1(\cdot)\, s_1(\cdot) \right] \frac{\partial \Pi_r}{\partial r_r} - r_r\, \Pi_r + \frac{\partial \Pi_r}{\partial t} \tag{1-19}$$

where $\overline{\psi}_1(\cdot)$ denotes the market price of the real interest rate risk in real terms. Obviously, this type of PDE is to be solved numerically because it has both a lower dimension than the PDE of the indexed bond in nominal terms and a regular terminal condition.

Comparing equations (1-18), (1-19), (1-8) and (1-9), we find the following relationships between the nominal and real instantaneous spot interest rates

$$r_n = r_r + m_3(\cdot) + \overline{\Psi}_3(\cdot)\, s_3(\cdot) \tag{1-20}$$

as well as between the market premia of factor risks

$$
\begin{aligned}
\bar{\Psi}_1(\cdot)\, s_1(\cdot) &= \bar{\psi}_1(\cdot)\, s_1(\cdot) - \rho_{13}\, s_1(\cdot)\, s_3(\cdot) \\
\bar{\Psi}_1(\cdot)\, s_1(\cdot) - \bar{\psi}_1(\cdot)\, s_1(\cdot) &= \Psi_1(\cdot)\, s_1(\cdot) - \psi_1(\cdot)\, s_1(\cdot) \\
\bar{\Psi}_3(\cdot)\, s_3(\cdot) &= \psi_3(\cdot)\, s_3(\cdot) - s_3(\cdot)^2
\end{aligned}
\tag{1-21}
$$

By the above equations, it is not possible to infer the market premium of the first factor risk which is associated with the indexed bond, $\bar{\psi}_1(\cdot)$, from the premium associated with the nominal bond, $\psi_1(\cdot)$. However, it will be identified through superposition later.

1-7 The SDEs of the Real Pure Discount Bond

In view of the empirical estimation of the model parameters, we derive the SDE of the price of a real pure discount bond in either real or nominal terms. Applying Ito's theorem to the function $P_r(\cdot)$ and substituting the partial derivative with respect to time from the PDE (1-17), then the SDE of the price of a real pure discount bond in nominal terms becomes the following expression.

$$
\begin{aligned}
dP_r(t) = &\left[-\bar{\Psi}_1(\cdot)\, s_1(\cdot) \frac{\partial P_r}{\partial r_r} - \bar{\Psi}_3(\cdot)\, s_3(\cdot)\, p(t) \frac{\partial P_r}{\partial p} + r_n(t)\, P_r(t) \right] dt \\
&+ s_1(\cdot) \frac{\partial P_r}{\partial r_r}\, dz_1(t) + s_3(\cdot)\, p(t) \frac{\partial P_r}{\partial p}\, dz_3(t)
\end{aligned}
\tag{1-22}
$$

Using the relationship $\Pi_r(\cdot) = P_r(\cdot) \,/\, p(t)$, the partial derivatives can be replaced by those of the indexed bond in real terms as follows.

$$
\begin{aligned}
dP_r(t) = &\left[-\bar{\Psi}_1(\cdot)\, s_1(\cdot)\, p(t) \frac{\partial \Pi_r}{\partial r_r} + \left\{ r_n(t) - \bar{\Psi}_3(\cdot)\, s_3(\cdot) \right\} P_r(t) \right] dt \\
&+ s_1(\cdot)\, p(t) \frac{\partial \Pi_r}{\partial r_r}\, dz_1(t) + s_3(\cdot)\, P_r(t)\, dz_3(t)
\end{aligned}
\tag{1-23}
$$

Next, the change of the price of the indexed pure discount bond in real terms can be written as

$$
d\Pi_r(t) = \frac{1}{p(t)}\, dP_r(t) - \frac{P_r(t)}{p(t)^2}\, dp(t) - \frac{1}{p(t)^2}\, dP_r(t)\, dp(t) + \frac{P_r(t)}{p(t)^3} \left[dp(t) \right]^2 \tag{1-24}
$$

by Ito's theorem. Substituting from equations (1-3), (1-20) and (1-22) into the above equation, the SDE of the price of a nominal pure discount bond in nominal terms can be written as follows.

$$
d\Pi_r(t) = \left[-\bar{\psi}_1(\cdot)\, s_1(\cdot) \frac{\partial \Pi_r}{\partial r_r} + r_r(t)\, \Pi_r(t) \right] dt + s_1(\cdot) \frac{\partial \Pi_r}{\partial r_r}\, dz_1(t) \tag{1-25}
$$

As it should be, the third Wiener process cancels off. The same result is obtained, if you, first, apply Ito's theorem to the indexed pure discount bond in real terms and, second, substitute the partial derivative with respect to time from the PDE (1-19).

1-8 Superposition

Suppose that we seek a solution for the price of a nominal pure discount bond in real terms which superposes on the price of an indexed bond the price of a "remainder" pure discount bond in real terms, that is, we seek a solution of the form

$$\begin{aligned} \Pi_n(t, T, r_r, \bar{r}_y, p) &= \Pi_r(t, T, r_r) + \Pi_n^*(t, T, r_r, \bar{r}_y, p) \\ \Pi_n^*(T, T, r_r, \bar{r}_y, p) &= \frac{1}{p(T)} - 1 \end{aligned} \tag{1-26}$$

where Π_n^* denotes the price of the "remainder" pure discount bond in real terms. The "remainder" bond depends on all the three factors considered. Clearly, a superposition is always possible. Substituting the additive function above into the PDE (1-5), we find the following expression.

$$\begin{aligned} 0 = &\left\{ \frac{1}{2} s_1(\cdot)^2 \frac{\partial^2 \Pi_r}{\partial r_r^2} + \left[m_1(\cdot) + \psi_1(\cdot) s_1(\cdot) \right] \frac{\partial \Pi_r}{\partial r_r} - r_r \Pi_r + \frac{\partial \Pi_r}{\partial t} \right\} \\ &+ \left\{ \frac{1}{2} s_1(\cdot)^2 \frac{\partial^2 \Pi_n^*}{\partial r_r^2} + \rho_{12} s_1(\cdot) s_2(\cdot) \frac{\partial^2 \Pi_n^*}{\partial r_r \partial \bar{r}_y} + \rho_{13} s_1(\cdot) s_3(\cdot) p \frac{\partial^2 \Pi_n^*}{\partial r_r \partial p} \right. \\ &+ \frac{1}{2} s_2(\cdot)^2 \frac{\partial^2 \Pi_n^*}{\partial \bar{r}_y^2} + \rho_{23} s_2(\cdot) s_3(\cdot) p \frac{\partial^2 \Pi_n^*}{\partial \bar{r}_y \partial p} + \frac{1}{2} s_3(\cdot)^2 p^2 \frac{\partial^2 \Pi_n^*}{\partial p^2} \\ &+ \left[m_1(\cdot) + \psi_1(\cdot) s_1(\cdot) \right] \frac{\partial \Pi_n^*}{\partial r_r} + \left[m_2(\cdot) + \psi_2(\cdot) s_2(\cdot) \right] \frac{\partial \Pi_n^*}{\partial \bar{r}_y} + \left[m_3(\cdot) + \psi_3(\cdot) s_3(\cdot) \right] p \frac{\partial \Pi_n^*}{\partial p} \\ &\left. - r_r \Pi_n^* + \frac{\partial \Pi_n^*}{\partial t} \right\} \end{aligned} \tag{1-27}$$

The expression between the second pair of curly brackets is the right-hand side of the PDE (1-5) where $\Pi_n(\cdot)$ is replaced with $\Pi_n^*(\cdot)$. Both the nominal pure discount bond in real terms and the "remainder" bond obey the same PDE, but have different terminal conditions. Hence, the expression between the second pair of curly brackets must be zero, which in turn implies that the expression between the first pair of curly brackets must also vanish. Comparing this expression with the PDE (1-19) of the indexed bond, we find that

$$\psi_1(\cdot) = \overline{\psi}_1(\cdot) \tag{1-28}$$

Hence, the (estimated) parameters of the nominal pure discount bond in real terms identify the parameters of the indexed pure discount bond in real terms.

1-9 Separability

Suppose that we seek a solution for the price of a nominal pure discount bond in real terms which separates the price of an indexed bond from the price of an "inflation" pure discount bond in real terms, that is, we seek a solution of the form

$$\begin{aligned} \Pi_n(t, T, r_r, \bar{r}_y, p) &= \Pi_r(t, T, r_r)\, \Pi_n^{\circ}(t, T, \bar{r}_y, p) \\ \Pi_n^{\circ}(T, T, \bar{r}_y, p) &= \frac{1}{p(T)} \end{aligned} \tag{1-29}$$

where Π_n° denotes the price of the "inflation" pure discount bond in real terms. The "inflation" bond does not depend on the real instantaneous spot interest rate. Separability implies additive spot interest rates. Obviously, a necessary and sufficient condition for separability is that the real instantaneous spot interest rate neither be correlated with the drift rate of the instantaneous spot inflation rate nor with the instantaneous spot inflation rate. Substituting the multiplicative function above into the PDE (1-5), we find the following expression.

$$\begin{aligned} 0 = &\left\{\left[\frac{1}{2}\, s_1(\cdot)^2 \frac{\partial^2 \Pi_r}{\partial r_r^2} + \left[m_1(\cdot) + \psi_1(\cdot)\, s_1(\cdot)\right] \frac{\partial \Pi_r}{\partial r_r} - r_r\, \Pi_r + \frac{\partial \Pi_r}{\partial t}\right] \frac{1}{\Pi_r}\right\} \\ &+ \left\{\left[\frac{1}{2}\, s_2(\cdot)^2 \frac{\partial^2 \Pi_n^{\circ}}{\partial \bar{r}_y^2} + \rho_{23}\, s_2(\cdot)\, s_3(\cdot)\, p\, \frac{\partial^2 \Pi_n^{\circ}}{\partial \bar{r}_y\, \partial p} + \frac{1}{2}\, s_3(\cdot)^2\, p^2\, \frac{\partial^2 \Pi_n^{\circ}}{\partial p^2}\right.\right. \\ &\left.\left. + \left[m_2(\cdot) + \psi_2(\cdot)\, s_2(\cdot)\right] \frac{\partial \Pi_n^{\circ}}{\partial \bar{r}_y} + \left[m_3(\cdot) + \psi_3(\cdot)\, s_3(\cdot)\right] p\, \frac{\partial \Pi_n^{\circ}}{\partial p} + \frac{\partial \Pi_n^{\circ}}{\partial t}\right] \frac{1}{\Pi_n^{\circ}}\right\} \\ &+ \left\{\left[\rho_{12}\, s_1(\cdot)\, s_2(\cdot) \frac{\partial \Pi_r}{\partial r_r} \frac{\partial \Pi_n^{\circ}}{\partial \bar{r}_y} + \rho_{13}\, s_1(\cdot)\, s_3(\cdot)\, p\, \frac{\partial \Pi_r}{\partial r_r} \frac{\partial \Pi_n^{\circ}}{\partial p}\right] \frac{1}{\Pi_r\, \Pi_n^{\circ}}\right\} \end{aligned} \tag{1-30}$$

If $\rho_{12} = \rho_{13} = 0$, then the expression between the third pair of curly brackets vanishes and the first two pairs of curly brackets contain the separated PDEs of the indexed bond and the "inflation" bond, respectively. This implies that each expression in curly brackets must be equal to a constant. Since an instantaneously riskless portfolio consisting of two real bonds with different terms to maturity earns the real instantaneous spot interest rate during a small period of time, the constant must be equal to zero. Hence we get the following two separate PDEs for the real bond and the "inflation" bond, respectively.

$$\begin{aligned} 0 &= \frac{1}{2}\, s_1(\cdot)^2 \frac{\partial^2 \Pi_r}{\partial r_r^2} + \left[m_1(\cdot) + \psi_1(\cdot)\, s_1(\cdot)\right] \frac{\partial \Pi_r}{\partial r_r} - r_r\, \Pi_r + \frac{\partial \Pi_r}{\partial t} \\ 0 &= \frac{1}{2}\, s_2(\cdot)^2 \frac{\partial^2 \Pi_n^{\circ}}{\partial \bar{r}_y^2} + \rho_{23}\, s_2(\cdot)\, s_3(\cdot)\, p\, \frac{\partial^2 \Pi_n^{\circ}}{\partial \bar{r}_y\, \partial p} + \frac{1}{2}\, s_3(\cdot)^2\, p^2\, \frac{\partial^2 \Pi_n^{\circ}}{\partial p^2} \\ &\quad + \left[m_2(\cdot) + \psi_2(\cdot)\, s_2(\cdot)\right] \frac{\partial \Pi_n^{\circ}}{\partial \bar{r}_y} + \left[m_3(\cdot) + \psi_3(\cdot)\, s_3(\cdot)\right] p\, \frac{\partial \Pi_n^{\circ}}{\partial p} + \frac{\partial \Pi_n^{\circ}}{\partial t} \end{aligned} \tag{1-31}$$

Comparing the first PDE above with the PDE (1-19), it follows again that $\psi_1 = \overline{\psi}_1$.

Since $P_n(\cdot) = \Pi_n(\cdot)\, p(t)$, we can define the price of an "inflation" bond in nominal terms as $P_n^\circ(\cdot) = \Pi_n^\circ(\cdot)\, p(t)$ by equation (1-29), where the terminal condition is $P_n^\circ(T, T, \cdot) = 1$ unit of money. Substituting the relationship $P_n^\circ(\cdot) = \Pi_n^\circ(\cdot)\, p(t)$ into the second PDE (1-31), we get the following PDE for the "inflation" bond in nominal terms.

$$0 = \frac{1}{2}\, s_2(\cdot)^2\, \frac{\partial^2 P_n^\circ}{\partial \bar{r}_y^2} + \left[m_2(\cdot) + \psi_2(\cdot)\, s_2(\cdot) - \rho_{23}\, s_2(\cdot)\, s_3(\cdot)\right] \frac{\partial P_n^\circ}{\partial \bar{r}_y} - \left[m_3(\cdot) + \psi_3(\cdot)\, s_3(\cdot) - s_3(\cdot)^2\right] P_n^\circ + \frac{\partial P_n^\circ}{\partial t} \quad (1\text{-}32)$$

The price of an "inflation" bond in nominal terms does not depend on the consumer price level. Hence, separability implies two separate PDEs for the indexed bond and the "inflation" bond each with one state variable only. Note that the three-factor model of the nominal pure discount bond in real terms proposed by CIR [1985b] is separable.

1-10 The Market Premia of Factor Risks

In an equilibrium model, the market premium of a factor risk is related to both the risk aversion of consumers and the covariance of changes of this factor with percentage changes in optimally invested real wealth (CIR, 1985b, Bakshi and Chen, 1996). Considering real wealth, w, as the stream of future real endowments discounted by the real instantaneous spot interest rate, real wealth is negatively correlated with the real instantaneous spot interest rate, that is, $\partial w / \partial r_r < 0$. Consider real endowments as a *known* function of time, we may write the SDE of the percentage change in real wealth as a function of the real instantaneous spot interest rate as follows.

$$\frac{dw(t)}{w(t)} = \frac{m_w(\cdot)}{w(t)}\, dt + s_1(\cdot)\, \frac{\left(\partial w / \partial r_r\right)}{w(t)}\, dz_1(t) \quad (1\text{-}33)$$

where $m_w(\cdot)$ denotes the drift function of the change in real wealth, $s_1(\cdot)$ the volatility function of the real instananeous spot interest rate and z_1 the Wiener process associated with the real instantaneous spot interest rate. Let $\psi_w(\cdot)$ denote the relative risk aversion of consumers multiplied by the negative value of the partial derivative of real wealth with respect to the real instantaneous spot interest rate, $-\,\partial w / \partial r_r$, and divided by real wealth, then we define the market premium of a particular factor as the negative value of the covariance of this factor with the percentage change in real wealth, mulitplied by the relative risk aversion of consumers, that is

$$\begin{aligned}
\psi_1(\cdot)\, s_1(\cdot) &\equiv \frac{\psi_w(\cdot)\, w(t)}{\left(\partial w/\partial r_r\right)} \frac{\mathscr{C}_t\left(\mathrm{d}r_r(t),\ \frac{\mathrm{d}w(t)}{w(t)}\right)}{\mathrm{d}t} = \psi_w(\cdot)\, s_1(\cdot)^2 > 0, \quad 0 < \psi_w(\cdot) < \infty, \\
\psi_2(\cdot)\, s_2(\cdot) &\equiv \frac{\psi_w(\cdot)\, w(t)}{\left(\partial w/\partial r_r\right)} \frac{\mathscr{C}_t\left(\mathrm{d}\bar{r}_y(t),\ \frac{\mathrm{d}w(t)}{w(t)}\right)}{\mathrm{d}t} = \psi_w(\cdot)\, \rho_{12}\, s_1(\cdot)\, s_2(\cdot) \gtreqqless 0, \\
\psi_3(\cdot)\, s_3(\cdot) &\equiv \frac{\psi_w(\cdot)\, w(t)}{\left(\partial w/\partial r_r\right)} \frac{\mathscr{C}_t\left(\frac{\mathrm{d}p(t)}{p(t)},\ \frac{\mathrm{d}w(t)}{w(t)}\right)}{\mathrm{d}t} = \psi_w(\cdot)\, \rho_{13}\, s_1(\cdot)\, s_3(\cdot) \gtreqqless 0.
\end{aligned} \tag{1-34}$$

We denote $\psi_w(\cdot)$ as the *modified* relative risk aversion of consumers. The expressions on the right-hand sides of the equality signs follow from equations (1-1) - (1-3) and (1-33). Note that the market premia of both the drift rate of the instantaneous spot inflation rate and the instantaneous inflation rate may have either sign.

1-11 The Domain of State Variables and Boundary Conditions

We assume that both the nominal and real spot interest rates remain non-negative, that is, $r_n, r_r \in \mathbb{R}_+$. Then the equation (1-9)

$$r_n = r_r + \bar{r}_y + \psi_3(t, r_r, \bar{r}_y)\, s_3(t, r_r, \bar{r}_y) - s_3(t, r_r, \bar{r}_y)^2 \equiv r_n(t, r_r, \bar{r}_y) \geqq 0 \tag{1-9}$$

defines the domain of the drift rate of the instantaneous spot inflation rate, $\bar{r}_y$. While we assume that the *inflation* rate may vary between zero and infinity, equation (1-9) imposes an upper limit on the *deflation* rate. Let $\bar{r}_y^{\min}$ denote the greatest real-valued zero for which equation (1-9) holds as an equality, then the domain of the drift rate of the instantaneous spot inflation rate is given by

$$\bar{r}_y^{\min}(t, r_r) \leqq \bar{r}_y \leqq +\infty, \quad \text{where } \bar{r}_y^{\min}\ (\leqq 0) \text{ is the greatest zero of } r_n(t, r_r, \bar{r}_y) = 0. \tag{1-35}$$

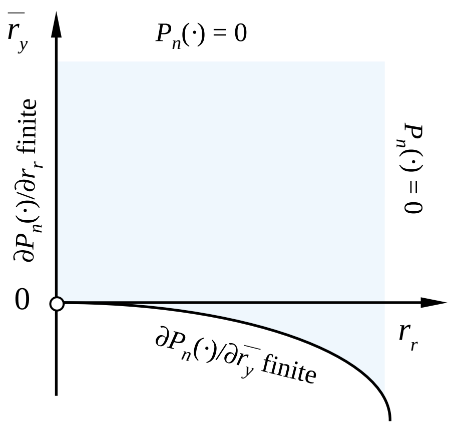

Fig. 1: Domain and Boundary Conditions

Note that we require the lower bound of the drift rate of the instantaneous spot inflation rate to be non-positive for all possible values of the real instantaneous spot interest rate. When the number of real-valued zeros is even, then the greatest zero is a maximum rather than a minimum. Since we do not wish to impose an upper limit on the "expected" inflation rate, any specification of both the market premium and the volatility function of the instantaneous inflation rate which leads to an even number of real-valued zeros must be excluded. When the number of real-valued zeros of the above equation is odd, the upper limit of the "expected" deflation rate is equal to the greatest zero. Hence, the market premium and the volatility function of the instantaneous inflation rate must be specified in a such a way that both an odd number of real-valued zeros and a non-positive lower bound of the drift rate of the instantaneous spot inflation rate are possible. Note that the lower bound of the drift rate of the instantaneous spot inflation rate is a function of both the real instantaneous spot interest rate and time. The domain of the two state variables $(r_r, \bar{r}_y)$ is depicted in Figure 1. Note that in the model of CIR, the drift rate of the instantaneous inflation rate is restricted to remain non-negative. Let us consider two examples.

EXAMPLE 1: In this example, we show that the consumer price level cannot follow a geometric Brownian motion with random drift if the lower bound of the drift rate of the instantaneous inflation rate should always be non-positive, given the assumption that the modified relative risk aversion of consumers is a positive constant, $\psi_w(\cdot) = \psi_{w0}$. Let σ_1 and σ_3 denote constant volatility parameters, then we assume the following two volatility functions of equations (1-1) and (1-3):

$$\begin{aligned} & s_1(\cdot) = \sigma_1 \sqrt{r_r}\,, \quad 0 < \sigma_1 < \infty\,, && s_3(\cdot) = \sigma_3\,, \quad 0 < \sigma_3 < \infty\,, \\ & \psi_w(\cdot) = \psi_{w0}\,, \quad 0 < \psi_{w0} < \infty\,, && \psi_3(\cdot)\, s_3(\cdot) = \psi_{w0}\, \rho_{13}\, \sigma_1 \sqrt{r_r}\, \sigma_3 \gtreqless 0\,, \\ & \mathrm{d}p(t) = \bar{r}_y(t)\, p(t)\, \mathrm{d}t + \sigma_3\, p(t)\, \mathrm{d}z_3(t)\,. \end{aligned} \tag{1-36}$$

The market premium of the instantaneous spot inflation rate is given by equation (1-34). It follows from equation (1-9) that there is a unique real-valued zero. The lower bound becomes

$$\bar{r}_y^{\min}(r_r) = -\left[r_r + \psi_{w0}\, \rho_{13}\, \sigma_1 \sqrt{r_r}\, \sigma_3 - \sigma_3^2\right], \quad \bar{r}_y^{\min}(0) = \sigma_3^2 > 0. \tag{1-37}$$

Hence, the lower bound is positive over a wide range of real instantaneous spot interest rates, in particular, when both the correlation coefficient is negative and the coefficient of the modified relative risk aversion is large.

EXAMPLE 2: In this example, we assume a particular function for the modified relative risk aversion of consumers in terms of the first two state variables. Then, the upper bound of the deflation rate turns out to be equal to the real instantaneous spot interest rate. Again, let σ_i denote various constant volatility parameters, then we assume the following two volatility functions of equations (1-1) and (1-3), and the following modified relative risk aversion of consumers.

$$\begin{aligned} & s_1(\cdot) = \sigma_1 \sqrt{r_r}\,, && s_3(\cdot) = \sigma_3 \sqrt{\bar{r}_y - \bar{r}_y^{\min}}\,, \\ & \psi_w(\cdot) = \psi_{w0} \sqrt{\frac{\bar{r}_y - \bar{r}_y^{\min}}{r_r}}\,, && \psi_3(\cdot)\, s_3(\cdot) = \psi_{w0}\, \rho_{13}\, \sigma_1\, \sigma_3 \left[\bar{r}_y - \bar{r}_y^{\min}\right] \gtreqless 0\,, \\ & \mathrm{d}p(t) = \bar{r}_y(t)\, p(t)\, \mathrm{d}t && 0 < \sigma_1, \sigma_3, \psi_{w0} < \infty\,. \\ & \qquad + \sigma_3 \sqrt{\bar{r}_y - \bar{r}_y^{\min}}\; p(t)\, \mathrm{d}z_3(t)\,, \end{aligned} \tag{1-38}$$

The market premium of the instantaneous spot inflation rate is given by equation (1-34). It follows from equation (1-9) that there is a unique real-valued zero. The lower bound becomes

$$\bar{r}_y^{\min}(r_r) = -\, r_r \tag{1-39}$$

Since the nominal instantaneous spot interest rate must be non-negative for all possible values of both the real instantaneous spot interest rate and the drift rate of the instantaneous spot inflation rate, the following condition must be satisfied by the parameters considered.

$$-\,\psi_{w0}\,\rho_{13}\,\sigma_1\,\sigma_3 + \sigma_3^2 \leqq 1 \tag{1-40}$$

In this example, there exists a one-to-one, time-independent inverse function $\bar{r}_y = \bar{r}_y(r_r,\, r_n)$ as given by equation (1-9).

We now turn to the boundary conditions of the PDE of the nominal pure discount bond in nominal terms as given in equation (1-6). Following Brennan and Schwarz [1977, 1978, 1979], the price of the nominal pure discount bond in nominal terms vanishes as the real instantaneous spot interest rate grows infinitely large. Similarly, the price vanishes as the drift rate of the instantaneous spot inflation rate grows infinitely large. Following Büttler and Waldvogel [1996], the partial derivative of the price of the nominal pure discount bond with respect to the real instantaneous spot interest rate must be finite as the real instantaneous spot interest rate vanishes. Similarly, the partial derivative of the price of the nominal pure discount bond with respect to the drift rate of the instantaneous spot inflation rate must be finite as the drift rate of the instantaneous inflation rate approaches its lower bound. Summarizing, we get the following boundary conditions.

$$\begin{aligned} &P_n(t, T, r_r, \bar{r}_y) \to 0 \text{ as } r_r \to +\infty, && P_n(t, T, r_r, \bar{r}_y) \to 0 \text{ as } \bar{r}_y \to +\infty, \\ &\left| \frac{\partial P_n(t, T, r_r, \bar{r}_y)}{\partial r_r} \right| < \infty \text{ as } r_r \to 0, && \left| \frac{\partial P_n(t, T, r_r, \bar{r}_y)}{\partial \bar{r}_y} \right| < \infty \text{ as } \bar{r}_y \to \bar{r}_y^{\min}(\cdot). \end{aligned} \tag{1-41}$$

The last two conditions admit regular solutions only. Since the terminal condition requires that the price of the nominal pure disount bond is equal to one unit of money at the maturity date T, there exist singularities along two of the four boundaries where the price falls from its terminal value of one to a value of zero during the first instant of time (when moving backwards in time).

1-12 The PDEs of the Nominal Pure Discount Bond in the (r_r, r_n) State Space

Suppose that there exists a one-to-one inverse function of equation (1-9), denoted as $\bar{r}_y = \bar{r}_y(t,\, r_r,\, r_n)$, then we can transform the PDE of the nominal pure discount bond (1-6) into the $(r_r,\, r_n)$ state space by the following transformation.

$$P_n(t, T, r_r, \bar{r}_y) = P_n^{\dagger}(t, T, r_r, r_n), \quad r_n = r_r + \bar{r}_y + \psi_3(\cdot)\, s_3(\cdot) - s_3(\cdot)^2, \tag{1-42}$$

where r_n is given by equation (1-9). Substituting the above equation into equation (1-6), we get the following PDE to value the nominal pure discount bond in nominal terms.

$$\begin{aligned}
0 = {} & \frac{1}{2}\, s_1(\cdot)^2 \frac{\partial^2 P_n^{\dagger}}{\partial r_r^2} + \left[s_1(\cdot)^2 \frac{\partial r_n}{\partial r_r} + \rho_{12}\, s_1(\cdot)\, s_2(\cdot) \frac{\partial r_n}{\partial \bar{r}_y} \right] \frac{\partial^2 P_n^{\dagger}}{\partial r_r\, \partial r_n} \\
& + \left[\frac{1}{2}\, s_1(\cdot)^2 \left(\frac{\partial r_n}{\partial r_r} \right)^2 + \rho_{12}\, s_1(\cdot)\, s_2(\cdot) \frac{\partial r_n}{\partial r_r} \frac{\partial r_n}{\partial \bar{r}_y} + \frac{1}{2}\, s_2(\cdot)^2 \left(\frac{\partial r_n}{\partial \bar{r}_y} \right)^2 \right] \frac{\partial^2 P_n^{\dagger}}{\partial r_n^2} \\
& + \left[m_1(\cdot) + \psi_1(\cdot)\, s_1(\cdot) - \rho_{13}\, s_1(\cdot)\, s_3(\cdot) \right] \frac{\partial P_n^{\dagger}}{\partial r_r} \\
& + \left\{ \left[m_2(\cdot) + \psi_2(\cdot)\, s_2(\cdot) - \rho_{23}\, s_2(\cdot)\, s_3(\cdot) \right] \frac{\partial r_n}{\partial \bar{r}_y} + \frac{1}{2}\, s_1(\cdot)^2 \frac{\partial^2 r_n}{\partial r_r^2} + \rho_{12}\, s_1(\cdot)\, s_2(\cdot) \frac{\partial^2 r_n}{\partial r_r\, \partial \bar{r}_y} + \frac{1}{2}\, s_2(\cdot)^2 \frac{\partial^2 r_n}{\partial \bar{r}_y^2} \right. \\
& \left. + \left[m_1(\cdot) + \psi_1(\cdot)\, s_1(\cdot) - \rho_{13}\, s_1(\cdot)\, s_3(\cdot) \right] \frac{\partial r_n}{\partial r_r} + \frac{\partial r_n}{\partial t} \right\} \frac{\partial P_n^{\dagger}}{\partial r_n} \\
& - \left[r_r + m_3(\cdot) + \psi_3(\cdot)\, s_3(\cdot) - s_3(\cdot)^2 \right] P_n^{\dagger} + \frac{\partial P_n^{\dagger}}{\partial t}
\end{aligned} \tag{1-43}$$

By the equation above, we can transform the partially wry $(r_r,\ \bar{r}_y)$ domain into the the first quadrant.

Let us consider a fourth Wiener process z_4, which is associated with the nominal instantaneous spot interest rate, and let ρ_{4j} $(j = 1, 2, 3)$ denote the corresponding correlation coefficients with the first three Wiener processes, then we define the SDE of the nominal instantaneous spot interest rate as

$$\begin{aligned}
& \mathrm{d}r_n(t) = m_4(\cdot)\, \mathrm{d}t + s_4(\cdot)\, \mathrm{d}z_4(t) \\
& m_4(\cdot) = \frac{\partial r_n}{\partial t} + m_1(\cdot) \frac{\partial r_n}{\partial r_r} + m_2(\cdot) \frac{\partial r_n}{\partial \bar{r}_y} + \frac{1}{2}\, s_1(\cdot)^2 \frac{\partial^2 r_n}{\partial r_r^2} + \rho_{12}\, s_1(\cdot)\, s_2(\cdot) \frac{\partial^2 r_n}{\partial r_r\, \partial \bar{r}_y} + \frac{1}{2}\, s_2(\cdot)^2 \frac{\partial^2 r_n}{\partial \bar{r}_y^2} \\
& s_4(\cdot)\, \mathrm{d}z_4(t) = s_1(\cdot) \frac{\partial r_n}{\partial r_r} \mathrm{d}z_1(t) + s_2(\cdot) \frac{\partial r_n}{\partial \bar{r}_y} \mathrm{d}z_2(t)
\end{aligned} \tag{1-44}$$

by equation (1-12). The covariances of the nominal instantaneous spot interest rate with the other factors become the following expressions

$$\begin{aligned}
\frac{\mathscr{C}_t(\mathrm{d}r_r, \mathrm{d}r_n)}{\mathrm{d}t} &= \rho_{14}\, s_1(\cdot)\, s_4(\cdot) = s_1(\cdot)^2 \frac{\partial r_n}{\partial r_r} + \rho_{12}\, s_1(\cdot)\, s_2(\cdot) \frac{\partial r_n}{\partial \bar{r}_y} \\
\frac{\mathscr{C}_t(\mathrm{d}\bar{r}_y, \mathrm{d}r_n)}{\mathrm{d}t} &= \rho_{24}\, s_2(\cdot)\, s_4(\cdot) = \rho_{12}\, s_1(\cdot)\, s_2(\cdot) \frac{\partial r_n}{\partial r_r} + s_2(\cdot)^2 \frac{\partial r_n}{\partial \bar{r}_y} \\
\frac{\mathscr{C}_t(\mathrm{d}p/p, \mathrm{d}r_n)}{\mathrm{d}t} &= \rho_{34}\, s_3(\cdot)\, s_4(\cdot) = \rho_{13}\, s_1(\cdot)\, s_3(\cdot) \frac{\partial r_n}{\partial r_r} + \rho_{23}\, s_2(\cdot)\, s_3(\cdot) \frac{\partial r_n}{\partial \bar{r}_y} \\
\frac{\mathscr{V}_t(\mathrm{d}r_n)}{\mathrm{d}t} &= s_4(\cdot)^2 = s_1(\cdot)^2 \left(\frac{\partial r_n}{\partial r_r} \right)^2 + 2\, \rho_{12}\, s_1(\cdot)\, s_2(\cdot) \frac{\partial r_n}{\partial r_r} \frac{\partial r_n}{\partial \bar{r}_y} + s_2(\cdot)^2 \left(\frac{\partial r_n}{\partial \bar{r}_y} \right)^2
\end{aligned} \tag{1-45}$$

by equations (1-1) - (1-3), (1-12) and (1-44). The above equations define consistently the various correlation coefficients ρ_{4j} (j = 1, 2, 3).

The PDE to value a nominal pure discount bond in *real* terms can be obtained directly when considering the fact that an instantaneously riskless portfolio consisting of four nominal pure discount bonds in real terms with different terms to maturity earns the real instantaneous spot interest rate during a small period of time. Let $\Pi_n^\dagger(t, T, r_r, r_n, p) = P_n^\dagger(t, T, r_r, r_n) \,/\, p(t)$ denote the nominal pure discount bond in *real* terms, then the PDE can be written as follows.

$$\begin{aligned}0 = &\frac{1}{2}\, s_1(\cdot)^2 \frac{\partial^2 \Pi_n^\dagger}{\partial r_r^2} + \rho_{13}\, s_1(\cdot)\, s_3(\cdot)\, p \frac{\partial^2 \Pi_n^\dagger}{\partial r_r\, \partial p} + \rho_{14}\, s_1(\cdot)\, s_4(\cdot) \frac{\partial^2 \Pi_n^\dagger}{\partial r_r\, \partial r_n} \\ &+ \frac{1}{2}\, s_4(\cdot)^2 \frac{\partial^2 \Pi_n^\dagger}{\partial r_n^2} + \rho_{34}\, s_3(\cdot)\, s_4(\cdot)\, p \frac{\partial^2 \Pi_n^\dagger}{\partial r_n\, \partial p} + \frac{1}{2}\, s_3(\cdot)^2\, p^2 \frac{\partial^2 \Pi_n^\dagger}{\partial p^2} \\ &+ \left[m_1(\cdot) + \psi_1^\dagger(\cdot)\, s_1(\cdot)\right] \frac{\partial \Pi_n^\dagger}{\partial r_r} + \left[m_4(\cdot) + \psi_4(\cdot)\, s_4(\cdot)\right] \frac{\partial \Pi_n^\dagger}{\partial r_n} + \left[m_3(\cdot) + \psi_3^\dagger(\cdot)\, s_3(\cdot)\right] p \frac{\partial \Pi_n^\dagger}{\partial p} \\ &- r_r\, \Pi_n^\dagger + \frac{\partial \Pi_n^\dagger}{\partial t}\end{aligned} \tag{1-46}$$

where $\psi_4(\cdot)$ and $\psi_j^\dagger(\cdot)$ (j = 1, 3) denote the various market prices of factor risks in real terms. Substituting the relationship $\Pi_n^\dagger(t, T, r_r, r_n, p) = P_n^\dagger(t, T, r_r, r_n) \,/\, p(t)$ into the equation above, we get the PDE to value the nominal pure discount bond in *nominal* terms in the (r_r, r_n) state space as follows.

$$\begin{aligned}0 = &\frac{1}{2}\, s_1(\cdot)^2 \frac{\partial^2 P_n^\dagger}{\partial r_r^2} + \rho_{14}\, s_1(\cdot)\, s_4(\cdot) \frac{\partial^2 P_n^\dagger}{\partial r_r\, \partial r_n} + \frac{1}{2}\, s_4(\cdot)^2 \frac{\partial^2 P_n^\dagger}{\partial r_n^2} \\ &+ \left[m_1(\cdot) + \psi_1^\dagger(\cdot)\, s_1(\cdot) - \rho_{13}\, s_1(\cdot)\, s_3(\cdot)\right] \frac{\partial P_n^\dagger}{\partial r_r} + \left[m_4(\cdot) + \psi_4(\cdot)\, s_4(\cdot) - \rho_{34}\, s_3(\cdot)\, s_4(\cdot)\right] \frac{\partial P_n^\dagger}{\partial r_n} \\ &- \left[r_r + m_3(\cdot) + \psi_3^\dagger(\cdot)\, s_3(\cdot) - s_3(\cdot)^2\right] P_n^\dagger + \frac{\partial P_n^\dagger}{\partial t}\end{aligned} \tag{1-47}$$

Comparing the equation above with equation (1-43), we find that $\psi_j^\dagger(\cdot) = \psi_j(\cdot)$ for j = 1 and 3, as well as the following relationships.

$$\begin{aligned}\rho_{14}\, s_1(\cdot)\, s_4(\cdot) &= s_1(\cdot)^2 \frac{\partial r_n}{\partial r_r} + \rho_{12}\, s_1(\cdot)\, s_2(\cdot) \frac{\partial r_n}{\partial \bar{r}_y} \\ s_4(\cdot)^2 &= s_1(\cdot)^2 \left(\frac{\partial r_n}{\partial r_r}\right)^2 + 2\, \rho_{12}\, s_1(\cdot)\, s_2(\cdot) \frac{\partial r_n}{\partial r_r} \frac{\partial r_n}{\partial \bar{r}_y} + s_2(\cdot)^2 \left(\frac{\partial r_n}{\partial \bar{r}_y}\right)^2 \\ \psi_4(\cdot)\, s_4(\cdot) &= \psi_1(\cdot)\, s_1(\cdot) \frac{\partial r_n}{\partial r_r} + \psi_2(\cdot)\, s_2(\cdot) \frac{\partial r_n}{\partial \bar{r}_y}\end{aligned} \tag{1-48}$$

The first two lines hold true by equation (1-45). The last line, which follows from equations (1-44) and (1-45), defines the market premium of the nominal instantaneous spot interest rate risk in terms of the market premia of the first two factor risks. Substituting the market premia of the first two factor risks from equation (1-34) into the last line of the equation above, we get

$$\begin{aligned} \psi_4(\cdot)\, s_4(\cdot) &= \psi_1(\cdot)\, s_1(\cdot)\, \frac{\partial r_n}{\partial r_r} + \psi_2(\cdot)\, s_2(\cdot)\, \frac{\partial r_n}{\partial \bar{r}_y} \\ &= \psi_w(\cdot) \left[s_1(\cdot)^2 \frac{\partial r_n}{\partial r_r} + \rho_{12}\, s_1(\cdot)\, s_2(\cdot)\, \frac{\partial r_n}{\partial \bar{r}_y} \right] \\ &= \psi_w(\cdot)\, \rho_{14}\, s_1(\cdot)\, s_4(\cdot) \\ &= \frac{\psi_w(\cdot)\, w(t)}{\left(\partial w / \partial r_r\right)}\, \frac{\mathscr{C}_t\left(\mathrm{d}r_n(t), \frac{\mathrm{d}w(t)}{w(t)} \right)}{\mathrm{d}t} \end{aligned} \tag{1-49}$$

The second line follows from equation (1-34), the third line from equation (1-45) and the last line defines consistently the market premium of the nominal instantaneous spot interest rate risk in the same way as for the other factors.

1-13 A Martingale of the Bond Price

The price of a nominal pure discount bond in nominal terms can be represented as a martingale if one considers the risk-adjusted processes of the two corresponding factors, denoted by an asterisk, as follows.

$$\begin{aligned} \mathrm{d}r_r^*(t) &= \left[m_1(\cdot) + \psi_1(\cdot)\, s_1(\cdot) - \rho_{13}\, s_1(\cdot)\, s_3(\cdot) \right] \mathrm{d}t + s_1(\cdot)\, \mathrm{d}z_1(t) \\ \mathrm{d}\bar{r}_y^*(t) &= \left[m_2(\cdot) + \psi_2(\cdot)\, s_2(\cdot) - \rho_{23}\, s_2(\cdot)\, s_3(\cdot) \right] \mathrm{d}t + s_2(\cdot)\, \mathrm{d}z_2(t) \end{aligned} \tag{1-50}$$

The risk-adjusted nominal instantaneous spot interest rate, again denoted by an asterisk, is given by equation (1-9) as follows.

$$r_n^* = r_r^* + \bar{r}_y^* + \psi_3(\cdot)\, s_3(\cdot) - s_3(\cdot)^2 \equiv r_n^*(t, r_r^*, \bar{r}_y^*) \tag{1-51}$$

Given equations (1-50) and (1-51), the price of a nominal pure discount bond in nominal terms, discounted by the risk-adjusted nominal instantaneous spot interest rate, is a martingale as follows

$$\left[\exp\left(-\int_t^s r_n^*\left(u, r_r^*(u), \bar{r}_y^*(u)\right) \mathrm{d}u \right) P_n\left(s, T, r_r^*(s), \bar{r}_y^*(s)\right) \right]_{t \leqq s \leqq T} \tag{1-52}$$

for fixed dates t and T as well as the running date s. As a corrolary, for $s = \{t, T\}$we obtain the result that the price of a nominal pure discount bond in nominal terms is equal to the expected value of one unit of money when discounted by the risk-adjusted nominal instantaneous spot interest rate as follows

$$P_n\left(t, T, r_r(t), \bar{r}_y(t)\right) = \mathscr{E}_t^* \exp\left(-\int_t^T r_n^*\left(u, r_r^*(u), \bar{r}_y^*(u)\right) \mathrm{d}u \right) \tag{1-53}$$

where $\mathscr{E}_t^*$ denotes the expectation operator under the risk-adjusted probability measure, given the information at date t.

2 The Numerical Solution of the PDE

The numerical solution of the partial differential equation of a particular function proceeds in three steps. First, the domain of the state variables is divided into regular or irregular meshes, depending on the shape of the domain and the purpose. For a finite-difference method, the state variables include the time variable, for the method of lines to be considered in this paper, the state variables exlude the time variable. Second, any partial derivative of the function under consideration at each mesh point is approximated by a finite difference of nth order, where the Taylor series representation of the function considered is truncated after the nth term. For instance, a second-order finite difference approximation to the first and second partial derivative of the function considered has no truncation error if this function is a second-degree polynomial. As a consequence of the finite differences, a partial derivative can be expressed in terms of the function *values* of neighbouring mesh points. Third, by applying the partial differential equation, approximated by finite differences, to each mesh point, a linear equation system for the function values at each mesh point is obtained, where the number of equations is equal to the number of mesh points.

We wish to solve numerically the parabolic PDE (1-6). Since the domain of the original state variables $(r_r, \bar{r}_y)$ is, in general, partially bent, this implies irregular meshes in any case. If there exists a one-to-one inverse function $\bar{r}_y = \bar{r}_y(t, r_r, r_n)$ obtained from equation (1-9), we can solve numerically the PDE (1-43) in the (r_r, r_n) state space, which is the first quadrant. In principal, regular meshes can be used. Here, we follow the second approach.

To make the PDE (1-43) an initial value problem, we introduce the time period remaining until maturity, denoted as τ, and a new function $\tilde{F}(\cdot)$ such that

$$P_n^{\dagger}(t, T, r_r, r_n) = \tilde{F}(\tau, T, r_r, r_n)\,, \quad \tau = T - t, \quad (T \text{ fixed}) \tag{2-1}$$

Then we can write the PDE (1-43) compactly as

$$\frac{\partial \tilde{F}}{\partial \tau} = \tilde{a}_{rr}(\cdot)\,\frac{\partial^2 \tilde{F}}{\partial r_r^2} + \tilde{a}_{rn}(\cdot)\,\frac{\partial^2 \tilde{F}}{\partial r_r\,\partial r_n} + \tilde{a}_{nn}(\cdot)\,\frac{\partial^2 \tilde{F}}{\partial r_n^2} + \tilde{a}_r(\cdot)\,\frac{\partial \tilde{F}}{\partial r_r} + \tilde{a}_n(\cdot)\,\frac{\partial \tilde{F}}{\partial r_n} + \tilde{a}_0(\cdot)\,\tilde{F} \tag{2-2a}$$

with the following abbreviations

$$
\begin{aligned}
\tilde{a}_{rr}(\cdot) &= \frac{1}{2}\, s_1(\cdot)^2 \,, \\
\tilde{a}_{rn}(\cdot) &= s_1(\cdot)^2 \frac{\partial r_n}{\partial r_r} + \rho_{12}\, s_1(\cdot)\, s_2(\cdot) \frac{\partial r_n}{\partial \bar{r}_y} \,, \\
\tilde{a}_{nn}(\cdot) &= \frac{1}{2}\, s_1(\cdot)^2 \left(\frac{\partial r_n}{\partial r_r}\right)^2 + \rho_{12}\, s_1(\cdot)\, s_2(\cdot) \frac{\partial r_n}{\partial r_r} \frac{\partial r_n}{\partial \bar{r}_y} + \frac{1}{2}\, s_2(\cdot)^2 \left(\frac{\partial r_n}{\partial \bar{r}_y}\right)^2 , \\
\tilde{a}_r(\cdot) &= m_1(\cdot) + \psi_1(\cdot)\, s_1(\cdot) - \rho_{13}\, s_1(\cdot)\, s_3(\cdot) \,, \\
\tilde{a}_n(\cdot) &= \left[m_2(\cdot) + \psi_2(\cdot)\, s_2(\cdot) - \rho_{23}\, s_2(\cdot)\, s_3(\cdot)\right] \frac{\partial r_n}{\partial \bar{r}_y} \\
&\quad + \frac{1}{2}\, s_1(\cdot)^2 \frac{\partial^2 r_n}{\partial r_r^2} + \rho_{12}\, s_1(\cdot)\, s_2(\cdot) \frac{\partial^2 r_n}{\partial r_r\, \partial \bar{r}_y} + \frac{1}{2}\, s_2(\cdot)^2 \frac{\partial^2 r_n}{\partial \bar{r}_y^2} \\
&\quad + \left[m_1(\cdot) + \psi_1(\cdot)\, s_1(\cdot) - \rho_{13}\, s_1(\cdot)\, s_3(\cdot)\right] \frac{\partial r_n}{\partial r_r} + \frac{\partial r_n}{\partial t} \,, \\
\tilde{a}_0(\cdot) &= -\left[r_r + m_3(\cdot) + \psi_3(\cdot)\, s_3(\cdot) - s_3(\cdot)^2\right] .
\end{aligned}
\tag{2-2b}
$$

It is common to reduce the domain of the state variables to the unit square by a transformation of variables as follows.

$$
\begin{aligned}
\tilde{F}(\tau, T, r_r, r_n) &= F(\tau, T, x_1, x_2) \\
x_1 &= \varphi_1(r_r)\,, \quad x_1 \in [0, 1]\,, \\
x_2 &= \varphi_2(r_n)\,, \quad x_2 \in [0, 1]\,,
\end{aligned}
\tag{2-3}
$$

where a suitable candidate for the functions φ_j ($j = 1, 2$) is either a rational function, e. g., $\varphi_1 = 1 / (1 + r_r)$, as proposed by Brennan and Schwarz [1977], or the hyperbolic tangent considered in this paper. The former transformation reverses the direction. Then, the PDE (2-2) can be written as

$$
\frac{\partial F}{\partial \tau} = a_{11}(\cdot) \frac{\partial^2 F}{\partial x_1^2} + a_{12}(\cdot) \frac{\partial^2 F}{\partial x_1\, \partial x_2} + a_{22}(\cdot) \frac{\partial^2 F}{\partial x_2^2} + a_1(\cdot) \frac{\partial F}{\partial x_1} + a_2(\cdot) \frac{\partial F}{\partial x_2} + a_0(\cdot)\, F
\tag{2-4a}
$$

with the following abbreviations

$$
\begin{aligned}
a_{11}(\cdot) &= \tilde{a}_{rr}(\cdot) \left(\frac{\mathrm{d}x_1}{\mathrm{d}r_r}\right)^2 , & a_{12}(\cdot) &= \tilde{a}_{rn}(\cdot) \frac{\mathrm{d}x_1}{\mathrm{d}r_r} \frac{\mathrm{d}x_2}{\mathrm{d}r_n} \,, \\
a_{22}(\cdot) &= \tilde{a}_{nn}(\cdot) \left(\frac{\mathrm{d}x_2}{\mathrm{d}r_n}\right)^2 , & a_1(\cdot) &= \tilde{a}_r(\cdot) \frac{\mathrm{d}x_1}{\mathrm{d}r_r} + \tilde{a}_{rr}(\cdot) \frac{\mathrm{d}^2 x_1}{\mathrm{d}r_r^2} \,, \\
a_2(\cdot) &= \tilde{a}_n(\cdot) \frac{\mathrm{d}x_2}{\mathrm{d}r_n} + \tilde{a}_{nn}(\cdot) \frac{\mathrm{d}^2 x_2}{\mathrm{d}r_n^2} \,, & a_0(\cdot) &= \tilde{a}_0(\cdot) \,.
\end{aligned}
\tag{2-4b}
$$

Assuming safely that

$$
\left|\frac{\partial r_n}{\partial r_r}\right| < \infty \text{ as } r_r \to 0, \qquad \left|\frac{\partial r_n}{\partial \bar{r}_y}\right| < \infty \text{ as } \bar{r}_y \to \bar{r}_y^{\min}(\cdot),
\tag{2-5a}
$$

and choosing the two functions φ_j (j = 1, 2) such that

$$\left|\frac{\mathrm{d}x_1}{\mathrm{d}r_r}\right| < \infty \text{ as } r_r \to 0, \qquad \left|\frac{\mathrm{d}x_2}{\mathrm{d}r_n}\right| < \infty \text{ as } r_n \to 0, \tag{2-5b}$$

then the boundary conditions as given in equation (1-41) translate into

$$\begin{aligned} &F(\tau, T, x_1, x_2) \to 0 \text{ as } x_1 \to \varphi_1(\infty), && F(\tau, T, x_1, x_2) \to 0 \text{ as } x_2 \to \varphi_2(\infty), \\ &\left|\frac{\partial F(\tau, T, x_1, x_2)}{\partial x_1}\right| < \infty \text{ as } x_1 \to \varphi_1(0), && \left|\frac{\partial F(\tau, T, x_1, x_2)}{\partial x_2}\right| < \infty \text{ as } x_2 \to \varphi_2(0), \end{aligned} \tag{2-5c}$$

Now, we are ready to apply the method of lines to the PDE (2-4) subject to the boundary conditions (2-5) and the initial condition $F(0, T, \cdot) = 1$.

2-1 The Method of Lines

Since the terminal condition requires that the price of the nominal pure disount bond is equal to one unit of money at the maturity date T, there exist singularities along two of the four boundaries where the price falls from its terminal value of one to a value of zero during the first instant of time. When solving numerically the PDE by means of a stable finite difference method such as the implicit finite difference method, these singularities may cause oscillations of prices (Ames, 1992, Smith, 1985). Indeed, option prices calculated by the implicit finite difference method often turn out to be negative for certain prices of the underlying asset (Hull, 2006, Büttler, 1995). Moreover, most of the finite difference methods (implicit, explicit, Crank-Nicolson) use a first-order finite difference approximation with respect to time, but a second-order finite difference approximation with respect to the state variables. A better approximation with respect to time requires the use of a finite-difference method with a fixed time extrapolation (Lawson and Morris, 1978, Gourlay and Morris, 1980). A variable rather than a fixed time extrapolation can be obtained by the method of lines which is considered in this paper (Ames, 1992, Smith, 1985).

The method of lines reduces a partial differential equation to a system of ordinary differential equations (ODEs), usually by finite difference approximations to the partial derivatives with respect to the state variables (exluding the time variable). Hence, the resulting system of ODEs can be solved by a variable time extrapolating technique based on the Runge-Kutta method.

As an example, we describe the finite difference approximation to the partial derivatives for *fixed* meshes. In many practical applications, however, one is interested in particular values of the state variables which, in turn, imply irregular meshes (Schwarz, 1988, Büttler, 1995).

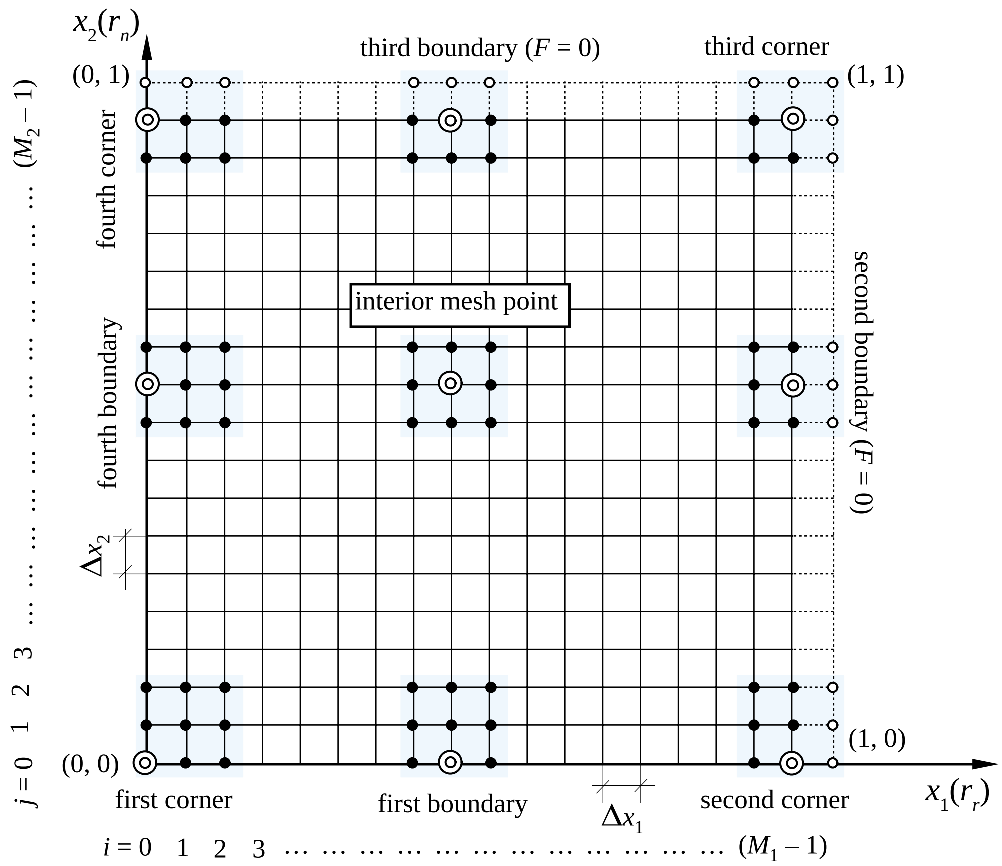

Figure 2: Meshes and Finite Differences

The grid of the domain is shown in Figure 2 where we underlie two functions of transformation φ_j (j = 1, 2) which preserve the direction. The x_k-axis is divided into M_k increments Δx_k, k = 1, 2. Let $F_{i,\,j}(\cdot)$ denote the price of the nominal pure discount bond evaluated at $x_1 = i\,\Delta x_1$ and $x_2 = j\,\Delta x_2$ for $i = 0, 1, 2, \ldots, (M_1 - 1)$ and $j = 0, 1, 2, \ldots, (M_2 - 1)$. We approximate all partial derivatives by second-order finite differences. For an *interior mesh point*, these are given by the following expressions.

$$\frac{\partial F_{i,j}}{\partial x_1} = \frac{1}{2\,\Delta x_1}\left[F_{i+1,\,j} - F_{i-1,\,j}\right], \qquad \frac{\partial F_{i,j}}{\partial x_2} = \frac{1}{2\,\Delta x_2}\left[F_{i,\,j+1} - F_{i,\,j-1}\right],$$

$$\frac{\partial^2 F_{i,j}}{\partial x_1^2} = \frac{1}{\Delta^2 x_1}\left[F_{i+1,\,j} - 2\,F_{i,j} + F_{i-1,\,j}\right], \qquad \frac{\partial^2 F_{i,j}}{\partial x_2^2} = \frac{1}{\Delta^2 x_2}\left[F_{i,\,j+1} - 2\,F_{i,j} + F_{i,\,j-1}\right],$$

$$\frac{\partial^2 F_{i,j}}{\partial x_1\,\partial x_2} = \frac{1}{4\,\Delta x_1\,\Delta x_2}\left[F_{i+1,\,j+1} - F_{i-1,j+1} + F_{i-1,\,j-1} - F_{i+1,\,j-1}\right], \quad \Delta^2 x_k \equiv \left(\Delta x_k\right)^2 \; (k = 1, 2). \tag{2-6}$$

The interior mesh point for which the PDE applies is indicated by a double circle in Figure 2. The neighbouring prices which determine the PDE at the location (i, j) are indicated by small, filled circles. Substituting the equations above into the PDE (2-4), we get the following linear equation for the interior mesh points.

Interior mesh point: $i = 1, 2, \ldots, M_1 - 2,\ j = 1, 2, \ldots, M_2 - 2,$

$$\begin{aligned}\frac{\partial F_{i,j}}{\partial \tau} = & \frac{a_{12}(\cdot)}{4\,\Delta x_1\,\Delta x_2} F_{i-1,\,j-1} + \left[\frac{a_{22}(\cdot)}{\Delta^2 x_2} - \frac{a_2(\cdot)}{2\,\Delta x_2}\right] F_{i,\,j-1} - \frac{a_{12}(\cdot)}{4\,\Delta x_1\,\Delta x_2} F_{i+1,\,j-1} \\ & + \left[\frac{a_{11}(\cdot)}{\Delta^2 x_1} - \frac{a_1(\cdot)}{2\,\Delta x_1}\right] F_{i-1,\,j} + \left[a_0(\cdot) - \frac{2\,a_{11}(\cdot)}{\Delta^2 x_1} - \frac{2\,a_{22}(\cdot)}{\Delta^2 x_2}\right] F_{i,j} + \left[\frac{a_{11}(\cdot)}{\Delta^2 x_1} + \frac{a_1(\cdot)}{2\,\Delta x_1}\right] F_{i+1,\,j} \\ & - \frac{a_{12}(\cdot)}{4\,\Delta x_1\,\Delta x_2} F_{i-1,\,j+1} + \left[\frac{a_{22}(\cdot)}{\Delta^2 x_2} + \frac{a_2(\cdot)}{2\,\Delta x_2}\right] F_{i,\,j+1} + \frac{a_{12}(\cdot)}{4\,\Delta x_1\,\Delta x_2} F_{i+1,j+1}\end{aligned} \tag{2-7}$$

For the mesh points along the *first boundary* (see Fig. 2), we use the following finite differences to replace the ones given in equation (2-6).

$$\begin{aligned}&\frac{\partial F_{i,j}}{\partial x_2} = \frac{1}{2\,\Delta x_2}\left[-3\,F_{i,j} + 4\,F_{i,\,j+1} - F_{i,\,j+2}\right], \quad \frac{\partial^2 F_{i,j}}{\partial x_2^2} = \frac{1}{\Delta^2 x_2}\left[F_{i,\,j+2} - 2\,F_{i,\,j+1} + F_{i,j}\right], \\ &\frac{\partial^2 F_{i,j}}{\partial x_1\,\partial x_2} = \frac{1}{4\,\Delta x_1\,\Delta x_2}\left\{-3\left[F_{i+1,\,j} - F_{i-1,\,j}\right] + 4\left[F_{i+1,\,j+1} - F_{i-1,\,j+1}\right] - \left[F_{i+1,\,j+2} - F_{i-1,\,j+2}\right]\right\}.\end{aligned} \tag{2-8}$$

The first expression is the second-order finite difference approximation to the first partial derivative with respect to x_2 at an initial point. The second expression sets the second partial derivative with respect to x_2 at the location (i, j) equal to the one at the location $(i, j + 1)$. If the second partial derivative is finite inside the domain, then it remains finite on the first boundary which in turn fulfils the fourth boundary condition as given in equation (2-5c). The third expression is the second-order finite difference approximation to the mixed partial derivative with respect to both x_1 and x_2 at an initial point. Substituting the equations above into the PDE (2-4), we get the following linear equation for the mesh points along the first boundary.

First boundary: $i = 1, 2, \ldots, M_1 - 2,\ j = 0,$

$$\begin{aligned}\frac{\partial F_{i,j}}{\partial \tau} = & \left[\frac{a_{11}(\cdot)}{\Delta^2 x_1} - \frac{a_1(\cdot)}{2\,\Delta x_1} + \frac{3\,a_{12}(\cdot)}{4\,\Delta x_1\,\Delta x_2}\right] F_{i-1,\,j} + \left[a_0(\cdot) - \frac{2\,a_{11}(\cdot)}{\Delta^2 x_1} + \frac{a_{22}(\cdot)}{\Delta^2 x_2} - \frac{3\,a_2(\cdot)}{2\,\Delta x_2}\right] F_{i,j} \\ & + \left[\frac{a_{11}(\cdot)}{\Delta^2 x_1} + \frac{a_1(\cdot)}{2\,\Delta x_1} - \frac{3\,a_{12}(\cdot)}{4\,\Delta x_1\,\Delta x_2}\right] F_{i+1,\,j} - \frac{a_{12}(\cdot)}{\Delta x_1\,\Delta x_2} F_{i-1,\,j+1} + \left[\frac{2\,a_2(\cdot)}{\Delta x_2} - \frac{2\,a_{22}(\cdot)}{\Delta^2 x_2}\right] F_{i,\,j+1} \\ & + \frac{a_{12}(\cdot)}{\Delta x_1\,\Delta x_2} F_{i+1,\,j+1} + \frac{a_{12}(\cdot)}{4\,\Delta x_1\,\Delta x_2} F_{i-1,\,j+2} + \left[\frac{a_{22}(\cdot)}{\Delta^2 x_2} - \frac{a_2(\cdot)}{2\,\Delta x_2}\right] F_{i,\,j+2} - \frac{a_{12}(\cdot)}{4\,\Delta x_1\,\Delta x_2} F_{i+1,j+2}\end{aligned} \tag{2-9}$$

For the mesh points along the vertical grid line next to the *second boundary* (see Fig. 2), we use the finite differences for an interior mesh point where we set the prices on the second boundary equal to zero according to the first boundary condition as given in equation (2-5c). Hence, we get the following linear equation for the mesh points along the vertical grid line next to the second boundary.

Second boundary: $i = M_1 - 1,\ j = 1, 2, \ldots, M_2 - 2$,

$$\frac{\partial F_{i,j}}{\partial \tau} = \frac{a_{12}(\cdot)}{4\,\Delta x_1\,\Delta x_2}\,F_{i-1,\,j-1} + \left[\frac{a_{22}(\cdot)}{\Delta^2 x_2} - \frac{a_2(\cdot)}{2\,\Delta x_2}\right] F_{i,\,j-1} + \left[\frac{a_{11}(\cdot)}{\Delta^2 x_1} - \frac{a_1(\cdot)}{2\,\Delta x_1}\right] F_{i-1,\,j}$$
$$+ \left[a_0(\cdot) - \frac{2\,a_{11}(\cdot)}{\Delta^2 x_1} - \frac{2\,a_{22}(\cdot)}{\Delta^2 x_2}\right] F_{i,j} - \frac{a_{12}(\cdot)}{4\,\Delta x_1\,\Delta x_2}\,F_{i-1,\,j+1} + \left[\frac{a_{22}(\cdot)}{\Delta^2 x_2} + \frac{a_2(\cdot)}{2\,\Delta x_2}\right] F_{i,\,j+1} \tag{2-10}$$

The mesh points with zero prices are indicated by small, empty circles in Figure 2. For the mesh points along the horizontal grid line next to the *third boundary* (see Fig. 2), we use the finite differences for an interior mesh point where we set the prices on the third boundary equal to zero according to the second boundary condition as given in equation (2-5c). Hence, we get the following linear equation for the mesh points along the horizontal grid line next to the third boundary.

Third boundary: $i = 1, 2, \ldots, M_1 - 2,\ j = M_2 - 1$,

$$\frac{\partial F_{i,j}}{\partial \tau} = \frac{a_{12}(\cdot)}{4\,\Delta x_1\,\Delta x_2}\,F_{i-1,\,j-1} + \left[\frac{a_{22}(\cdot)}{\Delta^2 x_2} - \frac{a_2(\cdot)}{2\,\Delta x_2}\right] F_{i,\,j-1} - \frac{a_{12}(\cdot)}{4\,\Delta x_1\,\Delta x_2}\,F_{i+1,\,j-1}$$
$$+ \left[\frac{a_{11}(\cdot)}{\Delta^2 x_1} - \frac{a_1(\cdot)}{2\,\Delta x_1}\right] F_{i-1,\,j} + \left[a_0(\cdot) - \frac{2\,a_{11}(\cdot)}{\Delta^2 x_1} - \frac{2\,a_{22}(\cdot)}{\Delta^2 x_2}\right] F_{i,j} + \left[\frac{a_{11}(\cdot)}{\Delta^2 x_1} + \frac{a_1(\cdot)}{2\,\Delta x_1}\right] F_{i+1,\,j} \tag{2-11}$$

For the mesh points along the *fourth boundary* (see Fig. 2), we use the following finite differences to replace the ones given in equation (2-6).

$$\frac{\partial F_{i,j}}{\partial x_1} = \frac{1}{2\,\Delta x_1}\left[-3\,F_{i,j} + 4\,F_{i+1,\,j} - F_{i+2,\,j}\right], \quad \frac{\partial^2 F_{i,j}}{\partial x_1^2} = \frac{1}{\Delta^2 x_1}\left[F_{i+2,\,j} - 2\,F_{i+1,\,j} + F_{i,j}\right],$$
$$\frac{\partial^2 F_{i,j}}{\partial x_1\,\partial x_2} = \frac{1}{4\,\Delta x_1\,\Delta x_2}\left\{-3\left[F_{i,\,j+1} - F_{i,\,j-1}\right] + 4\left[F_{i+1,\,j+1} - F_{i+1,\,j-1}\right] - \left[F_{i+2,\,j+1} - F_{i+2,\,j-1}\right]\right\}. \tag{2-12}$$

The first expression is the second-order finite difference approximation to the first partial derivative with respect to x_1 at an initial point. The second expression sets the second partial derivative with respect to x_1 at the location (i, j) equal to the one at the location $(i + 1, j)$. If the second partial derivative is finite inside the domain, then it remains finite on the first boundary which in turn fulfils the third boundary condition as given in equation (2-5c). The third expression is the second-order finite difference approximation to the mixed partial derivative with respect to both x_1 and x_2 at an initial point. Substituting the equations above into the PDE (2-4), we get the following linear equation for the mesh points along the fourth boundary.

Fourth boundary: $i = 0,\ j = 1, 2, \ldots, M_2 - 2$,

$$\frac{\partial F_{i,j}}{\partial \tau} = \left[\frac{a_{22}(\cdot)}{\Delta^2 x_2} - \frac{a_2(\cdot)}{2\,\Delta x_2} + \frac{3\,a_{12}(\cdot)}{4\,\Delta x_1\,\Delta x_2}\right] F_{i,\,j-1} - \frac{a_{12}(\cdot)}{\Delta x_1\,\Delta x_2} F_{i+1,\,j-1} + \frac{a_{12}(\cdot)}{4\,\Delta x_1\,\Delta x_2} F_{i+2,\,j-1}$$
$$+ \left[a_0(\cdot) + \frac{a_{11}(\cdot)}{\Delta^2 x_1} - \frac{2\,a_{22}(\cdot)}{\Delta^2 x_2} - \frac{3\,a_1(\cdot)}{2\,\Delta x_1}\right] F_{i,j} + \left[\frac{2\,a_1(\cdot)}{\Delta x_1} - \frac{2\,a_{11}(\cdot)}{\Delta^2 x_1}\right] F_{i+1,\,j}$$
$$+ \left[\frac{a_{11}(\cdot)}{\Delta^2 x_1} - \frac{a_1(\cdot)}{2\,\Delta x_1}\right] F_{i+2,\,j} + \left[\frac{a_{22}(\cdot)}{\Delta^2 x_2} + \frac{a_2(\cdot)}{2\,\Delta x_2} - \frac{3\,a_{12}(\cdot)}{4\,\Delta x_1\,\Delta x_2}\right] F_{i,\,j+1} + \frac{a_{12}(\cdot)}{\Delta x_1\,\Delta x_2} F_{i+1,\,j+1}$$
$$- \frac{a_{12}(\cdot)}{4\,\Delta x_1\,\Delta x_2} F_{i+2,j+1} \tag{2-13}$$

For the mesh point in the *first corner* (see Fig. 2), we use the finite difference approximations to the partial derivatives with respect to both directions at an initial point as given in equations (2-8) and (2-12). The finite difference approximation to the mixed partial derivative at an initial point now becomes the following expression.

$$\frac{\partial^2 F_{i,j}}{\partial x_1\,\partial x_2} = \frac{1}{4\,\Delta x_1\,\Delta x_2}\big[9\,F_{i,j} - 12\,F_{i+1,\,j} + 3\,F_{i+2,\,j} - 12\,F_{i,\,j+1} + 16\,F_{i+1,\,j+1} - 4\,F_{i+2,j+1}$$
$$+ 3\,F_{i,\,j+2} - 4\,F_{i+1,j+2} + F_{i+2,j+2}\big] \tag{2-14}$$

Substituting these equations into the PDE (2-4), we get the following linear equation for the mesh point in the first corner.

First corner: $i = 0,\ j = 0$,

$$\frac{\partial F_{i,j}}{\partial \tau} = \left[a_0(\cdot) + \frac{a_{11}(\cdot)}{\Delta^2 x_1} + \frac{9\,a_{12}(\cdot)}{4\,\Delta x_1\,\Delta x_2} + \frac{a_{22}(\cdot)}{\Delta^2 x_2} - \frac{3\,a_1(\cdot)}{2\,\Delta x_1} - \frac{3\,a_2(\cdot)}{2\,\Delta x_2}\right] F_{i,j}$$
$$+ \left[\frac{2\,a_1(\cdot)}{\Delta x_1} - \frac{2\,a_{11}(\cdot)}{\Delta^2 x_1} - \frac{3\,a_{12}(\cdot)}{\Delta x_1\,\Delta x_2}\right] F_{i+1,\,j} + \left[\frac{a_{11}(\cdot)}{\Delta^2 x_1} + \frac{3\,a_{12}(\cdot)}{4\,\Delta x_1\,\Delta x_2} - \frac{a_1(\cdot)}{2\,\Delta x_1}\right] F_{i+2,\,j}$$
$$+ \left[\frac{2\,a_2(\cdot)}{\Delta x_2} - \frac{2\,a_{22}(\cdot)}{\Delta^2 x_2} - \frac{3\,a_{12}(\cdot)}{\Delta x_1\,\Delta x_2}\right] F_{i,\,j+1} + \frac{4\,a_{12}(\cdot)}{\Delta x_1\,\Delta x_2} F_{i+1,\,j+1} - \frac{a_{12}(\cdot)}{\Delta x_1\,\Delta x_2} F_{i+2,j+1}$$
$$+ \left[\frac{a_{22}(\cdot)}{\Delta^2 x_2} + \frac{3\,a_{12}(\cdot)}{4\,\Delta x_1\,\Delta x_2} - \frac{a_2(\cdot)}{2\,\Delta x_2}\right] F_{i,\,j+2} - \frac{a_{12}(\cdot)}{\Delta x_1\,\Delta x_2} F_{i+1,j+2} + \frac{a_{12}(\cdot)}{4\,\Delta x_1\,\Delta x_2} F_{i+2,j+2} \tag{2-15}$$

For the mesh point in the *second corner* (see Fig. 2), we use the finite differences for a mesh point on the first boundary where we set the prices on the second boundary equal to zero according to the first boundary condition as given in equation (2-5c), that is,

Second corner: $i = M_1 - 1,\ j = 0$,

$$\frac{\partial F_{i,j}}{\partial \tau} = \left[\frac{a_{11}(\cdot)}{\Delta^2 x_1} - \frac{a_1(\cdot)}{2\,\Delta x_1} + \frac{3\,a_{12}(\cdot)}{4\,\Delta x_1\,\Delta x_2}\right] F_{i-1,\,j} + \left[a_0(\cdot) - \frac{2\,a_{11}(\cdot)}{\Delta^2 x_1} + \frac{a_{22}(\cdot)}{\Delta^2 x_2} - \frac{3\,a_2(\cdot)}{2\,\Delta x_2}\right] F_{i,j}$$
$$- \frac{a_{12}(\cdot)}{\Delta x_1\,\Delta x_2} F_{i-1,\,j+1} + \left[\frac{2\,a_2(\cdot)}{\Delta x_2} - \frac{2\,a_{22}(\cdot)}{\Delta^2 x_2}\right] F_{i,\,j+1} + \frac{a_{12}(\cdot)}{4\,\Delta x_1\,\Delta x_2} F_{i-1,\,j+2}$$
$$+ \left[\frac{a_{22}(\cdot)}{\Delta^2 x_2} - \frac{a_2(\cdot)}{2\,\Delta x_2}\right] F_{i,\,j+2} \tag{2-16}$$

For the mesh point in the *third corner* (see Fig. 2), we use the finite differences for an interior mesh point where we set the prices on both the second and third boundary equal to zero according to both the first and second boundary conditions as given in equation (2-5c), that is,

$$\text{Third corner: } i = M_1 - 1,\ j = M_2 - 1,$$
$$\frac{\partial F_{i,j}}{\partial \tau} = \frac{a_{12}(\cdot)}{4\,\Delta x_1\,\Delta x_2} F_{i-1,\,j-1} + \left[\frac{a_{22}(\cdot)}{\Delta^2 x_2} - \frac{a_2(\cdot)}{2\,\Delta x_2}\right] F_{i,\,j-1} + \left[\frac{a_{11}(\cdot)}{\Delta^2 x_1} - \frac{a_1(\cdot)}{2\,\Delta x_1}\right] F_{i-1,\,j} + \left[a_0(\cdot) - \frac{2\,a_{11}(\cdot)}{\Delta^2 x_1} - \frac{2\,a_{22}(\cdot)}{\Delta^2 x_2}\right] F_{i,j} \qquad (2\text{-}17)$$

Finally, for the mesh point in the *fourth corner* (see Fig. 2), we use the finite differences for mesh points along the fourth boundary where we set the prices on the third boundary equal to zero according to the second boundary condition as given in equation (2-5c), that is,

$$\text{Fourth corner: } i = 0,\ j = M_2 - 1,$$
$$\begin{aligned}\frac{\partial F_{i,j}}{\partial \tau} = {}& \left[\frac{a_{22}(\cdot)}{\Delta^2 x_2} - \frac{a_2(\cdot)}{2\,\Delta x_2} + \frac{3\,a_{12}(\cdot)}{4\,\Delta x_1\,\Delta x_2}\right] F_{i,\,j-1} - \frac{a_{12}(\cdot)}{\Delta x_1\,\Delta x_2} F_{i+1,\,j-1} + \frac{a_{12}(\cdot)}{4\,\Delta x_1\,\Delta x_2} F_{i+2,\,j-1} \\ & + \left[a_0(\cdot) + \frac{a_{11}(\cdot)}{\Delta^2 x_1} - \frac{2\,a_{22}(\cdot)}{\Delta^2 x_2} - \frac{3\,a_1(\cdot)}{2\,\Delta x_1}\right] F_{i,j} + \left[\frac{2\,a_1(\cdot)}{\Delta x_1} - \frac{2\,a_{11}(\cdot)}{\Delta^2 x_1}\right] F_{i+1,\,j} \\ & + \left[\frac{a_{11}(\cdot)}{\Delta^2 x_1} - \frac{a_1(\cdot)}{2\,\Delta x_1}\right] F_{i+2,\,j}\end{aligned} \qquad (2\text{-}18)$$

Now, we are ready to enumerate consecutively all the mesh points or equations, respectively, from left to right and bottom to top as follows.

$$k = (i + 1) + j\,M_1\,, \quad i = 0, 1, \ldots, M_1 - 1\,, \quad j = 0, 1, \ldots, M_2 - 1\,. \qquad (2\text{-}19)$$

Collecting all the coefficients of equations (2-7), (2-9), (2-11), (2-13), (2-15) - (2-18) in the $(M_1\,M_2) \times (M_1\,M_2)$ matrix $\mathbf{A}(\tau)$ and all the prices F_k, $k = 1, 2, \ldots, (M_1\,M_2)$, in the column vector $\mathbf{F}(\tau)$, then we can write the system of ODEs compactly as follows.

$$\frac{\partial \mathbf{F}}{\partial \tau} = \mathbf{A}(\tau)\,\mathbf{F}(\tau)\,. \qquad (2\text{-}20)$$

The matrix **A** has three main diagonal bands and four additional subdiagonals or superdiagonals, respectively. The structure of the matrix **A** is shown in the appendix B for $M_1 = 5$ and $M_2 = 6$. If the system of ODEs (2-20) is autonomous, then the computation time is considerably reduced.

2-2 The Numerical Accuracy

The numerical accuracy of the solution is determined by both the order of the finite difference approximation and the mesh size. Shrinking the mesh size beyond a certain size, however, does not necessarily improve the accuracy of the numerical solution for the following reason. It is a well-known fact from numerical mathematics that the number of significant decimal digits of the largest solution in absolute value of a linear equation system is approximately equal to the number of decimal digits used by the computer's floating-point arithmetic, minus the logarithm to the base 10 of the condition number of this equation system, minus one (Schwarz, 1988). For instance, if a particular computer uses 16 decimal digits for its floating-point calculations, and if the condition number of the linear equation system considered is equal to 10^8 then the number of significant digits of the largest solution in absolute value of this linear equation system is equal to 7 (= 16 – 8 – 1). The condition number of a linear equation system is defined as the ratio of the largest to the smallest singular value (Press et al., 1990). A matrix is singular if its condition number is infinite, and it is ill-conditioned if its condition number is too large, that is, if its reciprocal approaches the computer's floating-point precision, which is approximately 10^{-16} on a PC or 10^{-19} on a Macintosh. It is well known that increasing the size of a linear equation system, other things being equal, increases the condition number. Hence, shrinking the mesh size increases the number of equations which in turn increases the condition number. The only way out of this dilemma is to use a variable-precision arithmetic at the expense of an extensive computation time.

We study the numerical accuracy of the PDE (2-4) versus the analytical solution of the model 2 of CIR [1985b] which has the following specification of the SDEs (1-1) - (1-3) in our notation.

$$\begin{aligned}
m_1(\cdot) &= \kappa_1\left[\theta_1 - r_r\right], & s_1(\cdot) &= \sigma_1\sqrt{r_r}\,, & &\kappa_1 = 0.2;\ \theta_1 = 0.015;\ \sigma_1 = 0.5,\\
m_2(\cdot) &= \kappa_2\left[\theta_2 - \bar{r}_y\right], & s_2(\cdot) &= \sigma_2\sqrt{\bar{r}_y}\,, & &\kappa_2 = 0.2;\ \theta_2 = 0.035;\ \sigma_2 = 0.1,\\
m_3(\cdot) &= \bar{r}_y & s_3(\cdot) &= \sigma_3\sqrt{\bar{r}_y}\,, & &\sigma_3 = 0.5;\ \rho_{23} = 0.5,\\
\psi_1(\cdot)\, s_1(\cdot) &= -\lambda\, r_r\,, & & & &\lambda = -0.01,\\
\rho_{12} = \rho_{13} &\equiv 0\,, & \psi_2(\cdot) &= \psi_3(\cdot) \equiv 0\,,\ \bar{r}_y^{\min} = 0\,.
\end{aligned} \tag{2-21}$$

The κ's denote the speed of adjustment, the θ's the long-run equilibrium values, the σ's the volatility parameters and λ a constant associated with the market price of interest rate risk. As noted before, the model 2 proposed by CIR, first, is separable, i. e., $\rho_{12} = \rho_{13} \equiv 0$. Second, it does not allow for "expected" deflation rates, i. e., $\bar{r}_y^{\min} = 0$. Third, it has no market prices of factor risk with respect to both the drift rate of the instantaneous spot inflation rate and the instantaneous spot inflation rate, i. e., $\psi_2 = \psi_3 \equiv 0$. The parameter values used for the CIR model are shown in the third column of equation (2-21).

The nominal instantaneous spot interest rate as given in equation (1-9) becomes in the CIR model the following expression.

$$r_n = r_r + \left[1 - \sigma_3^2\right]\bar{r}_y\,, \quad \bar{r}_y \geqq 0\,, \quad r_n \geqq r_r\,. \tag{2-22}$$

from which it follows that the nominal instantaneous spot interest rate is always greater than or equal to the real instantaneous spot interest rate due to the fact that there is no correlation between the real interest rate and the inflation rate in the CIR model.

The domain of the $(r_r, \bar{r}_y)$ state space is now the first quadrant. Therefore, we use the PDE (1-6) rather than the transformed PDE (1-43), the domain of which is now an upper-left triangle.

In order to reduce the first quadrant to the unit square as shown in equation (2-3), we use the hyperbolic tangent as follows.

$$x_1 = \varphi_1(r_r) \equiv \tanh(k_1\, r_r)\,, \quad \text{where } \frac{dx_1}{dr_r} = \frac{k_1}{\cosh^2(k_1\, r_r)}\,, \quad \frac{d^2x_1}{dr_r^{\,2}} = -\frac{2\,k_1^2 \sinh(k_1\, r_r)}{\cosh^3(k_1\, r_r)}$$
$$x_2 = \varphi_2(\bar{r}_y) \equiv \tanh(k_2\, \bar{r}_y)\,, \quad \text{where } \frac{dx_2}{d\bar{r}_y} = \frac{k_2}{\cosh^2(k_2\, \bar{r}_y)}\,, \quad \frac{d^2x_2}{d\bar{r}_y^{\,2}} = -\frac{2\,k_2^2 \sinh(k_2\, \bar{r}_y)}{\cosh^3(k_2\, \bar{r}_y)} \tag{2-23a}$$

with the corresponding inverse functions as follows.

$$r_r = \frac{\operatorname{arctanh}(x_1)}{k_1} \quad \text{and} \quad \bar{r}_y = \frac{\operatorname{arctanh}(x_2)}{k_2} \tag{2-23b}$$

Hence, equal mesh sizes on the unit square, the (x_1, x_2) state space, translate into variable mesh sizes on the first quadrant, the $(r_r, \bar{r}_y)$ state space.

Table 2-1 summarizes the numerical accuracy of the method of lines against the analytical solution of the CIR model prices for moderate mesh sizes and for a term to maturity of the pure discount bond of 30 years.

Table 2-1: Numerical Accuracy of the Method of Lines

Meshes $M_1 \times M_2$	$k_1 = k_2 = 1$		$k_1 = k_2 = 1.5$		$k_1 = k_2 = 2$	
	RMSE	Max. error	RMSE	Max. error	RMSE	Max. error
4 × 4	3.40e–02	4.74e–02	1.76e–02	3.00e–02	1.26e–02	2.61e–02
8 × 8	7.66e–03	1.39e–02	3.41e–03	7.15e–03	1.95e–03	4.51e–03
16 × 16	1.48e–03	3.12e–03	6.69e–04	1.42e–03	3.15e–04	7.52e–04
32 × 32	2.62e–04	5.19e–04	1.05e–04	1.88e–04	3.45e–05	1.30e–04
64 × 64	3.24e–05	4.79e–05	5.47e–06	2.43e–05	3.12e–05	6.89e–05

Comments: The numerical accuracy is tested against the CIR model. The term to maturity of the discount bond considered is 30 years. RMSE denotes the root mean squared error. The maximum error refers to errors in absolute value. Both the root mean squared error and the maximum error are calculated for interest rates and inflation rates below 50%.

Table 2-1 shows that the numerical accuracy is slightly improved when setting the stretching parameters, k_1 and k_2, in equation (2-23) between 1.5 and 2 for large mesh sizes. Some experiments show that the numerical accuracy does not change considerably when the parameter values are varied within reasonable bounds. If the $(r_r, \bar{r}_y)$ state space is the first quadrant, 32 × 32 mesh points seem to be sufficient.

3 Two Specifications

In this section, the numerical solutions of the PDE to value a non-indexed pure discount bond are presented for two model specifications. The first model is an extended version of the CIR model considered in chapter 2 to test the numerical accuracy of the method of lines. It allows for real effects of monetary policy in the short run such that the correlation between the real instantaneous spot interest rate and the other two factors are taken into account. The second model is an extension of the first model such that it allows also for an "expected" *deflation* rate. Both models will be compared with the CIR model.

3-1 Model 1: Real Effects of Monetary Policy in the Short Run

In the Model 1, monetary policy has an impact on the real economy in the short run. A permanent monetary expansion lowers in general the real interest rate in the short run, while inflationary expectations are increasing the nominal interest rate in the long run (Niehans, 1978). Hence, we expect the correlation coefficient between the real instantaneous spot interest rate and the "expected" instantaneous spot inflation rate to be negative. The same applies to the correlation between the real interest rate and the percentage change of the consumer price level.

The following specification of the SDEs (1-1) - (1-3) are used in the Model 1, where the first four lines are the same as in equation (2-21) for the CIR model.

$$
\begin{aligned}
&m_1(\cdot) = \kappa_1\left[\theta_1 - r_r\right], && s_1(\cdot) = \sigma_1\sqrt{r_r}, && \kappa_1 = 0.2;\ \theta_1 = 0.015;\ \sigma_1 = 0.5,\\
&m_2(\cdot) = \kappa_2\left[\theta_2 - \bar{r}_y\right], && s_2(\cdot) = \sigma_2\sqrt{\bar{r}_y}, && \kappa_2 = 0.2;\ \theta_2 = 0.035;\ \sigma_2 = 0.1,\\
&m_3(\cdot) = \bar{r}_y && s_3(\cdot) = \sigma_3\sqrt{\bar{r}_y}, && \sigma_3 = 0.5;\ \rho_{23} = 0.5,\\
&\psi_1(\cdot)\, s_1(\cdot) = -\lambda\, r_r, && && \lambda = -0.01,\\
&\psi_2(\cdot)\, s_2(\cdot) = \psi_w(\cdot)\,\rho_{12}\, s_1(\cdot)\, s_2(\cdot), && \psi_w(\cdot) = \frac{-\lambda}{\sigma_1^2}, && \rho_{12} = -0.6,\\
&\psi_3(\cdot)\, s_3(\cdot) = \psi_w(\cdot)\,\rho_{13}\, s_1(\cdot)\, s_3(\cdot), && && \rho_{13} = -0.3.
\end{aligned}
\tag{3-1}
$$

Again, the κ's denote the speed of adjustment, the θ's the long-run equilibrium values, the σ's the volatility parameters and λ a constant associated with the market price of interest rate risk. The first four lines are taken from equation (2-21). The modified relative risk aversion of consumers, $\psi_w(\cdot)$, has been obtained from setting the market premium of the real interest rate risk in the CIR model, $\psi_1(\cdot)\ s_1(\cdot) = -\lambda\ r_r$, equal to the expression in the first line of equation (1-34). Consistent within our framework, the other two market premia of factor risks follow from equation (1-34), too. The parameter values used for the Model 1 are shown in the third column of equation (3-1).

Upon substituting the market premium of the consumer price risk from equation (3-1) into equation (1-9), the nominal instantaneous spot interest rate becomes in the Model 1 the following expression.

$$r_n = r_r + \bar{r}_y + \psi_w(\cdot)\,\rho_{13}\,s_1(\cdot)\,s_3(\cdot) - s_3(\cdot)^2$$
$$= r_r + (1 - \sigma_3^2)\,\bar{r}_y + \frac{-\lambda}{\sigma_1}\,\rho_{13}\,\sigma_3\,\sqrt{r_r}\,\sqrt{\bar{r}_y} \qquad (3\text{-}2)$$

Setting the nominal instantaneous spot interest rate equal to zero, we find a unique root for the drift rate of the instantaneous spot inflation rate for a large parameter space, that is, $\bar{r}_y^{\min}$ = 0. Hence, the domain of the $(r_r, \bar{r}_y)$ state space is still the first quadrant as in the CIR model. Therefore, we use the PDE (1-6) rather than the transformed PDE (1-43), the domain of which is now an upper-left triangle. Note that the nominal instantaneous spot interest rate could be less than the real instantaneous spot interest rate, if the following condition holds true.

$$r_n \begin{cases} > r_r\,, & \text{if } \rho_{13} > 0 \\ \leqq r_r\,, & \text{if } \rho_{13} < 0 \text{ and } \dfrac{\bar{r}_y}{r_r} \leqq \left(\dfrac{-\lambda\,\rho_{13}\,\sigma_3}{\sigma_1\,(1-\sigma_3^2)} \right)^2 \end{cases} \qquad (3\text{-}3)$$

The last condition in the equation above implies a very small ratio of the drift rate of the instantaneous spot inflation rate to the real instantaneous spot interest rate in the order of magnitude of $1.6 \cdot 10^{-7}$ for the parameter values given in equation (3-1).

In order to reduce the first quadrant to the unit square as shown in equation (2-3), we use again the hyperbolic tangent as given in equation (2-23). The following results are obtained from 32 × 32 mesh points and from setting $k_1 = k_2 = 2$.

Table 3-1-1 summarizes the deviation of pure discount bond prices obtained from the Model 1 from those of the CIR model for various terms to maturity in years or tenors, respectively. For the parameter values chosen in equation (3-1), the root mean squared deviation of the Model 1 prices from those of the CIR model is in the order of magnitude of $2 \cdot 10^{-2}$.

Table 3-1-1: Summary of Model 1

Term to maturity	RMSD	Max. deviation	Min. deviation
1	6.05e–03	1.37e–02	1.18e–06
2	1.31e–02	2.12e–02	3.75e–05
3	1.72e–02	2.41e–02	1.48e–04
4	1.96e–02	2.67e–02	3.35e–04
5	2.08e–02	2.83e–02	5.85e–04
6	2.14e–02	2.91e–02	8.84e–04
7	2.17e–02	2.95e–02	1.22e–03
8	2.17e–02	2.95e–02	1.57e–03
9	2.16e–02	2.94e–02	1.94e–03
10	2.14e–02	2.91e–02	2.32e–03
20	1.78e–02	2.40e–02	3.55e–03
30	1.45e–02	1.91e–02	3.36e–03

Comments: RMSD denotes the root mean squared deviation of the Model 1 prices from the CIR prices. The term to maturity is measured in years. Both the root mean squared deviation and the maximum deviation are calculated for interest rates and inflation rates below 50%. The results are obtained from 32 × 32 mesh points and from setting $k_1 = k_2 = 2$ in equation (2-23).

Table 3-1-2 shows the prices of a ten-year pure discount bond for some pairs of real spot interest rates and “expected” spot inflation rates. The selected values for the real instantaneous spot interest rate and the “expected” spot inflation rate are given by the selceted 32 × 32 mesh points, by the transformation of variables as given in equation (2-23) and by setting $k_1 = k_2 = 2$. The prices obtained from Model 1 are shown in the first row, the prices obtained from the CIR model are shown in the second row and the difference between these two prices are shown in the third row. Table 3-1-2 shows that the deviation of Model 1 from the CIR model increses with increasing values of the two factors. The deviation is in the order of magnitude of $1 \cdot 10^{-2}$.

Table 3-1-2: Prices of Model 1 for a Ten-Year Pure Discount Bond

$\bar{r}_y$	r_r					
	0	1.56	3.13	4.7	6.28	7.88
0	8.2358e–01	7.9548e–01	7.6831e–01	7.4197e–01	7.1640e–01	6.9153e–01
	8.2590e–01	7.9840e–01	7.7176e–01	7.4592e–01	7.2079e–01	6.9632e–01
	–2.3197e–03	–2.9178e–03	–3.4575e–03	–3.9464e–03	–4.3902e–03	–4.7932e–03
1.56	7.8689e–01	7.5883e–01	7.3187e–01	7.0585e–01	6.8068e–01	6.5627e–01
	7.8976e–01	7.6346e–01	7.3799e–01	7.1328e–01	6.8925e–01	6.6586e–01
	–2.8708e–03	–4.6371e–03	–6.1285e–03	–7.4267e–03	–8.5707e–03	–9.5852e–03
3.13	7.5188e–01	7.2426e–01	6.9786e–01	6.7247e–01	6.4798e–01	6.2427e–01
	7.5514e–01	7.2999e–01	7.0564e–01	6.8201e–01	6.5904e–01	6.3667e–01
	–3.2514e–03	–5.7378e–03	–7.7856e–03	–9.5384e–03	–1.1062e–02	–1.2398e–02
4.7	7.1836e–01	6.9133e–01	6.6562e–01	6.4096e–01	6.1722e–01	5.9428e–01
	7.2190e–01	6.9787e–01	6.7459e–01	6.5200e–01	6.3004e–01	6.0865e–01
	–3.5407e–03	–6.5372e–03	–8.9719e–03	–1.1037e–02	–1.2818e–02	–1.4369e–02
6.28	6.8619e–01	6.5983e–01	6.3486e–01	6.1098e–01	5.8802e–01	5.6587e–01
	6.8995e–01	6.6698e–01	6.4473e–01	6.2314e–01	6.0215e–01	5.8171e–01
	–3.7649e–03	–7.1454e–03	–9.8677e–03	–1.2162e–02	–1.4132e–02	–1.5840e–02
7.88	6.5524e–01	6.2961e–01	6.0541e–01	5.8231e–01	5.6015e–01	5.3880e–01
	6.5917e–01	6.3723e–01	6.1597e–01	5.9534e–01	5.7529e–01	5.5576e–01
	–3.9390e–03	–7.6165e–03	–1.0559e–02	–1.3027e–02	–1.5139e–02	–1.6962e–02

Comments: The prices obtained from Model 1 are shown in the first row, the prices obtained from the CIR model are shown in the second row and the difference between these two prices are shown in the third row. Interest rates and inflation rates are continuously compounded in percent per annum.The results are obtained from 32 × 32 mesh points and from setting $k_1 = k_2 = 2$.

Table 3-1-3 shows the three selected factor combinations C1 through C3 for Figures 3-1-1a through Figures 3-1-1d of Model 1 for a ten-year discount bond with a face value of 100. In Figures 3-1-1a and 3-1-1b, the real spot interest rates varies between zero and 103.58% per annum for six different values of the “expected” spot inflation rate, namely 0%, 6.28%, 12.77%, 19.71%, 25.42% ans 34.17% per annum.. In Figures 3-1-1c and 3-1-1d, the “expected” spot inflation rate varies between zero and 103.58% per annum for six different values of the real spot interest rate, namely 0%, 6.28%, 12.77%, 19.71%, 25.42% ans 34.17% per annum.

Table 3-1-3: Factor Combinations for Figures 3-1-1 of Model 1

	Figure 3-1-1a		Figure 3-1-1b		Figure 3-1-1c		Figure 3-1-1d	
Combination	r_r	$\bar{r}_y$	r_r	$\bar{r}_y$	r_r	$\bar{r}_y$	r_r	$\bar{r}_y$
C1	(0, 103.58)	0	(0, 103.58)	19.71	0	(0, 103.58)	19.71	(0, 103.58)
C2	(0, 103.58)	6.28	(0, 103.58)	25.42	6.28	(0, 103.58)	25.42	(0, 103.58)
C3	(0, 103.58)	12.77	(0, 103.58)	34.17	12.77	(0, 103.58)	34.17	(0, 103.58)

Comment: Interest rates and inflation rates are continuously compounded in percent per annum. Note that the long-run equilibrium value of the real instantaneous spot interest rate (θ_1) is assumed to be 1.5% per annum and the one for the instantaneous spot inflation rate (θ_2) is assumed to be 3.5% per annum.

Figure 3-1-1a shows the prices of a discount bond with a face value of 100 as a function of the real spot interest rate in percent per annum for three values of the "expected" spot inflation rate which are 0% per annum (combination C1), 6.28% per annum (combination C2) and 12.77% per annum (combination C3).

Figure 3-1-1b shows the prices of a discount bond with a face value of 100 as a function of the real spot interest rate in percent per annum for three values of the "expected" spot inflation rate which are 19.71% per annum (combination C1), 25.42% per annum (combination C2) and 34.17% per annum (combination C3). As a result, the higher the "expected" spot inflation rate for a given real spot interest rate, the greater is the deviation of the price of Model 1 from that of the CIR model.

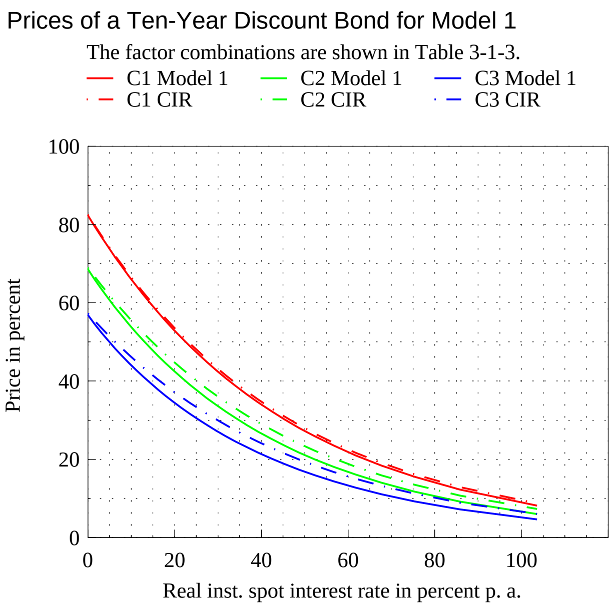

Figure 3-1-1a

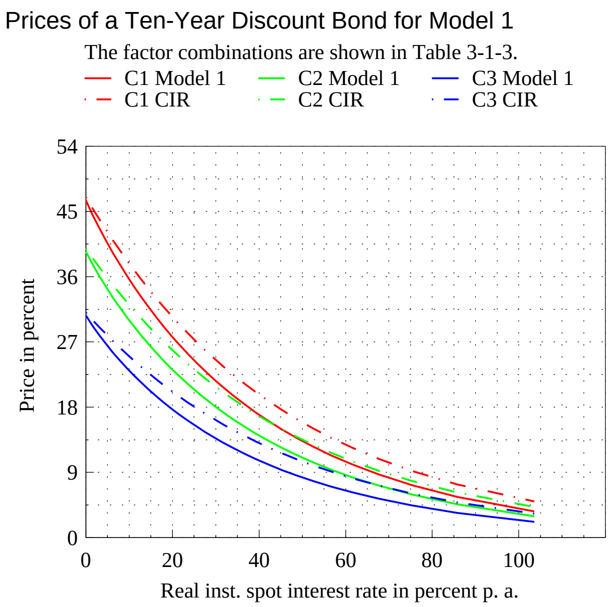

Figure 3-1-1b

Figure 3-1-1c shows the prices of a discount bond with a face value of 100 as a function of the "expected" spot inflation rate in percent per annum for three values of the real spot interest rate which are 0% per annum (combination C1), 6.28% per annum (combination C2) and 12.77% per annum (combination C3).

Figure 3-1-1d shows the prices of a discount bond with a face value of 100 as a function of the "expected" spot inflation rate in percent per annum for three values of the real spot interest rate which are 19.71% per annum (combination C1), 25.42% per annum (combination C2) and 34.17% per annum (combination C3). As a result, the higher the real spot interest rate for a given "expected" spot inflation rate, the greater is the deviation of the price of Model 1 from that of the CIR model.

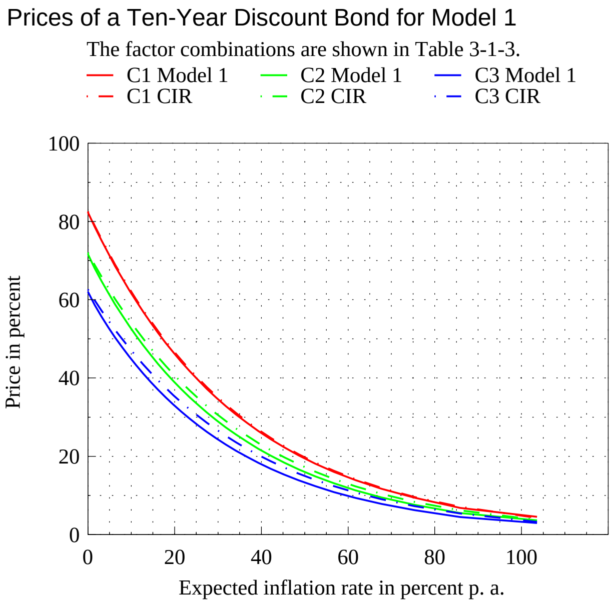

Figure 3-1-1c

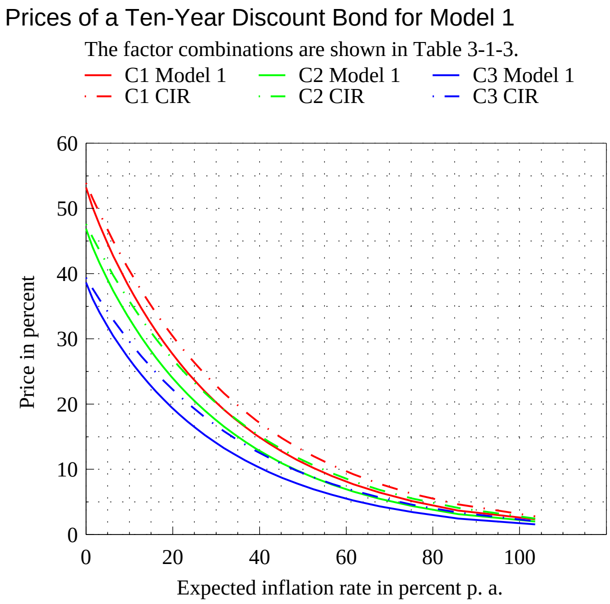

Figure 3-1-1d

Table 3-1-4 shows the three selected factor combinations C1 through C3 for Figures 3-1-2a through Figures 3-1-2d for the term structure of nominal spot interest rates obtained from Model 1. Both factors vary between zero and 6.28% per annum.

Table 3-1-4: Factor Combinations for Figures 3-1-2 of Model 1

	Figure 3-1-2a		Figure 3-1-2b		Figure 3-1-2c		Figure 3-1-2d	
Combination	r_r	$\bar{r}_y$	r_r	$\bar{r}_y$	r_r	$\bar{r}_y$	r_r	$\bar{r}_y$
C1	0	0	1.56	0	3.13	0	6.28	0
C2	0	3.13	1.56	3.13	3.13	3.13	6.28	3.13
C3	0	6.28	1.56	6.28	3.13	6.28	6.28	6.28

Comment: Interest rates and inflation rates are continuously compounded in percent per annum. Note that the long-run equilibrium value of the real instantaneous spot interest rate (θ_1) is assumed to be 1.5% per annum and the one for the instantaneous spot inflation rate (θ_2) is assumed to be 3.5% per annum.

Figure 3-1-2a shows the term structure of nominal spot interest rates when the real spot interest rate of 0% per annum is combined with three different values of the "expected" spot inflation rates which are 0% per annum (combination C1), 3.13% per annum (combination C2) and 6.28% per annum (combination C3). We obtain the three shapes given by the well-known one-factor term structure models, namely the normal shape (red curve for combination 1), the humped shape (green curve for combination 2) and the inverse shape (blue curve for combination 3). The differences between the term structure of nominal spot interest rates obtained from Model 1 and those obtained from the CIR model are moderate. They increase with increasing term to maturity or tenor, respectively.

Figure 3-1-2b shows the term structure of nominal spot interest rates when the real spot interest rate of 1.56% per annum is combined with three different values of the "expected" spot inflation rates which are 0% per annum (combination C1), 3.13% per annum (combination C2) and 6.28% per annum (combination C3). Again, we obtain the shapes given by the well-known one-factor term structure models with slightly more "curvature", namely the normal shape (red curve for combination 1), the humped shape (green curve for combination 2) and the inverse shape (blue curve for combination 3). The differences between the term structure of nominal spot interest rates obtained from Model 1 and those obtained from the CIR model are greater than in Figure 3-1-2a. They increase with increasing term to maturity or tenor, respectively.

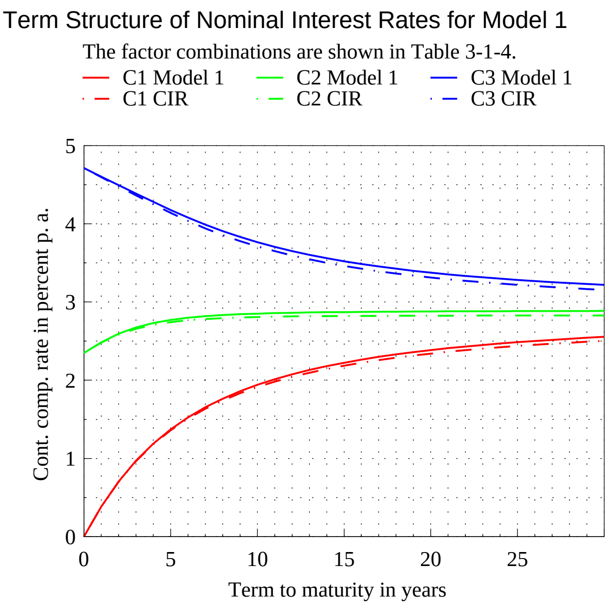

Figure 3-1-2a

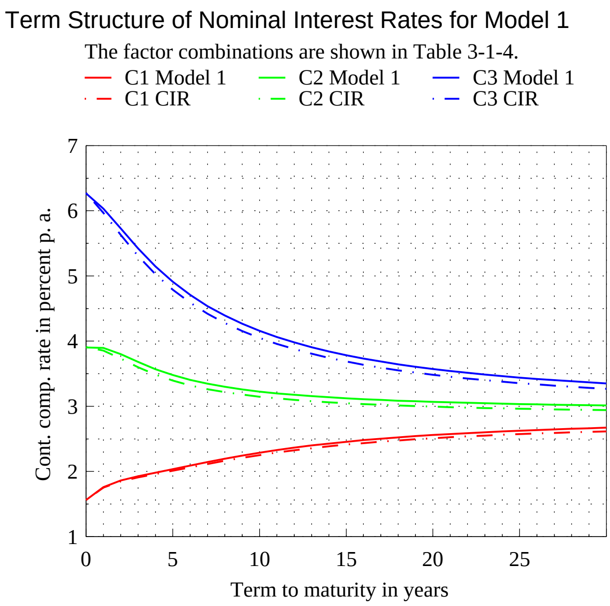

Figure 3-1-2b

Figure 3-1-2c shows the term structure of nominal spot interest rates when the real spot interest rate of 3.13% per annum is combined with three different values of the "expected" spot inflation rates which are 0% per annum (combination C1), 3.13% per annum (combination C2) and 6.28% per annum (combination C3). The shape of the red curve for combination 1 cannot be found with a one-factor model. It is humped, but approaches the long-run nominal spot interest rate from below The other two term structures are inverse with more "curvature" than in Figures 3-1-2a and 3-1-2b. The differences between the term structure of nominal spot interest rates obtained from Model 1 and those obtained from the CIR model are more accentuated. They increase with increasing term to maturity or tenor, respectively.

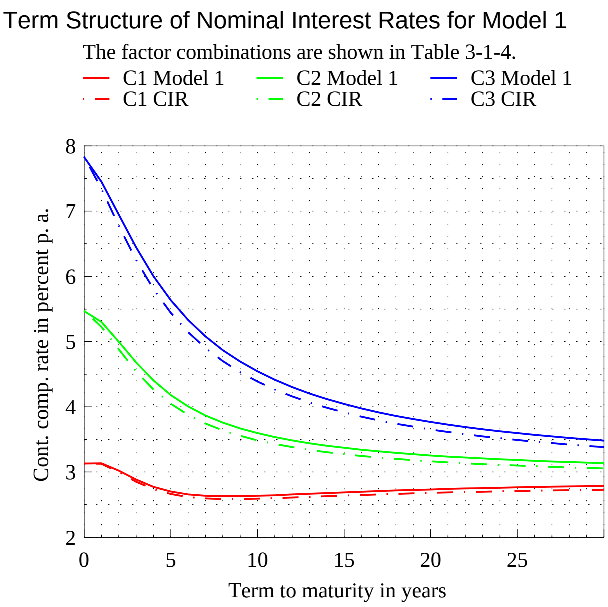

Figure 3-1-2c

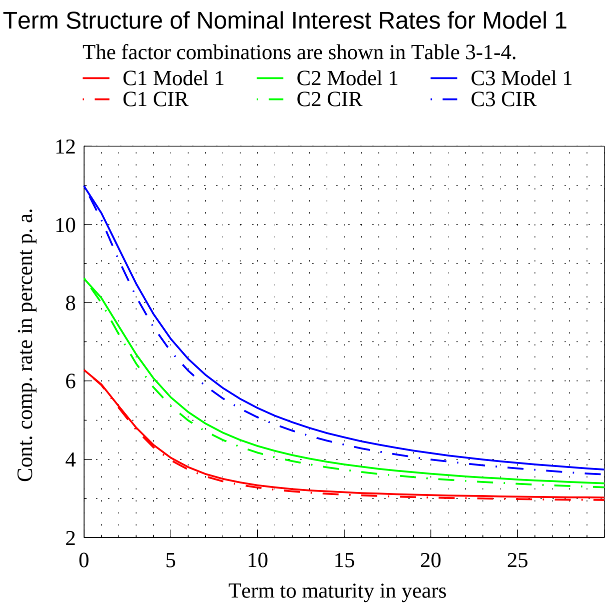

Figure 3-1-2d

Figure 3-1-2d shows the term structure of nominal spot interest rates when the real spot interest rate of 6.28% per annum is combined with three different values of the "expected" spot inflation rates which are 0% per annum (combination C1), 3.13% per annum (combination C2) and 6.28% per annum (combination C3). Since the actual values of the two factors are relatively big when compared with their long-run equilibrium values, all three term structures turn out to be inverse. The differences between the term structure of nominal spot interest rates obtained from Model 1 and those obtained from the CIR model are greater than in Figure 3-1-2a and Figure 3-1-2b. They increase with increasing term to maturity or tenor, respectively.

3-2 Model 2: Feasible "Expected" Deflation Rates

The well-known term structure models do not work properly when the real or nominal spot interest rate is low and/or when there is an "expected" deflation rate, a situation which has been encountered in Japan for the last twenty years (Hull, 2006). Hence, Model 2 allows for a negative drift rate of the instantaneous spot inflation rate or an "expected" deflation rate, respectively. We still retain the assumption that monetary policy can have an impact on the real economy in the short run.

We replace some specifications of Model 1 with those of Example 2 in section 1-11 as given in equations (1-38) through (1-40). Hence, Model 2 has the following drift functions, volatility functions, market premia and a modified relative risk aversion of consumers.

$$
\begin{aligned}
&m_1(\cdot) = \kappa_1 \left[\theta_1 - r_r\right] , && s_1(\cdot) = \sigma_1 \sqrt{r_r} \ , \\
&m_2(\cdot) = \kappa_2 \left[\theta_2 - \bar{r}_y\right] , && s_2(\cdot) = \sigma_2 \sqrt{\bar{r}_y - \bar{r}_y^{\min}} \ , \\
&m_3(\cdot) = \bar{r}_y && s_3(\cdot) = \sigma_3 \sqrt{\bar{r}_y - \bar{r}_y^{\min}} \ , \\
&\psi_1(\cdot)\, s_1(\cdot) = \psi_w(\cdot)\, s_1(\cdot)^2 , && \psi_w(\cdot) = \psi_{w0} \sqrt{\frac{\bar{r}_y - \bar{r}_y^{\min}}{r_r}} \\
&\psi_2(\cdot)\, s_2(\cdot) = \psi_w(\cdot)\, \rho_{12}\, s_1(\cdot)\, s_2(\cdot) , && \bar{r}_y^{\min}(r_r) = - r_r \ , \\
&\psi_3(\cdot)\, s_3(\cdot) = \psi_w(\cdot)\, \rho_{13}\, s_1(\cdot)\, s_3(\cdot) , && \theta_2 \geqq - \theta_1 \ .
\end{aligned}
\tag{3-4a}
$$

The parameter values given below are the same as the ones for the CIR model except for the modified relative risk aversion of consumers.

$$
\begin{aligned}
&\kappa_1 = 0.2, \quad \theta_1 = 0.015, \quad \sigma_1 = 0.5 \\
&\kappa_2 = 0.2, \quad \theta_2 = 0.035, \quad \sigma_2 = 0.1 \\
&\sigma_3 = 0.5, \quad \rho_{23} = 0.5, \quad \rho_{12} = 0 \\
&\rho_{13} = 0, \quad \lambda = - 0.01, \quad \psi_{w0} = \frac{- \lambda\ \theta_1}{\sigma_1^2 \sqrt{(\theta_2 + \theta_1)\ \theta_1}}
\end{aligned}
\tag{3-4b}
$$

The constant parameter of the modified relative risk aversion of consumers, ψ_{w0}, has been obtained from setting the market premium of the real interest rate risk in the CIR model, namely $\psi_1(\cdot)\ s_1(\cdot) = - \lambda\ r_r$, equal to the expression in the first line of equation (1-34) upon substituting the long-run euqilibrium values for the respective variables. For the parameter values given in equation (3-4b), $\psi_{w0} = 0.0219$.

The nominal instantaneous spot interest rate according to equation (1-9) becomes now the following expression upon substituting from equation (3-4a).

$$
\begin{aligned}
r_n &= r_r + m_3(\cdot) + \psi_3(\cdot)\, s_3(\cdot) - s_3(\cdot)^2 \\
&= (r_r + \bar{r}_y) \left[1 + \psi_{w0}\, \rho_{13}\, \sigma_1\, \sigma_3 - \sigma_3^2\right] \\
&= (r_r + \bar{r}_y)\ \alpha \ , \quad \text{where } \alpha \equiv 1 + \psi_{w0}\, \rho_{13}\, \sigma_1\, \sigma_3 - \sigma_3^2 > 0
\end{aligned}
\tag{3-5}
$$

Note that the nominal instantaneous spot interest rate may be less than the real instantaneous spot interest rate for an “expected” *deflation* rate. The inverse function $\bar{r}_y = \bar{r}_y(r_r, r_n)$ of equation (3-5), which is one-to-one in this case, is as follows.

$$
\begin{aligned}
\bar{r}_y &= \frac{r_n}{1 + \psi_{w0}\, \rho_{13}\, \sigma_1\, \sigma_3 - \sigma_3^2} - r_r = \frac{r_n}{\alpha} - r_r \ , \quad \text{and} \\
\bar{r}_y - \bar{r}_y^{\min} &= \frac{r_n}{1 + \psi_{w0}\, \rho_{13}\, \sigma_1\, \sigma_3 - \sigma_3^2} = \frac{r_n}{\alpha}
\end{aligned}
\tag{3-6}
$$

The arguments of the second and third volatility functions turn out to be functions of the nominal instantaneous spot interest rate solely.

The domain of the $(r_r, \bar{r}_y)$ state space is no longer the first quadrant as in the CIR model. Therefore, we use the transformed PDE (1-43) in the (r_r, r_n) state space rather than the PDE (1-6). The derivatives to be used with the transformed PDE (1-43) are as follows.

$$\frac{\partial r_n}{\partial r_r} = \frac{\partial r_n}{\partial \bar{r}_y} = 1 + \psi_{w0}\, \rho_{13}\, \sigma_1\, \sigma_3 - \sigma_3^2\,, \qquad (3\text{-}7)$$
$$\frac{\partial^2 r_n}{\partial r_r^2} = \frac{\partial^2 r_n}{\partial r_r\, \partial \bar{r}_y} = \frac{\partial^2 r_n}{\partial \bar{r}_y^2} = \frac{\partial r_n}{\partial t} = 0\,.$$

In order to reduce the first quadrant to the unit square as shown in equation (2-3), we use the hyperbolic tangent as given in equation (2-23) again. In the (r_r, r_n) state space, the drift rate of the instantaneous spot inflation rate, $\bar{r}_y$, is given by a M_1 by M_2 matrix according to equation (3-6). To make the results comparable with those for the $(r_r, \bar{r}_y)$ state space, we set

$$M_1 = M_2\,, \quad \text{and} \quad k_1 = \alpha\, k_2 \qquad (3\text{-}8)$$

from which it follows that along the diagonal $\bar{r}_y = 0$. Note that α is defined in equation (3-5). Since $[M_1\,(M_1 - 1) / 2 + M_1]$ of the mesh points refer to non-negative values of $\bar{r}_y$, we use 64 $\times$ 64 mesh points from which we get a 64 by 64 matrix of corresponding values of the drift rate of the instantaneous spot inflation rate, $\bar{r}_y$.

In a second example, we use a negative value for the long-run equilbrium value of the drift rate of the instantaneous spot inflation rate, θ_2, to account for a permanent deflationary monetary regime.

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Appendix A: List of Variables, Functions and Symbols

Variables and functions in Roman letters:

A	The $(M_1 \ M_2) \times (M_1 \ M_2)$ matrix of the system of ODEs which follows from the method of lines.
$\tilde{a}_{\cdot}$	Various coefficients of the PDE (2-2).
$a_{\cdot}$	Various coefficients of the PDE (2-4).
exp	$\equiv$ e. The exponential function.
$\tilde{F}(\tau, T, r_r, r_n)$	The spot price of a nominal pure discount bond in nominal terms in the (τ, r_r, r_n) space.
$F(\tau, T, x_1, x_2)$	The spot price of a nominal pure discount bond in nominal terms in the (τ, x_1, x_2) space.
F	The $(M_1 \ M_2) \times 1$ column vector of transformed spot prices of a nominal pure discount bond in nominal terms in the (τ, x_1, x_2) space.
ln	The natural logarithm.
M_j	Number of mesh points in the x_j-direction, $j = 1, 2$.
$m_j(\cdot)$	The drift functions of the three factors considered, $j = 1, 2, 3$. The three factors are the real instantaneous spot interest rate, the drift rate of the instantaneous spot inflation rate and the consumer price level.
$m_w(\cdot)$	The drift function of the percentage change in real wealth.
$p(t)$	The price level of consumer goods or the cost of living index, respectively, at date t.
$P_n(t, T, r_r, \bar{r}_y)$	The spot price of a nominal pure discount bond in nominal terms which is fixed and paid at the settlement date t. The debtor of the pure discount bond redeems one monetary unit when the bond matures at date T, but does not pay out any coupons during the bond's life. $P_n(T, T, \cdot) = 1$.
$P_r(t, T, r_r, p)$	$= \Pi_r(t, T, r_r)\, p(t)$. The spot price of an indexed or real pure discount bond, respectively, in nominal terms which is fixed and paid at the settlement date t. The debtor of the indexed pure discount bond redeems the price of one unit of consumer goods when the bond matures at date T, but does not pay out any coupons during the bond's life. $P_r(T, T, \cdot) = p(T)$.
$P_n^{\circ}(t, T, \bar{r}_y)$	$= \Pi_n^{\circ}(t, T, \bar{r}_y, p)\, p(t)$. The price of the "inflation" pure discount bond in nominal terms. $P_n^{\circ}(T, T, \cdot) = 1$.
$P_n^{\dagger}(t, T, r_r, r_n)$	$= P_n(t, T, r_r, \bar{r}_y)$. The spot price of a nominal pure discount bond in nominal terms in the (r_r, r_n) state space. $P_n^{\dagger}(t, T, \cdot) = 1$.
$r_n(t)$	The nominal instantaneous spot interest rate of a nominal pure disount bond in nominal terms whose price is fixed at date t and which matures at the same instant.

$r_r(t)$	The real instantaneous spot interest rate of an indexed pure disount bond in real terms whose price is fixed at date t and which matures at the same instant.
$r_y(t)$	The instantaneous spot inflation rate at date t.
$\bar{r}_y(t)$	The drift rate of the instantaneous spot inflation rate or the "expected" instantaneous spot inflation rate, respectively, at date t.
$\bar{r}_y^{\min}$	The lower bound of the drift rate of the instantaneous spot inflation rate which is determined by the condition that both the nominal and real instantaneous spot interest rates be non-negative.
$s_j(\cdot)$	The volatility functions of the three factors considered, $j = 1, 2, 3$. The three factors are the real instantaneous spot interest rate, the drift rate of the instantaneous spot inflation rate and the consumer price level.
t	Current date.
T	Maturity date.
$w(t)$	Real wealth at time t.
x_j	Transformed state variables (r_r, r_n), $j = 1, 2$.
$z_j(t)$	Wiener processes, $j = 1, \ldots, 3 + H$. The first three Wiener processes refer to the three factors considered.

Variables and functions in Greek letters:

κ_j	The speed of adjustment of the j-th state variable.
φ_j	Functions to transform the state variables (r_r, r_n) into (x_1, x_2).
$\Pi_n(t, T, r_r, \bar{r}_y, p)$	$= P_n(t, T, r_r, \bar{r}_y) / p(t)$. The spot price of a nominal pure nominal discount bond in real terms which is fixed and paid at the settlement date t. The debtor of the pure discount bond redeems the purchasing power of money when the bond matures at date T, but does not pay out any coupons during the bond's life. $\Pi_n(T, T, \cdot) = 1 / p(T)$.
$\Pi_r(t, T, r_r)$	The spot price of an indexed or real pure discount bond, respectively, in real terms which is fixed and paid at the settlement date t. The debtor of the indexed pure discount bond redeems one unit of consumer goods when the bond matures at date T, but does not pay out any coupons during the bond's life. $\Pi_r(T, T, \cdot) = 1$.
$\Pi_n^*(t, T, r_r, \bar{r}_y, p)$	The price of the "remainder" pure discount bond in real terms. $\Pi_n^*(T, T, \cdot) = 1 / p(T) - 1$.
$\Pi_n^\circ(t, T, \bar{r}_y, p)$	The price of the "inflation" pure discount bond in real terms. $\Pi_n^\circ(T, T, \cdot) = 1 / p(T)$.
$\Pi_n^\dagger(t, T, r_r, r_n, p)$	$= P_n^\dagger(t, T, r_r, r_n) / p(t)$. The spot price of a nominal pure discount bond in real terms in the (r_r, r_n, p) state space. $\Pi_n^\dagger(t, T, \cdot) = 1 / p(T)$.
$\psi_j(\cdot)$	The three market prices of factor risks in real terms associated with the nominal pure discount bond in real terms $\Pi_n(t, T, r_r, \bar{r}_y, p)$, $j = 1, 2, 3$.

$\psi_4(\cdot)$	The market price of the nominal instantaneous spot interest rate risk in real terms associated with the nominal pure discount bond in real terms in the (r_r, r_n, p) state space, $\Pi_n^\dagger(t, T, r_r, r_n, p)$.
$\bar{\psi}_1(\cdot)$	The market price of the real interest rate risk in real terms associated with the indexed pure discount bond in real terms $\Pi_r(t, T, r_r)$.
$\Psi_j(\cdot)$	The two market prices of factor risks in nominal terms associated with the nominal pure discount bond in nominal terms $P_n(t, T, r_r, \bar{r}_y)$, $j = 1, 2$.
$\psi_w(\cdot)$	The modified relative risk aversion of consumers defined as the relative risk aversion of consumers multiplied by the negative value of the partial derivative of real wealth with respect to the real instantaneous spot interest rate, $-\partial w / \partial r_r$, and divided by real wealth.
ψ_{w0}	A constant coefficient of modified relative risk aversion of consumers.
$\bar{\Psi}_j(\cdot)$	The two market prices of factor risks in nominal terms associated with the indexed pure discount bond in nominal terms $P_r(t, T, r_r, p)$, $j = 1, 3$.
$\psi_j^\dagger(\cdot)$	The two market prices of factor risks in real terms associated with the nominal pure discount bond in real terms $\Pi_n^\dagger(t, T, r_r, r_n, p)$ in the (r_r, r_n, p) state space, $j = 1, 3$.
ρ_{ij}	The correlation coefficients between the Wiener processes z_i and z_j.
σ_{ij}	Various volatility parameters.
τ	$= T - t$. The time period remaining until the maturity date of the bond.
θ_j	The long-run equilibrium values of the j-th state variable.

Symbols and operators:

$\mathscr{E}$	The expectation operator. For a variable x with probability density function $f(x)$, the expectation of x is defined as $\mathscr{E}x \equiv \int f(x)\,dx$.
$\mathscr{C}$	The covariance operator. For variables x and y, the covariance between x and y is defined as $\mathscr{C}(x, y) \equiv \mathscr{E}\{(x - \mathscr{E}x)(y - \mathscr{E}y)\}$.
$\mathscr{V}$	The variance operator. For a variable x, the variance is defined as $\mathscr{V}x \equiv \mathscr{E}\{(x - \mathscr{E}x)^2\}$.
$\mathbb{R}_+$	The non-negative real numbers.

Appendix B: The Structure of Matrix A

In this appendix, we show the structure of matrix A for $M_1 = 5$ and $M_2 = 6$.

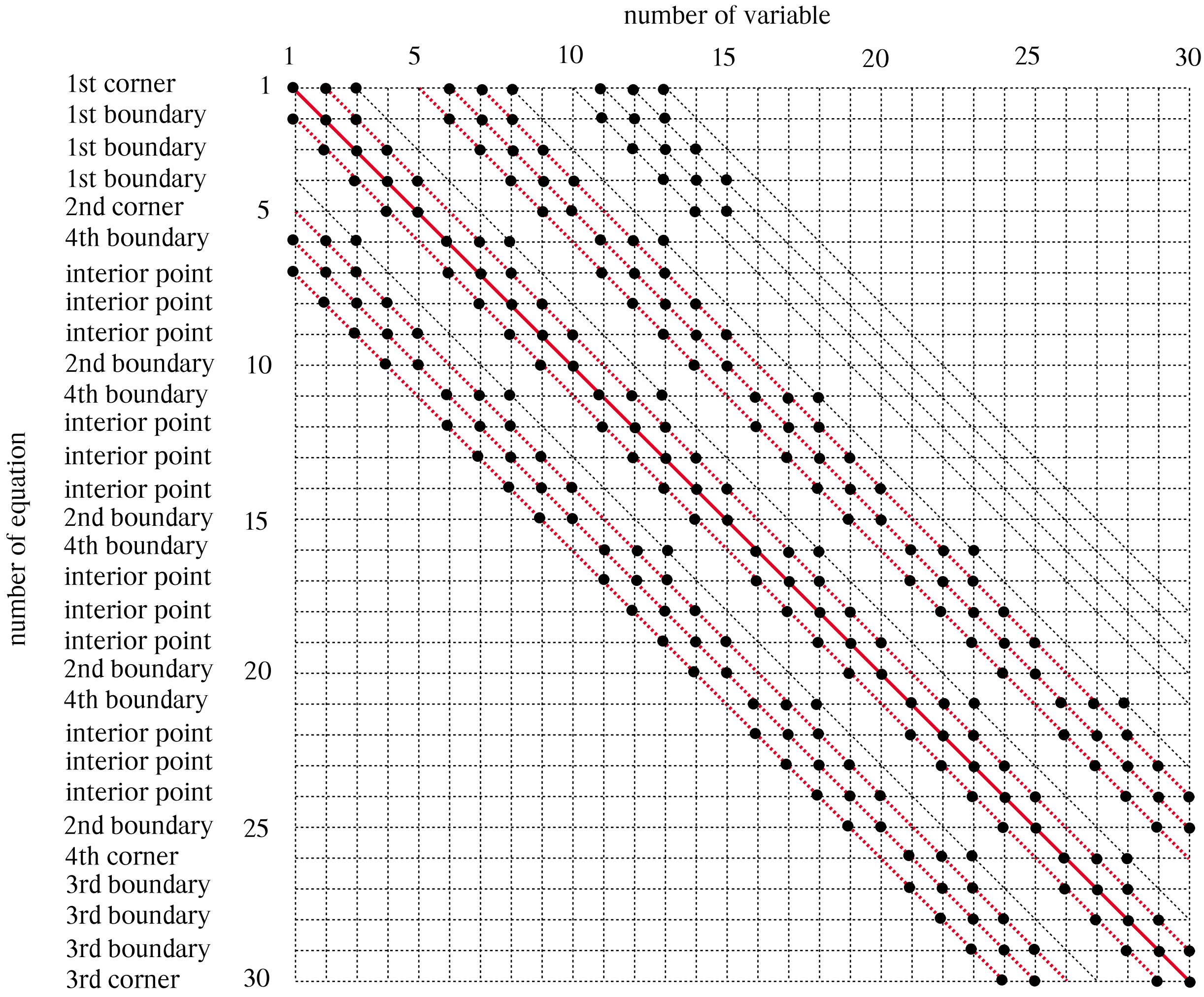

Figure B-1: The Structure of Matrix A